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A Memorial to the Founder
of the

Lockheed Aircraft
Corporation

A
NEW SYSTEM
OF
ARITHMETICK,
Theorical *and* Practical.

WHEREIN

The SCIENCE of NUMBERS

IS DEMONSTRATED

In a REGULAR COURSE from its FIRST PRINCIPLES, thro' all the PARTS and BRANCHES thereof;

Either known to the ANCIENTS, or owing to the *Improvements* of the MODERNS.

The PRACTICE and APPLICATION to the *Affairs* of *Life* and *Commerce* being alſo Fully Explained:

So as to make the Whole a

COMPLETE SYSTEM of THEORY,

For the Purpoſes of MEN of SCIENCE;

And of PRACTICE, for MEN of BUSINESS.

By *ALEXANDER MALCOLM*, A. M.
Teacher of the MATHEMATICKS at *Aberdeen*.

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M.DCC.XXX.

TO

THE RIGHT HONOURABLE

William Cruikſhank, *Eſq*;

LORD PROVOST:

James Moorifon,
John Gordon,
William Mowat,
Hugh Hay,
} *Baillies* :

Alexander Livingſton, *Dean of Guild*;
Alexander Robertſon, *Treaſurer* :

And the Remanent Members of the Town-Council of
ABERDEEN.

My LORD, &c.

THE Subject and Deſign of the following Work, with My Relation to the Town of *Aberdeen* as a publick Teacher, naturally directed me to its Governours for Patronage. As the Encouragement I have already met with from the Town, and in particular from its preſent

 Magiſtrates

Magiſtrates and Council, both with reſpect to this Work, and to my publick Employment, is but the Native Conſequence of that Noble Diſpoſition for promoting Learning and all good Education, which is the well-known Character of *Aberdeen*; ſo, I believe, I ſhould rather offend than pleaſe by attempting any Apology for this Addreſs; or running into the Common Way of flattering Dedications. I know how diſagreeable it is to Generous Minds, to hear their own juſt Praiſes: But I hope you will forgive me, if I avoid the Appearance of Ingratitude, by making this publick Acknowledgement of the Kindneſs and Civility with which You have uſed,

My LORD, &c.

Your Lordſhip's, &c. *Moſt*

Obedient Humble Servant,

ALEX^R. MALCOLM.

PREFACE.

WHEN a Subject has gone thro' so many Hands as Arithmetick has done, a new Book cannot want many Prejudices against it: and therefore to send it into the World without some introductory Account of it, is no better than laying it down at random; or, more properly, exposing it. 'Tis an unreasonable Neglect of something that equally concerns the Author and the World: for if an Author has endeavour'd to do something more useful and complete upon any Subject than has been already done, and thinks he has in some measure succeeded; as the telling the World so, may be done without any Breach of Modesty, so it appears to me equally just and necessary to explain particularly wherein the Improvements and Advantages of the Work lie; that every one may see how far it answers their Purpose, and deserves their Encouragement. It must stand upon its own Basis, no doubt; yet nothing seems more honest and reasonable than this kind of Invitation to look into it. It may be objected, I know, that here is only the Author's Word for this Account, which is a partial Testimony: But if it be consider'd, that he ventures his Credit as well as the Success of his Work, upon a fair Representation, this, I may reasonably hope, will incline the more Candid and Charitable, to believe that it is so. And upon this Hope I presume to give you the following Account of this Work.

ARITHMETICK is a Subject of that Extent, that in some Respects it can never be exhausted; and of that Value, as to deserve all the Study and Pains that can be bestow'd upon it. It is certain, there is no End in the Knowledge of Numbers; but as to a just and rational System of the Science, one would think that can't be a thing still wanted, after so many Books already written on this Subject: Nevertheless, in my Opinion, we are far from having any such thing, in our Language at least; and as to what may be in other Languages, I can only say, That I have not found it in the Books that have come to my hands.

But that I may express my Sentiments upon this Matter a little more particularly, as necessary to introduce an Account of the present Work, I shall first observe, That Arithmetick is to be consider'd in two Respects, *viz.* either in its *Theory*, which contains the Abstract and Speculative Knowledge of pure Numbers; or in its *Practice*, which contains the Application of that Theory to human Affairs. The Theory is first in order of Science; the Application supposing and depending upon it: So that there can be no Application without some previous abstract Knowledge of Numbers; that abstract Knowledge being the very thing to be apply'd. But then it is to be consider'd, that there is a great Difference betwixt understanding the Sense and Meaning of any Proposition, or of any Rule in Arithmetick, so as to be able to follow its Directions; and knowing the Reason and Demonstration of the Truth of that Proposition or Rule. Hence it is, that there are two very different ways of studying and knowing Arithmetick. The Generality who practise Arithmetick, and even many whose Business requires a Knowledge much above the more common Parts, yet understand little or nothing of the Reason or Demonstration

of

of what they do, becauſe they ſtudy not the Theory of Arithmetick, and ask no more than plain Rules for the Practice, ſo far as they have uſe for it. But others, conſidering Arithmetick as a Science founded upon Principles and Reaſon, require a Demonſtration for every thing.

Anſwerable to thoſe two different Demands, the Books of Arithmetick which we have, are of two kinds; the *Practical* and *Theorical*. The practical Books are moſt of them ſmall Treatiſes of the firſt and more ſimple Elements and Applications of Arithmetick: But beſides that they go a very ſhort way into the Science, they have alſo left us without the leaſt Reaſon for any thing they deliver, more than what is in ſome Caſes evident from the Nature of the thing: Taking all the reſt for granted, or leaving the Demonſtration to the Theoriſts.

The Theorical Writers have treated Arithmetick as a Science, by demonſtrating what they deliver: Some of them treat the Subject altogether abſtractly, without any particular Application, as *Parſons* in his *Clavis Arithmeticæ:* others with the Theory join alſo the Application, doing more or leſs in it as they have thought fit, as *Ward*, *Tacquet*, and others. Of this Claſs, again, ſome begin in the natural Order with the ſimple Elements: Others omit theſe, ſuppoſing them already underſtood, and fall in at once into a more advanc'd Theory. Such Elements of Arithmetick we have in *Euclid*'s 7th, 8th, and 9th Books; and this Method has been imitated both anciently and of late. *Tacquet* has given us thoſe three Books of *Euclid* for the Elements of Arithmetick; placing them before what he calls the practical Arithmetick, which contains the common Principles and Rules, and ſome things relating to Progreſſions, with the Extraction of the Square and Cube Roots; all very neatly explained and demonſtrated, as far as he carries the matter, [excepting one ſmall Miſtake I have occaſionally taken notice of in the following Work; and his Demonſtrations of the Square and Cube Roots, which appear to me deficient.] But I could never underſtand the Reaſon of this Order; he could not certainly mean that thoſe Elements of *Euclid* were to be ſtudied before the more ſimple Elements, which without doubt *Euclid* ſuppoſed as neceſſarily previous to his.

But of all the Works of this Claſs, I have found none which I can reckon a plain, rational and compleat Syſtem or Inſtitution of the Science of Arithmetick; either from the want of ſeveral things, even elementary and fundamental in the Science, (which is a common Fault with them all) or being too conciſe and ſhort in other things; or from ſome other Difficulty or Fault in the Method; owing, perhaps, to their particular Views and Deſigns; but which anſwers not to my Idea of the thing wanted. How unaccountable (for Example) is it in Mathematical Writers, to leave ſeveral things undemonſtrated, to ſend us to *Euclid* for others, or give us but very general and imperfect Hints of a Demonſtration? But I have done; for to be more particular, would not only be uſeleſs, but perhaps be miſconſtrued to a worſe Senſe.

FROM this general Account of Arithmetick, and the different Ways of treating it, the Thing wanted will eaſily appear to be this, *viz.* A Treatiſe, wherein the Science is deduced from its firſt Principles; and carried on with clear and accurate Demonſtration thro' all the fundamental Branches of its Theory and Practice, with the more conſiderable Improvements hitherto made in the Science; all diſpoſed according to the moſt eaſy and natural Connection and Dependence of the ſeveral Parts; hereby uniting the whole into one regular and complete Syſtem. Again, in ſuch a Syſtem Numbers muſt not only be conſider'd abſtractly, or purely as Numbers, but we muſt alſo conſider their Application to particular Subjects, that we may have a compleat Courſe of what we call the practical Arithmetick; which, beſides the more ſimple Elements of Practice, or fundamental Rules of Operation with pure and abſtract Numbers, explains the Application of thoſe Rules to the more common and ordinary Subjects of human Affairs.

Such

Such a complete and rational Syſtem of Arithmetick, accommodated to the Purpoſes both of the practical and ſpeculative People, I have endeavour'd to give in the following Work; of the Contents and Order of which, I ſhall give you a more particular Account immediately: But before this, it will be proper to make the following general Reflection upon the Syſtem of Arithmetick, both as to its Theory and Practice; which is this:

ARITHMETICK taken abſtractly, or in its Theory, being the firſt great Branch of the Mathematicks, its Application is to be found not only in the common Affairs of Life and Society, but in all the Sciences that are call'd Mathematical, (which have all their different Uſes in Society.) But then obſerve, that it is not to be expected that a Courſe of Arithmetick ſhould explain ſuch Applications as require the Knowledge of other Sciences; for then we ſhould be obliged to bring into it all the Mathematical Sciences; ſince to underſtand its Application to the Subjects of theſe Sciences, does neceſſarily require our underſtanding the Principles of them. For Example, if 'tis propoſed to find what Part or Parts any leſſer Sphere is of another, the Lengths of their Diameters being known? This is a Queſtion ſolvable by Arithmetick; yet the Reaſon of the Rule goes farther than Arithmetick, for it depends upon Geometry, *viz.* upon that Geometrical Truth, that Spheres are to one another in Proportion as the Cubes of their Diameters; and ſo belongs to Arithmetick only as this is applicable in Geometry, and ſuppoſes the Knowledge of this Science.

From this it is evident, that the Applications proper to be explained in a Syſtem of Arithmetick, are only ſuch as relate to the more ordinary Affairs of Life, which require the previous Knowledge of no other particular Science, and depend immediately and directly upon the Conſideration of the Numbers of things, and ſome other common Circumſtances. Such are all the ſimple Applications of the common fundamental Operations of Arithmetick, either in Whole Numbers or in Fractions; and the Applications of the general Rules of Proportion in the common Subjects of Trade and Commerce: For in all this there is no more requir'd, but a careful Attention to the Senſe of the Queſtion, and the true Effect of the Rules of Arithmetick.

Again it is to be obſerv'd, That as the Theory of Arithmetick is an abſtract Science, independent of all thoſe Subjects to which it may be apply'd, it is therefore neceſſary that we have a complete Syſtem of the Theory of Numbers, conſider'd purely and abſtractly by themſelves; this being preſuppoſed in the Solution of all Queſtions in other Sciences, which have any Dependence upon Numbers. The next thing I obſerve here, is, That tho' there be many Truths diſcovered in the Theory of Arithmetick, of which there has been no Uſe or Application yet found, this is no reaſon why thoſe things ſhould be neglected or kept out of the Syſtem; they are ſtill a Part of the Science, which we ought to enlarge more and more, as far as we can: One Age may find the Uſe of the Theory which a former has invented; as undoubtedly has been the Caſe, with reſpect to moſt part of the Theory both of Arithmetick and Geometry. I ſhall but add this one thing more, *viz.* That tho' many things in the Science of Numbers were ſuppos'd to be of no particular Uſe in human Affairs, yet as the Mind of Man is made for Knowledge and Contemplation, and the Pleaſure ariſing from the Perception of Beauty and Order in other things, is allow'd to be worthy of rational Natures; the Contemplation of the ſurprizing Connections, the beautiful Order and Harmony of Relations and Dependencies found among Numbers, is not leſs reaſonable: And if to this be join'd the vaſt Extent of the Uſe and Application of Arithmetick, the Reaſonableneſs and Neceſſity of explaining the Theory of Numbers ſo largely as I have done, will eaſily be allow'd.

I ſhould now come to the Contents of the following Work, but as ſome particular Circumſtances oblige me to take notice of two late and well-known Authors, Mr. *Hill* and Mr. *Hatton*, I ſhall firſt diſcuſs what I think neceſſary to ſay as to their Works.

Mr. *Hill*'s Book, which he calls *Arithmetick in Theory and Practice*, is remarkable chiefly for the very uncommon Recommendation it carries with it from a very conſiderable Maſter. We are told by Mr. *Ditton*, That take this Author purely as an Arithmetician,

metician, he has not only done more and much better than *Wingate*, *Cocker*, *Leyburn*, or any other of the Writers in our Tongue, but indeed all that can be done by Arithmetick; and therefore (says he) if no other Book on this Subject comes out till this Performance is really mended, I'm satisfy'd we shall have no new Book of Arithmetick very soon.

Now here was such a Defiance, and from such a Hand, that they were bold enough who ventur'd first to write after it, and even without the least Apology, or Notice taken of this Challenge, as several have done: Whatever Reason others thought they had for such a Conduct, I thought it necessary for my own Vindication to make the following Remarks on this Book and its Recommendation.

That Mr. *Hill* has several things that are not common, I do acknowledge; but for his having done much better than all that went before him, 'tis not my Business to determine; what I'm concern'd in chiefly, is the Assertion of his having done all that can be done by Arithmetick, a thing I was much surpriz'd to hear from so good a Judge as Mr. *Ditton*; and because if this be true, then what I offer to the World must be either impertinent or superfluous; it can't be thought out of my Road if I enquire a little into the Truth of this Assertion. By the mention Mr. *Ditton* makes of *Algebra*, it appears to me, that he would have nothing admitted into Arithmetick that is any way owing to *Algebra*. Now, supposing this were reasonable, yet the Book in question will still be found both very defective in what it ought to contain to answer so great a Character, and also to have many things that belong not to a pure Treatise of Arithmetick. In the first place, with what Truth and Justice can it be said, That a Book contains all that belongs to Arithmetick, and that one needs learn no more, (as he also says) which, (besides that there is no Demonstration, and consequently no Science) wants many things that are fundamental and necessary, and yet do not absolutely depend upon *Algebra*; (tho' they may be made easier in many things by its help) particularly in the Doctrine of Proportion: For tho' we have here several Propositions relating to this Subject, yet we are very far from having any thing like a just and orderly Treatise of Proportion: Nor (to mention no more of its Defects) have we any of the rest of the fundamental and curious Theory of Numbers, contain'd in *Euclid*'s 7th, 8th, and 9th Books. Again, if we must exclude what is any way owing to *Algebra*, then most of what is uncommon in Mr. *Hill*, as upon *Progressions*, *Interest*, *Logarithms*, *Combinations*, and *Extraction of Roots*, do not belong to Arithmetick: And if these belong to *Arithmetick*, notwithstanding their Dependence upon *Algebra*, then so must a great many other things not to be found in Mr. *Hill*'s Book. But I have said enough, and shall leave you to judge by the following Work, whether that Book contains all that can be done by *Arithmetick*, and consequently what to think of this extraordinary Recommendation, which indeed is more faulty than the Book itself.

Mr. *Hatton*'s Book, which I have here in my view, is his *Intire System of Arithmetick*: If this Book had answer'd the promising Title, my Labour had been prevented; but I could not help judging otherwise of a Book that not only leaves us without Demonstration in most things, and sometimes gives us a mere Proof of a particular Example, instead of a general Demonstration; but which, in a word, comes very far short both of the Contents and Order due to an *Intire System*. As I'm no further concern'd in the Criticism of another Man's Work, than it is necessary to vindicate my own, I shall be content with this general Reflection upon this Work, and leave it to an impartial Comparison to justify what I have alledg'd, and determine whether there was not yet wanting a *more Intire System*. There is one thing more I must say here, *viz.* That as I think it is every one's business who writes upon any Subject, to discover the Errors (especially if they are of any consequence) committed by others, so I hope there will be no Misconstruction made of my Design, in exposing some Errors I have found in this or in any other Author: It is the Treatment I expect my self, and shall receive without complaining.

plaining*. But I'm oblig'd for the sake of the Publick, to observe here, that as Mr. *Hatton* has found fault with Dr. *Harris*'s Rule for the discompting of *Simple Interest*, for Money paid before it is due; so he ought in justice to have told the World, That the Tables of Discompt in his own *Index to Interest*, printed *Anno* 1711, are calculated by the same false Rule; that no body may be longer impos'd upon by them. And that you may not think the Consequence inconsiderable, take this Example: The Discompt of 1000 *l.* paid 90 Days before due, is by his Tables *l.* 13.9571, the Discompt being at 6 *per Cent.* whereas, according to the True Rule, the Discompt ought to be *l.* 14.57882

I proceed now to a more particular Account of the following Work, which is divided into six Books.

BOOK I.

In this Book I have largely explain'd and demonstrated the first simple Principles and fundamental Operations of Arithmetick in *Integers* or *Whole Numbers*: In which, after the Principles and Rules for the Management of pure and abstract Numbers, I have separately explain'd the Use and Application of these Rules to particular Subjects, such as occur in human Affairs.

BOOK II.

Here you have fully handled the Doctrine of *Fractions*, where I have first explain'd (in a way I think very easy and demonstrative) the general Nature and Theory of Fractions, as a necessary Foundation for understanding the Reason of the Practice; which I have next fully explain'd both in *Vulgar* and *Decimal Fractions*, as they are distinguish'd. Only what we call *Infinite* or *Circulating Decimals*, are referr'd to *Book* 5. for the sake of the Demonstration.

Observe, As these two Books contain the first and fundamental Principles and Rules of Arithmetick; and as the right understanding of the Foundations of any Science is of great Importance, I have therefore enlarg'd and improv'd every Part almost with such particular Explications and Rules, as will, I hope, be of great use for attaining to a just and perfect Idea of this Science in its Fundamentals, and a more masterly Practice.

BOOK III.

Contains the Doctrine of the *Powers* and *Roots* of Numbers; wherein I have first particularly explain'd, the Nature and Theory of those Numbers call'd *Powers* and *Roots*. After this you have the Rules for raising or forming *Powers*, and *Extracting Roots* in Integers and Fractions, where I have explain'd Sir *Isaac Newton*'s famous Rule call'd the *Binomial Theorem*, and some other curious things relating to the *Extraction* of *Roots*. You have here also what is called the Arithmetick of *Surds*, which contains a more particular Application of the preceding Theory to *Roots*, especially those called *Surds*. Lastly, you have all the Propositions of the 2[d] Book of *Euclid*, which are applicable to *Numbers*, with some others of the same kind.

Observe, As to the Contents of this Book, that excepting the common Rules for extracting the *Square* and *Cube Roots*, all the rest of this curious Branch of Arithmetick is to be found only in our Books of *Algebra*: because the Use of it is above the common Affairs of Life, and occurs chiefly in the higher Applications of the *Algebraick* Art; and also because the Demonstration of it can hardly be made without the help of *Algebra*. But as it is directly and immediately a Part of the Theory of Numbers; which does indeed no otherwise belong to a Treatise of *Algebra*, than any other thing in Arithmetick,

* *The Particulars I have censured in Mr.* Hatton's *Book, you'll find in* Book 6. Chap. 6. Quest. 6. *and in* Chap. 11.

which may be demonstrated by the help of the *Algebraick* Method. Also, since I have taken that Method of Demonstration (of which I shall give you a particular Account afterwards) I have therefore given it its due place in the *System of Arithmetick*. I must also observe, that tho' the Writers of *Algebra* have taken this Part of the Theory of *Arithmetick* into their Province, yet it is not, in our Language, treated so fully and particularly as it ought to be; many things being left without Demonstration that to me seem far from being self-evident.

For the Extraction of *Roots*, especially those above the *Square* and *Cube*, there are easier Methods, owing also to the *Algebraick* Art; but as they exceed the Limits prescrib'd to this *System*, they must therefore be sought elsewhere.

BOOK IV.

Contains the *Doctrine of Proportion* in all its Branches, as distinguished into *Arithmetical*, *Geometrical* and *Harmonical*. In each of which, as I have endeavour'd to make the fundamental things clear and plain, so I have omitted nothing worth knowing, in this great and useful part of Arithmetick, that I could any where find, or that my own Study could furnish: Whereby, as you have all that our common Books contain, so you have many other things to be found only in such Authors as are not in every body's hands; and many things intirely new, for what I know. And, in both those last two kinds, besides what is mix'd here and there, there are some more considerable Additions; particularly upon *Arithmetical Progressions*, in *Chap.* 2. §. 2 All that is from *Schol.* 2. (after *Probl.* 2.) is intirely new. The *Chapters* 5, and 6. with the *Appendix* to this Book, contain things uncommon, and for the most part altogether new; (see *Contents* more particularly.) So that I dare presume to say, you have here a more compleat System of the Doctrine of Proportions than can be found else where, in our Language at least.

As to the Subject of *Chap.* 6. which is *Harmonical Proportion*, I have this Observation to make, That as Musick in its first Principles depends altogether upon Numbers, so the Knowledge of the Application of Numbers to Musick, which I may call the Arithmetical Theory of it, is so very useful and entertaining, that 'tis pity it were so little understood, as I doubt it is, both by the Practisers and Lovers of Musick. What was proper or necessary to be done in this Work, concerning that Application, I have done it; and if any one wants a particular Treatise upon this Subject, they will find it in a Book call'd, *A Treatise of Musick, Speculative, Practical, and Historical*; which is to be found with the Booksellers to whom the present Work belongs.

BOOK V.

This Book is a Miscellany of various things, which are not comprehended under one common Name; and consists of VI. Parts, in as many different Chapters; whose Contents are as follow.

1. The Doctrine of *Prime* and *Composite* Numbers; a fundamental and curious Branch of the Theory of Arithmetick.

This is a great Part of the Doctrine of *Euclid*'s 7th, 8th, and 9th Books of *Elements*; which contain, besides, many things relating to the Doctrine of *Proportion*; but those I have put in their due Place with the rest of that Doctrine, which is not so complete in *Euclid* as it has been made since: but as my Method of Demonstration is generally different from his, (tho' in some things there can't be a better than his, and perhaps no other;) so I have not only deliver'd this Theory in a different, and, as I think, a more natural Order; but by means of the *Algebraic* Method, I have gain'd these Advantages, *viz.* That several things are made *Corollaries* to others, which have a sufficiently tedious Demonstration in *Euclid*. Again, several Propositions are made universal, which are limited

mited in *Euclid* to a few particular Cases: And in others, which can be prov'd only by an Induction of Particulars, I have made the Universality of the Induction more clear and evident, by another Method of Reasoning.

There are here also many things which are not in *Euclid*; Part of which I met with in some rare Books, and others occurr'd to my own Study and Observation; particularly the 3[d] Section is intirely new. I shall mention but one thing more, that is, a *New* and very Easy Way of finding all the *Prime* and *Composite* Numbers within any given Limit: of which I have given an Example, extended only to 999. The Form of the Table in which they are collected, is much the same with that in Dr. *Pell's* Edition of *Brancker's Algebra*; tho' the Rule by which I have composed it be vastly more easy than what's given there.

2. The curious Theory of *Figurate Numbers*; a thing but just touch'd upon in any *English* Book, of my Acquaintance. I have met with more of it in some others, but either without Demonstration, or so much out of my Method, that I could make no use of it. And here the Advantage of the *Algebraick* Method was manifest, by which, several of those things are very simply and easily demonstrated, that otherwise have a very difficult and tedious Demonstration: and without which other things could not, I doubt, be demonstrated at all. To the same means also I owe several things here, that I found in none of my Authors; whereby I have carried this Part further, and, by putting the whole together in a just Order, have given it a more perfect Form than I have any where found it in.

Here you have a new *Canon* for the *Coefficients* of the *Powers* of a *Binomial* Root; and several curious Propositions, relating particularly to Square Numbers: With Rules for summing the Series of the Squares and Cubes of the *natural Progression* of Numbers 1. 2. 3. 4. *&c.* without actually raising these Powers and adding them together; and a Method of raising *Canons* for summing any of the higher Powers.

As to the Use of this Part, whatever else it may be (which in *Mathematical* Affairs is considerable) you have thro' the whole, remarkable Examples of what I formerly mention'd, *viz.* Of beautiful and surprizing Order and Connection among Numbers.

3. *Of Infinite Series* of Numbers; particularly of decreasing Geometrical *Progressions*, (some useful Applications of which you have in the next Chapter) and of those *Encreasing* Series, which are the chief and fundamental things of what the Mathematicians call the *Arithmetick* of *Infinites*; of which they have made a noble Use in *Geometry*; having hereby particularly found many useful practical Rules, for the Mensuration of *Solids* and gauging of Vessels. What I have done here belongs properly to *Arithmetick*. The Application of it to *Geometry* you'll find in *Sturmy's Mathesis Enucleata*, or *Ward's Introduction*. But the whole Doctrine and Application at large, is to be sought from the celebrated Author of it, Dr. *Wallis*.

4. The Theory and Practice of *Infinite* or *Circulating Decimals* (referr'd to this Place for the sake of the Demonstration) which, with what is already done in *Book* 2[d], *Chap.* 2[d], makes a compleat System of *Decimals*.

Dr. *Wallis* is probably the first, as he has himself observ'd, who has distinctly consider'd this curious Subject of Circulating Decimals. He has given us the fundamental Theory of it, but without Demonstration; nor has he meddled with the practical Part, or Way of managing Infinite *Decimals* in Arithmetical Operations. Mr. *Brown*, in his Decimal Arithmetick, has handled but one single Case of the Practice, and that not completely neither. Mr. *Cunn* (who is the last Author I know upon this Subject) in his *Treatise of Fractions*, has in his way given us all that Dr. *Wallis* says upon the Theory, yet without any Demonstration, and a few other obvious things, tending more immediately to the Practical Part; which he has handled at full length, giving us Rules for all Operations and all Cases: But as he demonstrates none of those Rules, (which perhaps he reserv'd for another Work) he has also chosen to express them in such a manner, as to set the

Reason as far out of view as possible, which has this Effect, that in the Rules of Multiplication and Division (which are the more complex and difficult Parts) his Directions are not so easily follow'd; and are besides much harder for the Memory than the Method I have chosen, which depends all upon the easy and natural Explication of one single Proposition, *viz.* The finding the finite Value of (or Vulgar Fraction equal to) any circulating Decimal: for tho' the Demonstrations are omitted, the Rule ought to be as simple and easy as possible. But I must observe this further Effect of Mr. *Cunn's* Way of delivering these Rules, That by themselves one could never, or very hardly, be led into the Reason of them, nor consequently into the way I have chosen; so that it will be the more easily believ'd that the Rules I have given, are the Effect of Speculations made upon this Subject, before I saw this Book; which I mention for this Reason only, that I may not be thought ungrateful to one whom I acknowledge the first Author upon this Practice, from whom therefore I might otherwise be supposed to have borrowed or deduced all that I say; and yet I do acknowledge I owe him one or two useful Hints. I have only one thing more to add, *viz.* That his Rule for the Addition of *Circulates* having compound *Repetends* is insufficient for a general Rule; it will bring out the true Answer in some Cases, but is not universally good for all Cases: the comparing it with the Rule I have here demonstrated will shew the Difference, and the Truth of what I say.

5. The *Logarithmick Arithmetick*; wherein the Nature, Construction, and Use of those admirable Numbers call'd LOGARITHMS are explain'd and demonstrated.

The Manner of constructing or making Logarithms, which I have explained here, is that of the Noble Inventor, the Lord *Neper*, because its Demonstration is more simple and easy, tho' the work itself vastly more tedious than other Methods which have been discover'd since, by means of a deeper Application of the *Algebraick Art* than my limits allowed me to use here. My Purpose is however sufficiently answer'd; for as every one who would understand the Reason and Use of *Logarithms*, is not under any necessity of constructing them, that being often done already; so I design'd chiefly what I think is most generally demanded, that is, (1.) To demonstrate the *Origin* and *Nature* of those Numbers, or shew that there are really such Numbers to be found, as we define *Logarithms*; which could not be better or more naturally done than by the Method of the Inventor. And then, (2.) to explain and demonstrate their Use and Application; which is the same, whatever way they are calculated or constructed.

I shall say but this one thing more, *viz.* That as those other Methods of Construction are chiefly owing to Sir *Isaac Newton's* Binomial *Theorem*; so far as I have explained that *Theorem*, (which is only so far as relates to *simple* or *proper Powers*, i. e. having Integral Indexes) I have so far also made their way easy, who would study those other Rules of Construction wherein that Theorem is also apply'd to *Roots*; which Rules they will find no where more easily and fully explain'd than in *Ronayne's* Algebra.

6. Of the *Combinations* of Numbers, a Part of Arithmetick which has been but very little and generally handled by our *English* Writers; and as little by others that have fallen in my way. We have indeed most of the fundamental Propositions of it, in *Hill's* Arithmetick, yet far short of the Length I have carried it to here. As the thing is in itself curious, and not without considerable Use, especially in the Calculations of Chances, I have explain'd it the more particularly. Here also you have another Demonstration of the *Binomial* Theorem for *Coefficients*.

BOOK VI.

Contains the Application of the Doctrine of *Proportion* to the Common Subjects of Human Affairs: Wherein I have gone thro' a large and complete Course of all the Common Rules and Branches of this Application. I have labour'd to make the Rules as plain and intelligible as possible; and at the same time express them so, as the vast Ex-

Extent of their Uſe may eaſily appear; and young People may not be ſo limited in their Notions of thoſe things, as not to be able to go further than the few Examples to which they have found them apply'd in Books, or by their Teachers; or ſuch Examples as are ſtrictly of the ſame kind, and propoſed in the ſame manner with theſe: as I have often found to be the Caſe in the Courſe of my Buſineſs and Experience in thoſe Matters. The beſt Remedy of which, is to underſtand the Reaſon and Demonſtration of every Rule, and ſee the Applications of it in a great Variety of Subjects and Circumſtances. Therefore I have ſhewn the Reaſons of all the Rules by their Dependence upon the preceding Theory. But leſt any thro' neglect, or ſome other fault, ſhould not underſtand that Theory, I have here given ſome other Demonſtration of the chief and moſt uſeful of thoſe Rules. And to make the Application complete, I have given you not only a ſufficient Number of Queſtions of common Uſe and Occurrence in every Branch; but alſo a great many that are uncommon and curious, the ſtudying of which will ſerve this very uſeful Purpoſe, *viz.* to lead one to a clearer and readier Apprehenſion of the Application of the Rules of Arithmetick, and eſpecially of Proportion, which is the moſt important and difficult thing in the Practical Arithmetick.

The Applications relating to the *Intereſt* of Money being of great Concernment to Society, I have explained and demonſtrated thoſe at large. And here I found myſelf neceſſarily engaged in the Examination of a Queſtion wherein Sir *Samuel Moreland* and Mr. *John Kerſey* have widely differ'd. The Queſtion is about the Calculation of the preſent Worth of an *Annuity* to continue any number of Years, diſcompting ſimple Intereſt. Whether Sir *Samuel*'s Book, which he calls *The Doctrine of Intereſt*, wherein he finds fault with Mr. *Kerſey*'s Rule, which is in his *Appendix* to *Wingate*'s Arithmetick, was written before *Kerſey*'s Death, or whether he ever ſaw it, or gave it any Anſwer, is what I know not; but the Difference ſeem'd to me too conſiderable to paſs over. Upon the moſt careful Examination, I was determin'd to Mr. *Kerſey*'s ſide: tho' I was very ſoon afterwards ſurpriz'd to find my Opinion contradicted by the Mathematical Writers, who have taken the other ſide, and form'd their Rules upon *Moreland*'s Foundation; as particularly by *Ward*, with this Remark, that "*Moreland* has detected ſeveral material "Errors committed by *Kerſey* upon *Wingate*." This put me upon a more narrow Examination of the Queſtion, which ended in a further Confirmation of my former Opinion; and yet what Mr. *Ward* and others have done merely as Mathematicians, is right; they have aſſum'd a certain State of the Queſtion, and upon that raiſed Rules which are good upon the Juſtice of that State of the Queſtion, but not otherwiſe: and therefore in the Defence of Mr. *Kerſey*'s Rule, I differ from *Ward*, and others who have taken that Method, not as to any pure Mathematical Truth in Numbers, but merely as to a point of Right and Equity, in the manner of ſtating a Queſtion betwixt Man and Man, according to the Conditions previouſly agreed to: But that every body may judge and chuſe for themſelves, I have given the Rules and Reaſonings upon both ſides.

As to the Rule of *Poſition*, or *Falſhood*, which is common enough in Books of Arithmetick, I have omitted it, becauſe it is of little or no Uſe in real Buſineſs, and very limited in its Application: Beſides, whoever has the leaſt Smattering of the Algebraick Method of ſolving Queſtions, can do all, and much more, than this Rule teaches.

Of the Method of Demonſtration uſed in the following Work.

I have every where endeavour'd to take the moſt eaſy and natural way the thing would admit of. In the firſt and ſimple Parts there is but one way almoſt to be taken; but in the more complex and difficult Parts, as there is room for a Choice, I have generally uſed the *Algebraick* Method, as what is natural and proper to *Arithmetick*, and the moſt eaſy and plain Method that can be uſed in moſt Parts of this Science, and without which many uſeful and curious things could not be demonſtrated. I have not ſuppoſed the Student of Arithmetick already acquainted with Algebra; but have gradually explain'd

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the Principles and Rules of it, as far as my Purpose requir'd. As *Algebra* is nothing else but an universal Method of representing Numbers, and reasoning about them, so it very naturally belongs to Arithmetick: And in the Opinion of the Great Sir *Isaac Newton*, who calls it the *Universal Arithmetick*, makes, with what in distinction from it he calls the *Vulgar Arithmetick*, but one complete Art of Computation. But my Design not reaching to a complete System of Arithmetick in this larger Sense, I have done no more as to *Algebra* than is necessary for demonstrating the System of Arithmetick in the more strict Sense.

I have indeed been ask'd, why any thing is brought into a Treatise of Arithmetick, which stands in need of the *Algebraick* Art, or can be better done by that means than otherwise, and not rather referr'd to a Treatise of *Algebra*? The Answer was obvious, *viz.* That wherever these things are placed, they belong to the System of Arithmetick: And for the *Algebra* requir'd to the Demonstration, if one has already learnt it in a more express and particular Study of that Art, it is well; but if not, 'tis just as proper and easy to learn it in a Course of Arithmetick, as naturally belonging to this Science. And if it is again ask'd, Why then I have not extended this Work to all the Parts of the *Algebraick* Art, and thereby made a System of Arithmetick more complete? I answer, That having the Choice of my Subject, I have given it such limits as I thought convenient, and done such a Work as I thought was most wanted: Those who incline to make a more particular Study of the *Algebraick* Art, must seek it elsewhere. But if what is done here, both as to the Principles and Application of *Algebra*, be well understood, it will, I believe, prove an useful Introduction to the higher Parts of this admirable Art, and a powerful Incitement to the further Study of it; when it is consider'd, how the most simple Elements of it are sufficient for acquiring such a Knowledge of Arithmetick as can't be obtain'd without it in many things, and in others not without much greater difficulty.

I hope then there will no Discouragement arise from a Prospect of Difficulty in this Method, by such as are willing to study Arithmetick in a reasonable manner: For tho' there are difficult and abstruse things in the *Algebraick* Art, yet all the Principles and Rules of it used in this Work, are in effect no more than a particular kind of Language; or rather a compendious way of representing and comparing Numbers and the Effects of their Operations: which may be learnt with a little pains, in two Lessons, or three at most; and as they are explain'd and apply'd by degrees, it will become easy and familiar as you proceed.

For those who would study only the Practical Part, without the Theory and Reasons of Things; they will find what they want in the first, second, and sixth Books, with the second Chapter of Book III. In all which, let them pass over the *Demonstrations*. And if they would go further, they may read the *Problems* in the fourth and fifth Books.

It remains that I explain the Meaning of a few Names used in the following Work, for different kinds of Propositions.

A DEFINITION is the Explication of the Use and Meaning of any Word or Term of Art; as of this itself and the following.

An AXIOM is a Proposition whose Truth is self-evident.

A THEOREM is a Proposition whose Truth is to be demonstrated.

A LEMMA is a Proposition to be demonstrated; and which is premised to some other, to serve as a Principle for the more easy Demonstration of this other.

A PROBLEM is a Proposition of something to be done or discover'd.

A COROLLARY is a Proposition gain'd in consequence of another, whose Truth is evident from the Truth or Demonstration of that other.

A SCHOLIUM is some further Explication relating to what precedes.

Observe, In the *Demonstrations* of the following Work, when any former Proposition is quoted, it's understood to be in the same Book and Chapter in which it is quoted, unless it is otherwise expressed.

A SHORT HISTORY OF ARITHMETICK.

THAT Arithmetick was very early in the World, no body can doubt, becaufe the Idea of Number arifes from all things about us. In the beginning, while the Way of Living was fimple, and things were in a manner common, the Knowledge of Numbers would make a fmall Progrefs: But when *Property* and *Commerce* began to be eftablifhed, Men would foon find the Neceffity of enquiring into the Nature of Numbers, and contriving an *Art* of *Numbering*; without which no Bufinefs can be carried on. This was, no doubt, very rude at the firft, and improved by degrees; as all our Knowledge is: But where, and by whom, Arithmetick received its firft Form of an *Art* or *Science*, we know little about it. If the *Phœnicians* were, as it is conjectured, the firft Merchants after the Flood, (and before that we know nothing of the Affairs of Mankind) then it is probable, the *Art* began among them; by whom *Trade* and *Arithmetick* were carried into *Egypt*; and here, 'tis thought, began the *myftical* Application of Numbers: For the *Egyptians* explained every thing by thefe; the Nature of the *Gods*, of *Human Souls*, the *Virtues*; in fhort, for every thing *divine* and *human*, they found fome Symbol or Reprefentation in Numbers: Hence we hear of the wonderful Virtues and Properties of fome particular Numbers, as *One*, *Two*, *Three*, *Four*, *Seven*, and *Nine*. From *Egypt* this Knowledge paffed into *Greece*, which added its own Improvements to the myfterious Part; of which a great deal is to be feen in *Plato*; the Life of *Pythagoras* by *Jamblichus*; and more lately in the Commentators upon *Boethius*'s Arithmetick. Now we are come to the Country where we may expect to find the firft diftinct Rudiments of the Science.

The firft thing Men were obliged to do to make their *Ideas* and Knowledge of Numbers ufeful in Society, was to eftablifh fome Method of *Notation*, and then upon this found an Art of *Computation*: after this they would gradually enquire into the *Relations* and *Properties* of Numbers; and fo the Science went on.

The *Greeks*, *Hebrews*, and other Eaftern Nations, ufed a Notation by the Letters of their Alphabet. The *Greeks*, particularly, had two different Methods; the firft was much the fame with the *Roman* Notation, explain'd in *Chap.* 2. *Book* 1. of the following Work, which is a very imperfect Method. Afterwards they had a better Method, in which the firft nine Letters of their Alphabet reprefented the firft Numbers from One to Nine, and the next nine Letters reprefented any Number of Tens from One to Nine, that is, 10, 20, 30, *&c.* to 90. Any Number of Hundreds they expreffed by other Letters,

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supplying what they wanted with some other Marks: And in this Order they went on, using the same Letters again with some different Marks to express *Thousands, Tens of Thousands, Hundreds of Thousands*, &c. As to this Method, 'tis to be observ'd, that they were upon the very Point of discovering the *Arabian* Notation: For, as they made the Progression to 9, they wanted but one Step further, *viz.* Instead of using other 9 Letters, to make the same 9 change their Values in a decuple Progression according to their Places, which would in course discover the Necessity of a Character that of itself signifies Nothing, only fills up a Place.

The Manner of their *Computation*, (*i. e.* of *Addition, Subtraction*, &c.) and the Difficulty of it, especially in great Numbers, we may easily discover from the *Notation*. As to any express Treatises upon the *Art* of *Computation*, they have left us none. There is a Commentary by *Eutocius*, upon *Archimedes*'s Treatise of the Dimensions of a Circle; and some Fragments of *Pappus*, in Dr. *Wallis*'s Works, which relate particularly to the Work of *Multiplication*, and shew us the great Difficulty of their Practice, owing to the imperfect Notation.

The most perfect Method of *Notation*, which we now use, was owing to the Genius of the *Eastern* Nations; the *Indians* being reckoned the Inventors of our Notation; which we call the *Arabian*, because we had it from them, and they from the *Indians*, as themselves acknowledge. When the *Indians* invented this Method, and how long it was before the *Arabs* got it, we cannot tell: These things only we know, 1. That we have no ground to believe, the antient *Greeks* or *Romans* knew any thing about it: For *Maximus Planudes*, the first *Greek* Writer who treats of Arithmetick according to this Notation, lived about the Year of Christ 1370, as *Vossius* says; or about 1270, according to *Kircher*; long after the *Arabian* Notation was known in *Europe*: And owns it for his Opinion, that the *Indians* were the Inventors, from whom the *Arabs* got it, as the *Europeans* from them. 2. That the *Moors* brought it into *Spain*; whither many learned Men from other Parts of *Europe* went to seek that, and the rest of the *Arabick* Learning (and even the *Greek* Learning, from *Arabick* Versions; before they got the Originals themselves) imported there by the *Saracens*. As to the Time when this new Art of Computation was first known in *Europe*, *Vossius* thinks it was not before the Year 1250; but Dr. *Wallis* has, by many good Authorities, proved that it was before the Year 1000; particularly that a Monk called *Gerbertus*, afterwards Pope by the Name of *Sylvester* II. who died in the Year 1003, was acquainted with this Art, and brought it from *Spain* into *France*, long before his Death. The Doctor shews also, that it was known in *Britain* before the Year 1150, and brought a considerable length, even in common Use, before 1250, as appears by the Treatise of Arithmetick of *Joannes de Sacro Bosco*, who died about 1256.

Tho' the numeral Figures which we now have are a little different from what the *Arabians* use, having been changed since they came first among us; yet the *Art* of *Computation* by them is still the same.

Having said all that's necessary about the *Notation* of *Numbers*, we shall go back again, and see what kind of *Science* of *Arithmetick* is to be found among the Antients, with the Progress of it till now.

The oldest Treatise extant upon the *Theory* of *Arithmetick*, is *Euclid*'s 7th, 8th, and 9th Books of Elements; wherein he gives us the Doctrine of *Proportion*, and that of *Prime* and *Composite* Numbers. Both of which have received Improvements since his time, especially the former. The next, of whom we know any thing, is *Nicomachus* the *Pythagorean*, who wrote a Treatise of the Theory of Arithmetick, which consisted chiefly of the Distinctions and Divisions of Numbers into certain Kinds and Classes, as *Plain* and *Solid*, *Triangular*, *Quadrangular*, and the rest of the Species of *Figurate* Numbers (as they called them) Numbers *Odd* and *Even*, &c. with some of the more general Properties of the

the several kinds. As to the time in which *Nicomachus* lived, some place him before *Euclid*; others long after. His Arithmetick was published at *Paris* 1538. What kind of Work it is, we may guess by the *Latin* Treatise of Arithmetick of *Boethius* the Philosopher, who lived at *Rome* in the time of *Theodorick* the *Goth*; and is the next remarkable Writer extant upon this Subject. He is supposed to have seen and copied most of his Work from *Nicomachus*.

From this Work of *Boethius*, with a few small Abstracts of the same nature, made very long after his Time, as that of *Psellus*, and *Jodochus Willichius*, both in *Greek*; some have said that the antient Arithmetick consisted of nothing else but these Divisions and Distinctions of Numbers. I confess I was surprized to find this Account from such an Author as *Wolfius*, to whom *Euclid* is no Stranger; whose Books contain things much more important in the Science of Arithmetick than these Distinctions; and want many of them, that are in *Boethius*: For *Euclid* speaks nothing of the *Figurate Numbers*, and their various Species and Classes; except what relates to Squares and Cubes. And, on the other hand, *Boethius* has very little of *Euclid*'s Doctrine.

We must come next to the Times when the *Arabian* Notation was known in *Europe*; after which we find many Writers both upon the *Theory* and *Practice*. The eldest of them, who is very considerable, is *Jordanus* of *Namur*, who flourish'd about 1200. His *Arithmetick* (from which I have taken several things) was published and demonstrated by *Joannes Faber Stapulensis* in the fifteenth Century, (who has given us himself a Compendium of *Boethius*) soon after the Invention of Printing. It's altogether upon the Theory; and contains most of what *Euclid* and *Boethius* have, and many other curious Theorems. The same Author wrote also upon the new Art of Computation by the *Arabick* Figures, and called this Book *Algorismus Demonstratus*; the Manuscript of which, Dr. *Wallis* says, is in the *Savilian* Library at *Oxford*. But it has never been printed, as I know.

As Learning advanced in *Europe*, so did the Knowledge of Numbers; which by degrees received large Improvements both in the *Theory* and *Practice*, owing in a great measure to a more perfect Notation. To trace out every Step in that Improvement, is impossible; therefore I shall only name a few of the remarkable Writers after *Jordanus* and *Sacro-Bosco*, both named already. As to the Writers, these were most remarkable in *Italy*, viz. *Lucas de Burgo*, about the Year 1499, whose Arithmetick, which is both Theorical and Practical, Dr. *Wallis* commends much: *Nicholas Tartaglia*, whose Work is chiefly *Practical*. He is called by some the *Prince* of the *Practitioners*; which must be understood only for his own Times. In *France*, there were *Clavius* and *Ramus*. In *Germany*, *Stifelius* and *Henischius*. In *England*, *Buckley*, *Diggs*, and *Record*. All these, and many more, were before the Year 1600. But since that, our Writers are almost innumerable.

As to the Improvements made since the *Arabick* Notation was known in *Europe*; besides many things in the Theory, particularly in the Nature of *Progression*, both Arithmetical and Geometrical, in the Nature of *Powers*, and in the *Extraction of Roots* and the *Combinations of Numbers*, which we do not so well know the History of; there are a few very considerable Improvements, in the *practical* Part, of which we can give a better Account. But that I may connect the Antient and Modern History, we must go back to the second Century of *Christianity*, in which *Claudius Ptolomeus* lived, who is supposed to be the Inventor of the *Sexagesimal Arithmetick*; which was a new Method of *Notation*, and consequently of Computation, designed to remedy the Difficulty of the common Method, especially with regard to *Fractions*. The Nature of it was this: Every Unit was supposed to be divided into 60 Parts, and each of these Parts into 60 Parts, and so on; hence any Number of such Parts were called *Sexagesimal* Fractions. And to make the Computation in Integers also more easy, he made the Progression in these also *Sexagesimal*. Thus, From one to fifty-nine were marked in the common way; then sixty was called a *Sexagena prima*, (or first *Sexagesimal* Integer) and marked with the Sign of *Unity*

and one single Dash over; so sixty was thus expressed I′. Two sixtys, or 120, thus II′; and so on to 59 times 60, (or 3540) which is LIX′. Then for 60 times 60, (or 3600) this he called a *Sexagena secunda*, (or second *Sexagesimal* Integer,) and marked any Number of them less than 60, by the Signs of Numbers less than 60, with two Dashes: Thus, 60 times 60 (or 3600) was marked I″; two times 3600, thus II″; ten times 3600, thus X″; and so on to 59 times 3600. In this manner the Notation went on: And when a Number less than 60 was joined with any of these *Sexagesimal* Integers, their proper Expression was annexed without the Dash: Thus, the Sum of 4 times 60 and 25 is expressed thus, IV′,XXV. The Sum of twice 60, ten times 3600, and 15 is expressed X″,II′,XV; the highest *Sexagesimal* being set next the Left-hand. As for the *Sexagesimal* Fractions, they were marked the same way, their Numerators by the Signs of Numbers less than 60, and their Denominators by one or more Dashes (according as they were Primes, Seconds, *&c.* *i. e.* 60, 3600, and so on in the order of the Powers of 60) set either over the Numerator upon the Left-hand, or under it upon the right. Thus five sixty Parts are marked ′V or V͵. And fourteen 3600 Parts ′XIV or XIV͵. The Practice by this Notation would be easier than their common Method; yet still very difficult, especially in Multiplication and Division, as appears by the Work of *Barlaamus Monachus*, called *Logistica*; wrote in *Greek* about 1350; translated into *Latin*, and published 1600. And here it is remarkable how very near this Method is in the general Nature of it to the *Arabick*. He wanted no more, but instead of *Sexagesimal* Progression, to make it Decimal; to make the Signs of Numbers from one to nine simple Characters; and lastly, to make a Character which signifies nothing by itself, serving only to fill up Places. But every Age and Nation has its Genius; and therefore we owe this to others.

As this *Sexagesimal* Notation was used chiefly in the Astronomical Tables, so for the sake of these, it was not laid aside immediately after the Introduction of the *Arabick* Notation. The *Sexagenæ Integrorum* went first out; but the *Sexagesimal* Fractions continued till the Invention of the *Decimals*. *Regiomontanus* about the Year 1464, is the first we know who in his *Triangular Tables* divided the Radius into 10,000 Parts instead of 60,000; and so tacitly introduced decimal Parts in place of *Sexagesimals*. *Ramus* in his Arithmetick, written about 1550, (and published by *Lazarus Schonerus* in 1586) uses decimal Periods in carrying on the Extraction of Square and Cube Roots to Fractions. The same did our Country-men *Bucklæus*, before *Ramus*; and *Record* about the same time. But the first who wrote an express Treatise of Decimals, was *Simon Stevinus*, about 1582.

As to the *Circulating Decimals*, Dr. *Wallis* was the first among us who took much notice of them. But I have spoke of this already.

Another most wonderful Improvement that the Art of Computation has received from the Moderns, is the *Logarithms*; the unquestionable Invention of the Lord *Neper*, Baron of *Merchiston* in *Scotland*, towards the end of the sixteenth Century, or beginning of the seventeenth.

Dr. *Wallis* is the Author of the *Arithmetick of Infinites*; which has been very usefully applied in *Geometry*.

But the Consummation of the Art is in the *Algebraick* Method of resolving Questions: The particular History of which, I have said nothing of; because, tho' *Algebra* belongs to *Arithmetick* in a larger view, yet I have limited myself to Arithmetick taken in a more strict sense, as it is distinguished from *Algebra*: Therefore I shall only say, that most of the Authors mentioned have also wrote upon the Algebraick Art, which came into *Europe* at the same time, and by the same hands, as the Numeral *Notation*: *Lucas de Burgo* being reckoned the first *European* Writer on this Subject.

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ARITHMETICK.

BOOK I.

Of WHOLE NUMBERS.

CHAP. I.

Of the Nature of ARITHMETICK *in general, as to its Object and Operations; with the Division and Order of the Science.*

§. 1. *DEFINITIONS.*

I. ARITHMETICK is the Science, or Knowledge of Number; which is either *Unity*, or a *Multitude* of Units.

II. *Of* UNITY. When we consider any thing by itself alone, either as indivisible, or at least undivided; or also considering several things as connected in some certain manner, thereby making up a whole, neglecting what differences may be among them in other respects; the Idea we have of this thing, or Collection of things, consider'd in this manner, is called *Unity*, or *One*, *i.e.* an individual thing of a particular Kind and Name; as one Man, one Stone, one Kingdom, one Army.

III. *Of* MULTITUDE. When we consider several things as really distinct Individuals, and which separately taken we would call Units, whether they are of the same, or of different kinds and natures of things; or whether they are really separated from one another, or only distinguished by the Imagination, as the conceivable Parts of any continuous Body (for Example, a Rod) looking at no more in them but that they are not the same individual thing; the Idea acquir'd by this way of considering them, is called *Multitude* (or *many*, in distinction from one;) so we say, a Multitude of Men, of Horses, of Trees.

SCHOLIUM I. The Units, or Individuals that make a Multitude, may either be of the same Kind, or Species and Denomination of Things, or they may not be so: for however different several things are in Name and Nature, the Idea of Multitude arising from them

 is

is the ſame: So, for Example, a Man, a Tree, and a Horſe, make as truly a Multitude, as if they were all Trees, or Men, or Horſes: They are at leaſt a Multitude of Beings, or Things; for the Idea of Multitude has no dependence upon the Likeneſs of the Things from which it is formed, but only upon their different and diſtinct Being or Exiſtence, in whatever manner they are connected together, or under whatever other differences they really exiſt, or are conceived to exiſt.

SCHOLIUM 2. We may alſo conceive Multitudes of things under the Notion of one, or many; and ſo we may ſay one Multitude, or a Multitude of Multitudes. But it is to be obſerv'd, that in this caſe, the Multitudes which make the Parts of one Multitude, are conceiv'd each as an Unit, or one of its own Species, (*viz.* Multitude,) to diſtinguiſh it from the Multitude of which it is one conſtituent Part: ſo that Multitude in its general Nature is ſtill a Collection of Units, which are in all caſes ſimple Units in reſpect of the Multitude which they compoſe; tho' they may be themſelves Multitudes compoſed of more ſimple Units. And this Diſtinction of Units may be very well diſtinguiſh'd by the Names, *Simple and Collective Units.*

IV. *Of* NUMBER. *Unity* and *Multitude* comprehend the whole Object of Arithmetick, and are both comprehended under one general Name, *Number*; whoſe Definition does therefore take in the other two; and may be made thus, *viz. Number* is the Name of that Idea or Notion under which things being conſider'd, they are ſaid to be *One* or *Many*: Every particular *Multitude* having a diſtinct Name; as two, three, four, *&c.* As afterwards will be taught.

SCHOLIUM. We muſt make a little Stand here, and take notice of an old Diſpute among *Arithmeticians* about the Definition of Number; ſome denying Unity to be a Number, and others affirming it: about which there has been a great deal of Argument fill'd with abundance of idle and nonſenſical Jargon, to the ſhame even of ſome late Writers: For after all the learned Contention, it dwindles into a meer Diſpute about the Name, or what ſhall be the Uſe of the word *Number*; which no doubt each Party has a Right to eſtabliſh for themſelves at pleaſure; but no Right to impoſe it upon others: And ſo then where is the ground of a Diſpute? For if any Man asks me whether Unity is a Number, I muſt firſt know of him what he calls a Number, and then I anſwer him according to his own Definition; or I firſt give him my definition of the word Number, and then anſwer his Queſtion out of that. But we ſhall hear their different Definitions: Some define *Number* a *Multitude* of Units; and according to them it is plain, *Unity* is not a *Number* in that ſenſe in which *Unity* and *Multitude* are diſtinguiſhed, (for we have obſerved already how *Unity* and *Multitude* may be applied to the ſame Subject in different ſenſes;) ſo that theſe by denying *Unity* to be a *Number*, do only deny it to be a *Multitude* in the ſame ſenſe or application in which it is *Unity*, which no body will affirm. Others adhere to the former Definition which comprehends *Unity* and *Multitude*; but ſome of them are as much in the wrong, becauſe they contend about it as if they had the only right to ſettle the Uſe of Words; and ſtill they are more ridiculous to pretend they are arguing about the Nature of Things themſelves, when it's only about a Word: and if it does not yet appear that there can be no more in the Diſpute, let this be conſider'd, *viz.* That *Unity* and *Multitude* are agreed upon to ſignify different things. And I believe it muſt be yielded, that theſe comprehend the whole Object of Arithmetick; therefore *Number* muſt either ſignify the ſame with one of theſe, or be apply'd as a general Name to both; and then the only remaining Queſtion will be, Which is moſt reaſonable? And this, I think, will be eaſily decided by conſidering, that of two Words merely ſynonimous, one is ſuperfluous; but it's often very convenient to comprehend ſeveral things, which have alſo their different Names, under one general Name, becauſe of ſome common thing in which they agree, as it is in this preſent caſe. And thoſe who would make *Number* equivocal with a *Multitude*, are preſſed alſo with this other Difficulty, *viz.* That if they retain their Defi-

Definition of *Arithmetick*, *viz.* the Science of *Numbers*, then *Unity* will be no part of the Object of *Arithmetick*, ſince it is not a *Number*. But this, I believe, they will not ſay; for whatever can be any part of the *Data*, or Means by which a Queſtion in *Arithmetick* is ſolved, or be itſelf a real and poſitive Anſwer, muſt belong to the Science, as a part of its Object. And indeed, tho' *Euclid* defines *Number* to be a *Multitude* of *Units*, yet all along he treats of *Unity* under the ſame Name.

V. *Of* NUMBERS, *Abſtract and Applicate*. 1. When in things number'd we conſider their Number, abſtracting from (*i.e.* not attending to) their other particular Properties and Differences; the Idea or Notion we hereby form of Numbers, is called *abſtract* or *general*; or, we are ſaid to conſider Number *abſtractly*: Becauſe whatever is true of that Number of things conſider'd ſimply and purely in the Number, is true of the ſame Number wherever it is found, or in whatever things it exiſts.

SCHOLIUM. We can form no Idea of Numbers, without that of things number'd; becauſe it is an Idea form'd by compariſon of things: Yet while we conſider a Number of particular things, tho' we ſtill know that the Number is inſeparable from other Ideas that make up the complex Idea of theſe things, it's in our power to conſider and compare only the Numbers of things together, and examine their various Properties and Differences; and the Mind can perceive at the ſame time, that whatever is true of the Number of theſe things, muſt neceſſarily be true of the ſame or equal Number of whatever other things, wherein the Number only is what we conſider and compare. From whence it is that we ſpeak of Numbers without naming any particular things; by barely naming the particular Number, or joining it with the general word Thing, (which is always ſuppoſed, when not mentioned.) So we ſpeak of the Number, Two, Three, *&c.* *i.e.* two, three things, without pointing out any particular thing: Becauſe where nothing is taken into conſideration (as the Subject of Compariſon and Reaſoning) but the Number of things, then any things may be ſuppoſed; and ſo tho' the Names *Two* or *Three* are Names of particular Numbers, inſeparable from particular things, yet becauſe the ſame Numbers in every other thing muſt have the ſame Names and Properties, we make uſe of the Name without mentioning particular things: not becauſe that Name belongs to (or repreſents) an Idea of that particular Number which is not connected with any particular things; but becauſe it is a general Name applicable to the ſame Number of whatever particular things; and is uſed in this manner without mentioning any thing, when it's indifferent which things are ſuppoſed, (*i.e.* when the Number only is the matter in queſtion;) in the ſame manner as we have here uſed the word *Number* itſelf, without mentioning a particular Number, as One, or Two, *&c.* Not as if the word Number repreſented an Idea different from all Particulars; but as it is a general Name comprehending them all.

2. When we conſider Number not in its general Nature, as above explained, but as it is a Number of certain particular things, as two Years, two Men, or two Yards; then we call it an *Applicate* Number: which Name I chuſe, for its obvious Meaning, rather than the word *Contract* or *Concrete*, which ſome Authors uſe.

SCHOLIUM. When particular things are mentioned, there is always ſomething more conſidered, than barely their Numbers; ſo that what is true when Numbers are compared in their abſtract or general Nature, (*i. e.* when nothing but the Number of things is conſider'd) will not be true, when the Queſtion is limited to particular things: So, for example, the Number Two is leſs than Three; yet two Yards is a greater quantity than three Inches: for the Compariſon here is not ſimply of the Number of things two, and three; but of the Numbers joined with another Conſideration, *viz.* that of their lengths. And when things are of quite different Species, then tho' we can compare their Numbers *abſtractly*, yet we cannot compare them in any *applicate* Senſe. And this Difference is neceſſary to be conſider'd, becauſe upon it the true Senſe, and the Poſſibility or Impoſſibility of ſome Queſtions depends; as we ſhall learn more particularly afterwards.

COROLLARIES.

1. Number is unlimited in respect of Increase; so that beginning at Unity, and adding to it another Unit, and to this last Collection another Unit, and so on, we may proceed *in infinitum*, *i. e.* we can never come to an end, or never conceive a Number, but still there is a greater. But on the side of Decrease it's limited; Unity being the first and least Number, below which therefore it cannot descend. In what sense Unity is said to be divided into Parts, which make a greater Number, shall be consider'd in its place. Also we may not only begin at Unity, but at any other Number, and increase it *in infinitum*, by the continual joining of Unit after Unit; or diminish it to nothing, by continually retracting or taking away Unit after Unit.

2. Any Number may be increased by any other Number, or by any Number of Numbers; for every Number is either Unity, or a Collection of Units, which can be joined separately to another Number till they be all joined. Also it may be decreased by any Number not greater than itself; or by any two or more Numbers, which taken all together do not exceed it, *i. e.* such a Number or Numbers may be taken out of it.

3. Every greater Number may be consider'd as composed not only of Units, (which are its most simple constituent Parts) but also variously of other Numbers lesser than itself, according to the variety of lesser Numbers, whose Units taken all together make a Collection equal to that Number; or, according to the various Distributions that may be made of its Units, by putting them together in separate Collections: where also every lesser Number may be conceived as a Part of every greater; which is as a Whole with respect to all the lesser Numbers, which, being joined together, make up that Number.

§. 2. *Of the general Division, and Order of this Science.*

THE most general Division of Arithmetick is that of *Theory* and *Practice*.

The *Theory*, or Speculative Part, is that Science which considers and explains the Properties and Relations of *Pure and Abstract Numbers*; consisting of such Propositions as exhibit to the Understanding certain Truths concerning Numbers, either more general or more particular; as *Axioms* and *Theorems*.

The *Practical* Part is the Art of Numbering, or applying the Theory to the Solution of Questions, either in abstract or applicate Numbers; consisting of *Problems*, or such Propositions as require something to be done or effected: and gives us a Rule for the Performance; teaching how, by means of certain known Numbers, to discover other Numbers connected and related to them, according to the Conditions proposed in the Question; or at least to find that from the given Numbers, compared and applied to one another, as the nature of the Question requires, there arises no Number.

Observe, Some consider as Theory all that is proposed in abstract Numbers, whether Theorems or Problems; and the Application to Questions in applicate Numbers only, they call the Practical Part. Others define the Theory as above; and confine the Practical Part to Problems of Abstract Numbers: and Problems of Applicate Numbers they call the *Effective* Part.

As the Truth and Reasons of the Practical Rules are contained in *Theorems*, with other more general Principles, as *Definitions* and *Axioms*, so they are to be reckon'd Deductions from them, or rather their Applications: And therefore in the natural Order, the Theory ought to precede the Practical Part. But yet these two Parts ought not, and cannot be treated entirely separate from one another; *i. e.* all of the first Kind together, and afterwards all of the other: But they must be mix'd together according to their Dependence. It's certain that *Theory* must precede *Practice*, because that contains the Grounds and Reasons of this; yet 'tis as true, that we can make but a small Progress in Theory, till we under-

understand the fundamental Elements and Rules of the Practical Part: These being indispensibly necessary both for understanding the Sense, and illustrating, or proving by Examples, the Truth of Theorems.

Wherefore the *Division* that must be followed in explaining this Science, is not that of *Theory* and *Practice*, (tho' these must also be disposed according to Reason, and their natural Connection and Dependence;) but the most proper and reasonable *Division* is into two other Parts, under the Titles of the *Simple* and *Comparative Elements*.

If we reflect upon the Definitions and Corollaries already explained; these are so many of the first general Principles and Axioms of the Science: And from these we shall easily understand the Reason of this Division, and what in general belongs to each Part. For it's plain, that the most general, and what we may call the only *absolute* Property of Number, is, *a Capacity of Increase* in infinitum, *or Decrease to nothing*: all other particular Properties are *relative*, depending upon the Comparison of *Numbers* together. And since there is nothing in *Numbers*, but different Collections of Units, or different Compositions of lesser *Numbers* in greater, these particular Properties must all depend upon the Effect of different Applications of *Numbers* to one another, whereby they are variously compounded together or resolved, according to certain Conditions. And for Arithmetical Problems, or Questions, in which an unknown Number is to be found by means of certain Connections and Relations it has to some known Numbers; these Connections can consist in nothing else but this, *viz.* That the Number sought is the Result of variously increasing and decreasing the known Numbers by one another, according to the Conditions proposed; so that all that can be known or done in Arithmetick does evidently relate to, and depend upon, the Application of Numbers to one another by Composition and Resolution, or Increasing and Decreasing them.

Therefore the first and fundamental Part of Arithmetick is the Knowledge of the various Rules and Operations (with the Principles upon which their several Reasons depend) by which Numbers are compounded and resolved; *i.e.* increased and decreased by one another, which are the fundamental Elements of Practice; including in general all that can be done with Numbers; and indispensibly necessary also for understanding and proving the more particular Theory; which does all relate to the Effect of these Operations: Which I have therefore justly, I think, consider'd as the *Simple* and Primitive Elements of Arithmetick. What further Subdivision of this is necessary, shall be shewn in its proper place.

All the rest of the *Science of Arithmetick* I comprehend under the general Name of *Comparative Elements*; because it consists of such relative Properties as arise from the comparing of Numbers together, and applying them to one another by the various Methods of Compounding and Resolving, taught in the first Part; as also the Solution of such Questions as depend upon these Relative Properties.

In the remaining Part of this Book, with the second and third, you have the first Branch, or Simple Elements explained; and the Comparative Elements in the remaining Books.

§. 3. *Of the Operations of* ARITHMETICK *in general.*

BY what has been already explained, it will be obvious, That all the *Operations* in Numbers are of two Kinds in General, *viz. Augmenting* and *Diminishing*. Each of these are performed after two different ways, and thereby come under two different Names: Thus, *Augmenting* is divided into ADDITION, and MULTIPLICATION; *Diminishing* into SUBSTRACTION, and DIVISION: Which shall be explain'd in order. Some add a third Class, whose Branches are call'd *Involution*, and *Evolution*; or also *Raising of Powers*, and *Extracting of Roots*. But these may be comprehended under *Multiplication* and *Division*; for the Operation is of the same general Kind, only under certain Limitations. They will deserve however to be explain'd distinctly by themselves.

But

But there is yet something previous to all this, *viz.* The Knowledge of the Signs whereby our Ideas of Numbers are expressed or represented; for without some easy Method and Art of representing them, so as they may be clearly and distinctly compared, there could be little or nothing known or done in Numbers. This Art we call the *Notation* of Numbers; which, if we consider by itself, and in its primary Design, is only a necessary Instrument for the better and more easy Comparison of Numbers, and performing their Operations; and is therefore rather a Handmaid, than an essential Part: For we can call nothing essential but what belongs to the very Nature of a Thing; and which being taken away, the thing would be destroy'd: not that which is arbitrary, and may be changed, and another thing put in its place at pleasure, as it is in the Notation of Numbers. However, as there must first be a Method of Expression instituted, whatever that is, so far as the Rules of Operation depend upon it, it is the Foundation of them; and therefore it is commonly look'd upon as the first Rule or Operation in Arithmetick, making in all five fundamental Operations, *viz. Notation*, *Addition*, *Substraction*, *Multiplication*, and *Division*. But still it ought to be consider'd only as an arbitrary Rule and Foundation, which requires and supposes no other Principle but this, *viz.* That *any of our Ideas may be represented by any Marks or Signs we please to institute*: whereas the other Operations, besides what they owe to the Notation, have also a dependence upon Reasonings from the Nature of Numbers themselves.

Before we enter upon these Operations, we must here repeat an Observation which has been already made, *viz.* That all Science must begin with *Theory* as a Foundation for *Practice*. Now this Order we have in effect followed; for the Definitions and Corollaries explained in §. 1. are the first and more general Principles of this Science; to which if we join this general *Axiom*, *viz.* That *the Whole is equal to all its Parts*, we have all that is necessary for entring upon these practical Elements. What other Principles are employ'd in particular Rules, shall be explain'd in order as we go on; for they are gained most part by consequence in the progress of the Science.

CHAP. II.

Of the Notation, or Expression of NUMBERS, *with their Distinction into Integral and Fractional.*

§. 1. *Of the Notation of Numbers.*

DEFINITION.

NOTATION *is the Method or Art of Expressing Numbers:* which is done two ways; by certain *Words* or *Names*, and also by certain *Signs* or *Characters*, called *Figures*; the one corresponding to the other in the Representation of the same Numbers, and both equally necessary: The Figures being contrived for the easy management of Operations, whereby the greatest Numbers are compared, and the Operations performed with the greatest ease and readiness; without which, our Knowledge in Numbers had reached a very short way. And so much does the Science owe to these, that upon this Account some call it, *The Art of Figuring*; but we might better call it, *The Science*

of

of Numbers, as they are repreſented and managed by Figures: So that this is the principal Branch of *Notation*; which yet cannot be without the other, the Names of Numbers being neceſſary for our converſing or ſpeaking to one another about Numbers, and the Reſult of the Operations made by the Figures, and for the actual numbering or telling over things, by the Application of Names in an orderly Progreſſion from Unity, ſtill joining one Unit after another to the Collection; or telling things out one by one. For without ſome Signs whereby the Number, as it increaſes, is continually diſtinguiſhed, we could make nothing of the Numbers of things, nor compare one Number with another: and Words are the moſt proper and convenient Signs for this purpoſe; which are alſo of good and neceſſary Uſe in making the Operations with the Figures.

The Syſtem of the Names of Numbers is indeed a part of our Language, and therefore the Writers on Arithmetick ſuppoſe them to be known, and reckon it their buſineſs only to explain the Repreſentation of Numbers by Figures, and their Correſpondence to the Names: But it will be a more regular and juſt Method to explain both the Syſtems of Names and Figures by themſelves; and then ſhew their mutual Correſpondence. Theſe two Syſtems I ſhall explain under the Titles of the *Nominal* and *Figural Notation*; which being compared, their mutual Correſpondence will be eaſily underſtood. But this I ſhall more particularly explain in the Solution of a *Problem*, teaching how from the Expreſſion of any Number in one manner, to find its correſpondent Expreſſion in the other.

(1.) NOMINAL NOTATION, *or the Expreſſion of Numbers by Words or Names.*

A DIFFERENT ſimple Name for every Number, or even for as many particular Numbers as we have occaſion to conſider in human Affairs, would be a Burden altogether inſupportable; but it is more happily contrived, that a few ſimple Names, and theſe compounded together in a very eaſy manner, anſwer all the Ends and Purpoſes both of Speculation and Practice.

The Simple Names of Numbers are theſe,

One, Two, Three, Four, Five, Six, Seven, Eight, Nine,	Ten, A Hundred, A Thouſand, A Million, A Billion, A Trillion, A Quadrillion, *&c.*	*Explanation of this TABLE.* *One* is another Name for *Unity*: the reſt of the Names to *Ten* expreſs Numbers in a Series from Unity, by the continual joining Unit after Unit; ſo *Two* expreſſes *One* more *One*; *Three* expreſſes *Two* and *One*, and ſo on to *Ten*: a *Hundred* expreſſes *Ten-Tens*; a *Thouſand* expreſſes *Ten-Hundreds*; a *Million* is a *Thouſand-Thouſands*; a *Billion* is a *Million* of *Millions*; a *Trillion* is *a Million* of *Billions*; a *Quadrillion* is a *Million* of *Trillions*.

So that we have here Names anſwering to the natural Series of Numbers from *Unity* to *Ten*. But after this, the Series is interrupted, aud we paſs to the Names of greater Numbers: And all that remains to be explain'd is, how the intermediate Numbers are named from *Ten* to a *Hundred*, and from a *Hundred* to a *Thouſand*, and ſo on. Which is done thus, after *Ten* the Names are compounded of *Ten* and the preceding to a *Hundred*. Firſt, from *Ten* to two *Tens* we proceed thus;

Ten,		
Eleven,	Equal to	Ten and One,
Twelve,		Ten and Two,
Thirteen,		Ten and Three,
Fourteen,		Ten and Four,
Fifteen,		Ten and Five,
Sixteen,		Ten and Six,
Seventeen,		Ten and Seven,
Eighteen,		Ten and Eight,
Nineteen,		Ten and Nine,
Twenty,		Two Tens.

Then from *Twenty* we ſimply join the Names of the firſt *nine* Numbers thus, *Twenty-one*, *Twenty-two*, &c. to *Twenty-nine*: then the next Number is *Three Tens*, called *Thirty*; and the ſame Compoſition of Names we uſe from *Thirty* to *four Tens*, called *Forty*: and from this to *Fifty*, (or *five Tens*:) and ſo on to *Sixty*, (or *ſix Tens*:) *Seventy*, (or *ſeven Tens*:) *Eighty*, (or *eight Tens*:) *Ninety*, (or *nine Tens*:) and after *Ninety* and *nine*, *One* added makes *ten Tens*, or a *Hundred*. Then from a *Hundred* (or *one Hundred*) we proceed by joining with it all the preceeding Names from *One* to *Ninety* and *nine*, thus, *One Hundred* and *one*, *One Hundred* and *two*, &c. to *One Hundred* and *Ninety nine*, to which one added makes *Two Hundred*. In the ſame manner we proceed from *Two Hundred*, to *Three Hundred*; and ſo on to *Nine Hundred Ninety* and *nine*, and then *one* added makes *Ten Hundred*, or *One Thouſand*. In like manner we proceed from *One Thouſand*, joining with it all the preceding from *Unity* till we come to *Two thouſand*, and ſo on to a *Thouſand-thouſands*, or a *Million*; and with this alſo we join all the preceding Names from *Unity* to *two Millions*, &c. to a *Million* of *Millions*, or a *Billion*; and ſo on to a *Trillion* and *Quadrillion*.

Obſerve, If we would proceed farther, we may uſe theſe Names, *Quintillions*, (or a Million of Quadrillions;) *Sextillions*, (or a Million of Quintillions,) and ſo on; filling up the intermediate Numbers as before: But for any real Uſe in human Affairs, we need no Names above Millions. In Mathematical Work, greater Numbers occur; but they are manag'd by Figures, and can be compared without Names; which, if required, may be contriv'd in the manner now mention'd. *Obſerve* alſo, that ſome inſtead of the ſimple Names Billions, Trillions, Quadrillions, *&c.* chuſe the compound Names *Millions of Millions*; *Millions of Millions of Millions*; *Millions of Millions of Millions of Millions*; and ſo on, compounding the word Million once more gradually: But the other ſimple Names ſeem more convenient, tho' the contrivance and way of making the complex Names is more obvious and eaſy; however, ſince we have little uſe for Names above Millions, we need not diſpute about the difference.

(2.) FIGURAL NOTATION.

AS a few ſimple Names ſerve all our Purpoſes in Arithmetick, ſo yet fewer ſimple Figures are found ſufficient, not only for common Uſe, but even to carry us thro' the Infinity of Number: Which Figures, with their correſponding Names, are theſe;

Figures	0	1	2	3	4	5	6	7	8	9.
Names	Nought	One	Two	Three	Four	Five	Six	Seven	Eight	Nine

That a Figure ſignifying of itſelf *Nothing*, (or no Number) is neceſſary, we ſhall preſently ſee. In the mean time *obſerve*, that the Number of ſimple Figures being *Ten*, they are hence called the Digits, from the Number of *Fingers* (*Digiti*) on our Hands. How this Number came to be choſen, we ſhall afterwards conſider.

All other Numbers greater than Nine, (9,) are expreſſed by Combinations of theſe Digits; placing them together in a Line in various Orders; every Figure changing its Value according to the place it ſtands in. By this

GENERAL RULE.

In a Rank of Figures placed together in a Line, (reckoning the Order of Places from the Right to the Left-hand; that being the first which is first on the Right-hand; and the second, third, &c. being in order from that on the Left) any Figure in the first place represents the primitive simple Value above expressed, as if it stood alone; and in every place gradually towards the Left-hand, it signifies ten-times as many as it would do in the preceding place.

Thus 1, 2, 3, *&c.* in the first place signify simply so many Units as above named; but in the second place of a Rank, any one of them signifies ten times so many Units, (or so many times ten.) *Ex.* 48 signifies forty and eight, [*i. e.* four tens for the value of 4 in the second place, and 8 in the first place.] Again, 60 signifies sixty, [or six tens, which is the value of the 6 in the second place; the 0 in the first place signifying nothing of itself.] Again, in the third place they signify ten times their Value in the second place; or ten times so many Tens, (*i. e.* so many hundreds:) and so on gradually towards the Left-hand, still increasing their value tenfold of that of the preceding place. So 600 is six hundred; 642, is six hundred, forty and two.

Now this general Institution being well conceiv'd, it will, I suppose, be found evidently sufficient for the expression of all Numbers; or that all Numbers in a gradual Series from Unity may be represented thereby. And if this is not evident, it may be made so---Thus; Every Number is either less than Ten; which is expressed by one of the nine significant Digits: or it is a Number of Tens less than ten; which is therefore expressed by some Digit in the second place, and 0 in the first: or it is such a Number joined with a Number less than ten; which last Number is expressed by a Digit in the first place: or it is a Number of Ten-tens (or hundreds) less than ten; which is therefore expressed by some Digit in the third place, and 0 in the first and second places: But if with this Number is joined any lesser Number, (*i. e.* any of the preceding Classes) that is expressed by Digits in the first or second, or in both places. The Progression to greater Numbers, I think, will be now plain enough. Or we may consider it in this manner---Any Combination of Figures expresses some one determinate Number, according to the instituted Value of Places; and if Unity is joined to this Number, the Sum can be expressed: for this is done by changing the Figure in the first place, and taking the next greater, *i. e.* for 1, taking 2; for 2, taking 3, *&c.* and if 9 is already in the first place, then because nine and one make ten, we must (according to the Institution) put 0 in the first place, and change the Figure of the second place, taking the next greater; and if that is also 9, we put 0 also in the second place, and change the Figure of the third; and so on till we come to a place in which there is a Figure less than 9. So that if all the Figures of the given Example are 9's, we set 0 in all their places, and set 1 on the left of all. *Example* I. 47 and 1 is 48. *Ex.* II. 29 and 1 is 30. *Ex.* III. 499 and 1 is 500. *Ex.* IV. 999 and 1 is 1000.

Now it's plain, that if to any Number we can add one and express the Sum, by the same Rule we can add 1 to the Sum; and 1 to this last Sum, and so on. And because 1 may be the given Number, it follows that we can by this Institution express any Number in the natural Series from Unity *in infinitum*; and the way of doing it is here also made evident.

SCHOLIUMS.

1. We see now that tho' the Figure 0 signifies nought of itself, yet it is not useless; but indispensibly necessary to fill up places, that other Figures may possess such places as they ought to be in, for the expression of certain Numbers which could not be expressed by this fundamental Rule, without the help of this Character. One Example is enough to shew

it, and we ſhall chuſe the Number *Ten*, which could not otherways be expreſs'd: for by this Inſtitution, 1 in the ſecond place is ten; but there cannot be a ſecond place unleſs there be a firſt; and if any ſignificant Figure ſtand in the firſt place, the whole will make a Number greater than Ten: ſo 14 is ten and four; therefore the Figure in the firſt place muſt ſignify *nothing*, and ſerves only to make two places, that the 1 may be put in the ſecond, where it ſignifies Ten: thus, 10.

2. The Figure 0 ſtanding on the Left-hand, or in the laſt place of any Rank of Figures is altogether uſeleſs; ſo 04 is no more than 4, becauſe the Value of Figures riſes from the Right-hand to the Left.

3. According to this Inſtitution, each of the nine ſimple ſignificant Figures may be conſider'd as having two Values; the one certain and determinate, known by its Form, which is that it ſignifies ſimply by itſelf, (as 4 is *four*) and may be called its *ſimple* or *primitive* Value: the other is uncertain and variable, depending upon its place in a Rank with others; ſo any Figure in the ſecond place is ſo many Tens; in the third place it is ſo many Ten-tens, (or hundreds) and ſo on in a ten-fold Increaſe: and this may be called the *ſecondary* or *local* Value, *i. e.* the Value of the *Place*; and theſe two Values compounded (or the one repeated as oft as the other contains Unity) makes the compleat Value of that Figure in that place. For Example, 4 in the ſecond place is four-times Ten, (or forty;) and in the third place it is four-times Ten-tens, or four hundred. But in the firſt place (and in no other) theſe two Values coincide; for here there is no Value but the *Simple*.

4. If theſe two Syſtems of *Names* and *Figures* are duly compared, their mutual Correſpondence will be eaſily underſtood; ſo that you'll find little difficulty in expreſſing any Number in the one manner which is firſt expreſſed in the other. But that no body may complain, I ſhall explain this more particularly in the following *Problem*.

PROBLEM.

Having any Number expreſſed by Figures, *how to read or expreſs it in* Words; *or, having it expreſſed in* Words, *how to write or expreſs it in* Figures.

CASE 1. *From a given Expreſſion of any Number in* Figures, *how to read it in* Words.

We have already conſider'd how that any Number being expreſſed by Figures, every Figure may be conſider'd in a double view, *i. e.* according to its *ſimple* and *local* Value. Again, the Syſtem of Names as above explain'd, is ſo contrived, that Ten times Ten; Ten times Ten times Ten; Ten times Ten times Ten times Ten; [and ſo on, which are the Value of places after the ſecond, which is Ten] have diſtinct Names, either ſimple or compound; and ſuch Values taken any number of times leſs than Ten, are named no other way than by expreſſing the number of times (leſs than Ten) that Value is repeated; ſo that if we know the Names of both values for every Figure in any Rank (or Expreſſion of a Number, which has more than one place) and expreſs each according to the Compoſition of theſe two Values, then we have the Expreſſion ſought, for the whole Rank or Number propoſed. The Names of the ſimple Values of Figures we have already learnt; what remains is to know the Names for the local Values, which is the Deſign of what they commonly call the Table of Notation (or Numeration) which I ſhall put in a more convenient Form.

TABLE of NAMES *for the Local Values of Figures (in which* X *stands for Ten;* C *for Hundred;* Th *for Thousand.)*

Places	Names
18th	C, th, Bill.
17th	X, th, Bill.
16th	th, Bill.
15th	C, Bill.
15th	X, Bill.
13th	Billions.
12th	C, th, Mill.
11th	X, th, Mill.
10th	th, Mill.
9th	C, Mill.
8th	X, Mill.
7th	Millions.
6th	C, Thousands.
5th	X, Thousands.
4th	Thousands.
3d	Hundreds.
2d	Tens.
1st	Units.

If you would carry this Table farther, it is to be done by the Names *Trillions*, &c. (already explained) the same way used, as here *Millions* and *Billions* are.

Now the Use of the Table is plainly this. Apply each of these Names in Order, to the several Figures of a Rank, according to their Places (the Name Units signifying only, that the Figure in that Place is taken in its simple Value) and prefix the simple Value, making a Composition of both (observing how any simple Value compounded with Ten is contracted, *viz.* For two Tens, say Twenty, and so on to Thirty, Forty, *&c.* as above explained) and thus read all the Figures from the Left to the Right. Observe also, that where any Word belongs to the *Names* of *several Places*, which stand all next together, as do the Words, Thousand, and Million, and Billion, that needs not be repeated; but only once expressed, after all the Places to which it belongs are read according to what is more in the Name, as the following Examples will make plain.

EXAMPLES.

86, is Eighty Six

254, is Two Hundred, Fifty, and Four.

7408, is Seven Thousand, Four Hundred, and Eight.

67040, is Sixty and Seven Thousand, and Forty (*i. e.*) Sixty Thousand, Seven Thousand, and Forty.

46258300, is Forty Six Millions, Two Hundred and Fifty Eight Thousand, and Three Hundred, (*i. e.*) Forty Millions, Six Millions, Two Hundred Thousand, Fifty Thousand, and Eight Thousand, and Three Hundred.

24800540362, is Twenty Four Thousand and Eight Hundred Millions, Five Hundred and Forty Thousand, Three Hundred Sixty Two, (*i. e.*) Twenty Four Thousand Millions, and Eight Hundred Millions, *&c.*

But this may be made yet easier, by considering the several adjacent Places that have any one or more Words common in their Names; from which we have this other

TABLE.

th-Bill.	Bill.	th-Mill.	Mill.	Thousd.	Un
c : x : Un.	c : x : Un.	c : x : Un.	c : x : Un.	c : x : Un.	c : x : Un.

Observe, If you neglect the Word Un. (or Units) which stands first in the upper Line of this Table; and where the Word Unit stands after the three first Places in the under

 Line,

Line, put the Name that ſtands over every Repetition of c : x : Un. repeating it alſo with the other two, *viz.* c : x. then you make the ſame Series of Names as in the preceding Table. And for the Advantage of this Form, we ſhall preſently ſee it.

The Application of the TABLE.

Begin at the Right Hand of any Number, and divide or ſeparate the whole Figures into Claſſes or Periods of three Figures each, as long as there are as many: then each Period muſt firſt be read as if it were alone by *Units, Tens, Hundreds, i. e.* the firſt place on the right of the Period by its ſimple Value, the ſecond place as ſo many Tens, and the third place ſo many Hundreds as the ſimple Value expreſſes [which Practice is expreſſed by theſe Signs c : x : Un. conſtantly repeated in the under Line of the Table.] Again, becauſe Figures increaſe their Value towards the left, the ſame Figures in different Periods are of different Values, and therefore every Period has a common Name that compleats the Expreſſion; which are theſe in the upper Line of the Table; (in which *Units* is made the Name of the firſt Period merely for a diſtinction, but is never expreſſed) therefore to apply this Table more eaſily, get the Names of the Periods by heart backwards and forwards; and then applying them to the Periods from the right to the left, that you may find the Name of the higheſt Period in any Example; and remembring exactly the Names of Periods in order from the left; you muſt begin from the left and firſt read the Figures of each Period, as c : x : Un. then join the Name of the Period, which is ſuppoſed to be applied to each Figure of it.

Obſerve, That in ſome Caſes you'll have a broken or incomplete Period of one or two Figures next the Left-hand, which muſt be read by itſelf, juſt as it happens to be; but the reſt will each have three Figures: yet theſe may be alſo in a true ſenſe but broken Periods, *i. e.* not have ſignificant Figures in all their places, as in this 24,048. Again *Obſerve*, That in the Periods of Mill. and th-Mill. becauſe the word Mill. is common, you need not repeat it: but in the th-Mill. uſe only the word Thouſand, ſuppoſing the word Mill. which you need only expreſs after the Figures in the Period of Mill. unleſs that Period be fill'd with o's, as here, 24,000,468,350. which is 24 th-Mill. *&c.* but in this, 24,360,579,200, I read 24 th. 360 Mill. *&c.*

See theſe few more Examples.

278,307,000	read thus	278 Mill. 307 Th.
348,026,000,123		348 Th. 26 Mill. 123.
7,200,809,867,345		7 Bill. 200 Th. 809 Mill. 867 Th. 345.
326,009,478,205,723		326 Bill. 9 Th. 478 Mill. 205 Th. 723.

CASE 2. *Any Number being expreſſed in* Words, *to write it down in* Figures.

If the preceding Table of Names correſponding to the ſeveral places of a Rank of Figures be conſider'd, there can be no difficulty in this part of the Problem, which is but the reverſe of the former; yet perhaps it may not be uſeleſs to ſome to point them out a Method, which is this——Remember exactly the Names and Order of the Periods from Left to Right; and of the three places in each Period: Then obſerve what is the Period firſt named in the Example, alſo what is the Number (*viz.* of c : x : Un.) applied to that Period; ſet down that Number, with a Point after it: then conſider what is the Name of the next Period below that in order, and what Number is applied to it in the Example; ſet down that Number on the right of the former, with a Point after it; and whatever be the Number, it muſt be ſet down ſo as to poſſeſs three places, by ſetting o in thoſe places of the Period to which no Number is applied; (except the firſt or higheſt Period where this is not neceſſary:) ſo that if there is no Number applied to any whole Period, (after that which is the higheſt in the Example,) ſet three o's in its

three places: thus proceed to the loweſt Period or firſt on the right. *Obſerve* alſo, that if you find the Name Thouſand mention'd before Million or Billion, it ſignifies Th-Mill. or Th-Bill. and is to be diſtinguiſhed from the Period of ſimple Mill. or Bill. as in the preceding Table. One or two Examples will ſufficiently illuſtrate this Practice.

Ex. 1. To write in Figures this Number; *Forty Million Two Hundred Thouſand and Eight*, I proceed thus; *Million* is the higheſt Period, and *Forty* the Number applied, which I write down thus, 40; the next Period is *Thouſand*, and the Number here applied is *two Hundred*, which I ſet on the right of the former, thus 40,200; then follows the Period of the *Units*, and the Number here applied is only *Eight*; which, according to the Rule, I ſet on the right of the former thus, 40,200,008.

Ex. 2. *Twenty-four thouſand and ſixty-two Millions*. Here the name *Thouſand* ſtanding before *Million* ſignifies *th-Mill.* which is the higheſt Period in the Example, and the Number applied is *Twenty-four*, thus written 24; then follows the Period of *Millions*, and the Number is *Sixty-two*, joined to the former thus, 24,062; and becauſe there is no more in the Example, and yet there are two Periods remaining, *viz. Thouſands* and *Units*, I fill them up with 0's thus, 24,062,000,000. For without theſe, the other Figures could not expreſs above Thouſands. If you compare the Examples given for the firſt part reverſely, you have enow for this laſt Purpoſe.

COROLLARIES to the FIGURAL NOTATION.

1. Of two Numbers, expreſſed by Figures, (by the Rule and Inſtitution explain'd) that which has feweſt Figures (or Places) is the leaſt Number. *Examp.* 99 is leſs than 100. For tho' the Figures of that which has feweſt be all the greateſt poſſible, (*i. e.* all 9's) and thoſe of the other be all the leaſt poſſible, (*i. e.* 1 in the higheſt place, and the reſt 0's) yet the other will want at leaſt 1 to make it equal to this; becauſe 10 is equal to 9 and 1, and 100 equal to 99 and 1, and ſo on. Hence the value of an Unit in any place of a Rank is greater than the value of all the preceding Figures on the Right-hand; becauſe taking that Unit in its true value, it is a Number having one place more than are upon its Right-hand.

2. If two Numbers expreſſed by Figures have an equal Number of places, that is the leaſt Number which has the leaſt Figure in the higheſt place; or in any other place, all the preceding places on the left being equal. *Ex.* 199 is leſs than 200; and 232 is leſs than 233; and 469 is leſs than 472.

3. If one Number is greater than another, and if you ſet before each of them an equal Number of whatever Figures, that which was greateſt before will ſtill be the greater.

4. If before, or on the Right-hand of any Rank of Figures, (or Number expreſſed by Figures) be placed any one Figure, the given Rank expreſſes thereby ten-times what it did before (or without that Figure;) and if two Figures are ſet before it, it expreſſes an hundred-times what it did before; and ſo on in a ten-fold Progreſſion, according to the fundamental Rule. *Ex.* 480 is ten-times 48; and 4800 is 100 times 48, *&c.* And obſerve, That with reſpect to the raiſing the value of the given Rank, it is the ſame whether 0, or any other Figure be prefixed; it is certain that another Figure will make a greater Number of the whole Rank as it is now encreaſed; but will not make that part of it greater, which was the given Rank. So here in 486, the 48 is equal in value to 480, whatever Figure ſtands before it; tho' 486 is greater than 484 or 480.

5. Hence again; Any part of a Number taken from the Left-hand, may be valued thus, *viz.* we may conſider what Number it makes taken by itſelf; and then conſider the place in which its firſt Figure (on the right) ſtands; and make the *Name* of that place a common Name or local Value to the whole Rank: Thus, all the Figures of any Rank, excluding the firſt on the right, or place of Units, being read by themſelves, (as if there were no Figure before them) is ſo many Tens; or ſuch a Number taken 10 times. Again, excluding the two firſt places, the reſt read by themſelves is ſo many Hundreds, and ſo

ſo on. *Ex.* 246873: here the 24687 is equal to 24687 Tens, (taking 7 as it were in the place of Units) *i. e.* twenty-four thouſand ſix hundred and eighty-ſeven Tens. The 2468 is 2468 Hundreds; the 246 is 246 Thouſands, and the 24 is 24 Ten-thouſands: as the 2 is 2 Hundred Thouſands.

SCHOLIUMS.

1. Both in the General Rule of the Figural Notation, and theſe two laſt Corollaries, I have ſuppoſed this for a Truth, *viz.* That Ten-times, (or a Hundred, *&c.*) any Number is the ſame as that Number of times Ten, (or a Hundred, *&c.*) For Example, that Ten-times 7, is 7 times Ten; which is a Truth I believe will be eaſily granted: yet it is of that kind that admits a Demonſtration; but I ſhall refer it till we come to *Multiplication*, to which it properly belongs.

2. By the Figural Notation now explained, all Numbers above Nine are expreſſed in a compound Form; repreſenting either a certain Compoſition of the Number Ten, or ſuch a Compoſition with the addition of a Number leſs than Ten; which are notably diſtinguiſhed by this method of Expreſſion. But there are various Degrees of the Compoſitions of Ten; for a Number compounded of Tens is either a Number of Tens leſs than Ten, which is expreſſed by a Digit in the ſecond place and 0 in the firſt, as 10, 20, *&c.* or it is ſuch a Number taken ten-times, which is expreſſed by a Digit in the third place, and 0's in the ſecond and firſt, as 300, 400, *&c.* or ſuch a Number as the laſt taken ten-times, as 2000, 5000, *&c.* and ſo on. All which different Degrees are expreſs'd by a Digit in different places. From which it's clear, that if ſeveral of theſe degrees be joined together, *i. e.* if there is a Rank conſiſting of more than one ſignificant Figure, and having at leaſt one 0 on the Right-hand, as 460, which is 46 Tens, that expreſſes a Number which is alſo a Compoſition of Tens; but if there is a ſignificant Figure in the place of Units, that Number contains ſo much odd or over a certain number of Tens, as 468, which is 46 Tens and Eight. Now there are two things to be remarked from this Method of Notation: Firſt, That every Number is diſtinguiſhed by the very Expreſſion into as many parts as there are ſignificant Figures in it; each of which is expreſſed ſeparately by ſetting as many 0's before it as there are Figures before it in the given Expreſſion; as in the following Example. And ſecondly, That each of theſe Parts is a certain Compoſition of the Number Ten (as above explained) except that which is in the place of Units, which is always leſs than Ten. *Thus*, the Number 4682 is equal to 4000 and 600 and 80 and 2, which is more ſimply and conveniently written all in one Rank 4682; whereby all the Parts are as clearly and intelligibly marked out by the meer ſituation of the Figures, (according to the Rule.) Now tho' this is really a compound Form, expreſſing ſeveral Numbers diſtinctly from one another; yet becauſe it is the moſt ſimple way of expreſſing that Number, or Term in the natural progreſſion, which is equal to the Sum of all theſe leſſer Numbers (according to the Inſtitution) therefore it is ſaid to be a ſimple Expreſſion of one Number, in compariſon of the other ways of expreſſing the Parts ſeparately; or of the Expreſſion of a Number by any other of its component Parts, ſeparately written each in their moſt ſimple Form, (in each of which therefore there are Parts, or Figures that have the ſame local Value) as if inſtead of 8 we ſhould write 5 more 3, (which together make 8) or inſtead of 74 we ſhould write 43 more 31, which therefore in diſtinction from the other we may call complex Expreſſions repreſenting ſeparately two different Numbers, the leſſer of which has Parts of the ſame local Value with the greater. Or we may explain this diſtinction of ſimple and complex Expreſſions, thus: That Expreſſion is called ſimple, or one Number, which is one of thoſe whereby the natural Series of Numbers is expreſſed in a continued Progreſſion from Unity; but when two different Expreſſions of that Series are ſeparately written in two diſtinct Ranks, they are ſaid to be two different Numbers: and with reſpect to that Number to which they are both together equal, they are ſaid to be a complex Expreſſion

of it; especially with some Mark or Word betwixt them to signify their being joined together. For Example, 46 is a simple Expression of one distinct Number in the natural Series: But 32 and 14 are two Numbers, which being both together equal to 46, therefore 32 more 14, is a complex Expression of the Number 46. Again, in the same manner any Number which is equal to the difference of two Numbers may be complexly expressed by these two, with a Mark of Substraction betwixt them: As for Example, since 32 more 14 is equal to 46, therefore if 14 is taken out of 46, the remainder is 32; and hence 32 may be expressed thus, 46 less 14.

The same things are applicable to the Parts by which any Number in its simple Form is expressed; as if for 46 we should write 40 more 6; or for 6 we should write 46 less 40.

In the last place observe, That the compound Form of this Notation is the Perfection of it; not merely as it is compound, but the particular Manner of it; because of the certain relation that each part has to one Number, *viz. Ten*, by the constant, regular and uniform Progression in the Value of Places: For by this Similarity or Likeness of some Parts in different Numbers, (*i. e.* of Parts expressed by Figures in the same or like places;) and the Connection or Relation of all their Parts by means of the common Number Ten, to which they have all a relation; Numbers can be compared together, and their Operations performed in a very easy, clear and distinct manner. And more particularly we have from the Method of Notation this Principle of Operation, *viz.* That some Operations may be performed by the few simple Characters, taken and applied in their primitive Values; and what is wrong or deficient by the neglect of their local Values may be again made up, by the order or due placing the several Figures arising from the Operation: And in other cases where we cannot work by single Figures, yet we can take the Parts of Numbers, (*i. e.* two or more of their Figures) and consider them in the Value they would have by themselves; correcting the Defects the same way as now mentioned. Which Principle we shall see applied in the following Rules; and here only I shall further observe, That in this it is that the Rule of Notation is the ground of all other Operations; affording regular and easy Rules for expeditious and certain Work, as we shall learn.

As to the History and Invention of this admirable NOTATION, see what is said in the *PREFACE*.

3. We might here make Comparisons betwixt this Method of Notation, and others that have been or may be used, in order to shew the Excellency of this Method; but this will be better understood after you see the Application of it in the Operations, where I have allotted a Place for some general Reflections upon these Operations. However, I shall here explain the *Notation of Numbers* instituted by the *Romans*; which will be proper, because it is also practised for Marking of Chapters and Sections, and such things; and this being compared with what we use, the difference in favour of our Method will be sufficiently evident.

Of the ROMAN NOTATION.

The Characters whereby the *Romans* marked Numbers were taken out of their Alphabet of Capital Letters; thus:

The Simple Characters.

	I.	V.	X.	L.	C.	D.	M.
Equal to	1.	5.	10.	50.	100.	500.	1000.

The

The intermediate Numbers betwixt these are expressed by a repetition of the same; setting them together in a Line, thereby expressing the Sum of all their Values (above expressed) joined to one another, (for they have no different Values from their Places) the Characters of greatest Value being set next the Left, as for Example; II is 2. III is 3. VI is 6. VII is 7. XI is 11. XV is 15. XX is 20. LX is 60. LXV is 65. DX is 510. DC is 600. DCCCC is 900. DCCCCLXXXXVIIII is 999. These Examples will sufficiently shew how all the rest are made.

But to prevent too great a Repetition of the same Characters, they sometimes set the lesser Character before the greater; and then it represented the difference of these two, or the effect of taking the one Number away from the other; thus IV is 4. IIX is 8. IX is 9. XL is 40. CD is 400. CM is 900. And when a Number is expressed by more than two Characters, if any part of it is thus expressed, it is fit to distinguish it from the Characters on the Left of it, by a Point; thus, 140 may be expressed C,XL, (for CXXXX) and 148 thus, C,XL,IIX (for CXXXXVIII.) Again, 499 thus CD,XC,IX, instead of (CCCCLXXXXVIIII) which are more convenient.

Again, for Numbers greater than 1000 or M, they are expressed after the same manner. But there are other things in their System, both for some Numbers less than 1000, and especially for greater, which I shall briefly explain. *Thus*, for D or 500 they write IↃ. and then by adding another Ↄ it gradually expresses ten-times as much; so IↃↃ is 5000. IↃↃↃ is 50000, and so on. Again for M or 1000 they write CIↃ, and by joining another such Mark as C and Ↄ one on each hand, it expresses ten-times as much; so CCIↃↃ is 10000. CCCIↃↃↃ is 100000. But lastly, they had a more convenient way of expressing any Number of Thousands, which was by drawing a Line over any Expression of a Number less than a Thousand, whereby it expressed so many Thousands: so $\overline{V}$ is 5000. $\overline{VI}$ is 6000. $\overline{X}$ is 10000. $\overline{LX}$ is 60000. $\overline{C}$ is 100000. and $\overline{M}$. is a Thousand-thousands, or a Million, 1000000. $\overline{MM}$ is 2000000. But I shall insist no more.

If we now compare this Method of Notation with ours, it presently appears by the preceding Examples, that some Numbers are more shortly expressed by the *Roman* Way; but these are very few in respect of what are otherwise: And then there is here no such regular Progression in the Value of the same simple Characters as in the other Method. But we must learn the worth of this from its Application.

Of the UNIVERSAL NOTATION.

By this Name is not meant any constant or established Method used every where for the Expression of particular Numbers; but a Method of representing any Number indefinitely; in order to the more easy and general Expression and Demonstration of certain Truths in Numbers, which tho' they be limited to particular Conditions, yet not to particular Numbers, but extend to all Numbers, wherein the same Conditions are found.

The fundamental Principle of this Notation is the same as the last, *viz.* That any Mark or Sign may be instituted for the Representation of any of our Ideas; and here it is done by Letters, making the same Letter stand indifferently for any Number, upon this Condition, That through the same Proposition and Demonstration it be supposed to stand for the same Number; *i. e.* when particular Examples are applied, we must apply the same Number always to the same Letter. Thus the Letter A or B, or any other, may represent any Number we please to suppose.

Observe, This Notation, and the consequent Operations are called *The* LITERAL or SPECIOUS *Arithmetick*, which is in part the Foundation of the *Algebraick Art*; the designed Use of which in the following Work is to make easy and universal Demonstrations.

For when any Truth is proposed which is not limited to particular Numbers, but only to certain Conditions; it is not a sufficient Demonstration to shew that it holds in one, or any

any Number of particular Examples; it muſt be ſhewn that it will hold good in all Caſes poſſible: and as this muſt be done by an univerſal Method of Reaſoning abſtracted from all particular Examples; ſo there muſt neceſſarily be an univerſal Notation for Numbers. It is true indeed, that the Univerſality of a Truth may, in ſome Propoſitions, be made to appear through one, or a few particular Examples; but for the moſt part this would prove very tedious, and require many Words, which would render the Demonſtration more difficult and obſcure; and in very many Caſes could not be done at all. Which makes the *Algebraick* Method of Demonſtration neceſſary in Arithmetick, (as I have more fully repreſented in the *Preface*.) The Principles and Rules of which, as far as this Work requires, you'll find explained gradually as we proceed.

Before we enter particularly upon the other Rules and Operations of Arithmetick, there is another Diſtinction of Numbers muſt firſt be explained; *viz.*

§. 2. *Of Numbers* INTEGRAL *and* FRACTIONAL.

THIS Diſtinction proceeds from the Compariſon of leſſer Quantities with greater; and to underſtand the Nature of it aright, we muſt conſider the different Notions of *Parts* and *Whole*; thus, Every leſſer Quantity is called a Part with reſpect to a greater of the ſame Species, which is called an *Integer* or *Whole* with reſpect to the leſſer.

But there is a more general, and alſo a particular Senſe in which one Quantity is called a Part of another. In the general, no more is meant but that it is a leſſer Quantity, which with ſome one or more Quantities, alſo leſſer, make up a Quantity equal to that other; and into which therefore that Greater may be reſolved. But in a more particular Senſe, a Part ſignifies ſuch a leſſer Quantity as is contained a number of times preciſely in a greater, (or Whole:) *i.e.* it is one of thoſe leſſer Quantities, all equal among themſelves, into which any Quantity may be reſolved or ſeparated. Or we may alſo conceive it thus, *viz.* As a leſſer Quantity, of which a certain Number joined together makes up a greater Quantity or Whole. Hence ſuch a Part is called an Equal or *Aliquot* Part; and the number of times it is contained in the Whole, or the Number of equal Parts contained in the Whole, gives a Denomination to the Part, and is called its *Denominator*: ſo if any Quantity is contained in a greater 6 times, it is called a ſixth part of it. Now it is in this ſenſe only that a Part can be underſtood in Arithmetick: for in order to compare two different Quantities together by the means of Numbers, we muſt conſider them as compoſed of ſome common Element, or equal Part; by the Number of which contained in each, the Compariſon may be made; and the Value of theſe Quantities with reſpect to one another be determined. If the leſſer is an *Aliquot Part* of the greater, there is no more to be done; but otherwiſe they muſt both be conceived as compoſed of, and reducible into, ſome common Element or equal Part; ſo that if the leſſer is not one of theſe *Aliquot* Parts of the greater, yet it is equal to a greater Number of ſuch Parts; [for otherwiſe they cannot be compared together; at leaſt the Relation cannot be expreſſed in Numbers.] Such a leſſer Quantity, to diſtinguiſh it from an *Aliquot* Part, is called an *Aliquant* Part. For Example, a leſſer Quantity equal to 2 of 3 Parts of another, is called an Aliquant Part thereof.

I have hitherto ſpoken of Quantities and their Parts in general; but what is ſaid is applicable both to what is called *Continued* Quantity, (as Length, Weight, Time, &c.) or to pure *Number*. For every leſſer Number is a part of a greater; and is either an *Aliquot* or *Aliquant* Part, becauſe Unity is the common Element or *Aliquot* Part of all Numbers; ſo that every Number is a Number of ſuch *Aliquot* Parts (as Units are) of every other Number. But this difference is very remarkable, That the *Aliquot* Part of a continued Quantity, conſidered properly by itſelf in the nature of a continued Quantity,

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is only one single individual thing, or an Unit; whereas in pure Numbers, one *Aliquot* Part may be a Number greater than Unity. *Ex.* The 3^d^ part of 12 is 4 (for 3 times 4 is 12.) It is true indeed, that conceiving any continued Quantity to be resolved into 12 equal Parts, one 4^th^ part of it is equal to 3 of these parts; yet the Whole and Part are here consider'd only as pure Numbers: for to say 3 is a 4^th^ part of 12, whatever things we speak of, it is but a pure Arithmetical Expression; whereas in continued Quantities, because the Part of any thing must be of the same nature with the Whole, therefore a 4^th^ of any Length (for *Ex.*) must be one certain Length; which, in so far as is necessary to constitute a 4^th^ part, is not conceived to be any further divided, but to be an entire Length equal to a 4^th^ part of another. But a more remarkable thing is, that any continued Quantity may have any *aliquot* Part, for that we can conceive at pleasure; but pure Numbers cannot: For some have no other *aliquot* Part but Unity; as 5 and 7; and others have different parts according to their Compositions: so 6 has a half and a third part, but not a 4^th^ part or a 5^th^ part. Again, no Number can have such a Part as is denominated by a Number greater than itself; for Unity is the least part of any Number, and is the part denominated by that Number itself; so 3 has not a 4^th^ part in pure Numbers: But if we consider any Number applicately as signifying a Number of things which are divisible into any conceivable Parts, then any *aliquot* Parts of one, or of any number of these things is possible. For *Ex.* Tho' a 4^th^ part of 3 is impossible in pure Numbers, it is possible when the 3 is applied to things divisible into any Number of Parts as 4; yet here it must be carefully remarked, that this is not the 4^th^ part of the Number 3, but of a greater Number into which 3 things are resolved: Therefore every such Expression as a 4^th^ of 3, or 3 4^ths^ of 2, must be conceived with this qualification, *i. e.* as possible only in *applicate* Numbers.

The same also is to be understood, tho' the Denominator is less than the Numerator, which is considered as the *Whole*, when this Number has no such Part as is expressed; as in this *Ex.* a 3^d^ of 5.

We shall now gather together these Definitions; and from them you'll see the ground of the Distinction proposed with the Definition of the Terms.

DEFINITIONS:

1. Every lesser Quantity or Number is in a more general Sense, a Part of every greater (of the same Kind,) which is called a *Whole* or *Integer* with respect to the *Part.* But more particularly,

2. An *Aliquot Part* is that which is contained a certain number of times precisely in the Whole; and that number of times is the Name or Denominator of the Part. *Ex.* If any Quantity or Number is resolved into 3 equal Parts, one of them is an *aliquot* Part, called a 3^d^ part; so 2 is a 3^d^ part of 6. And from the nature of Numbers we have this *Corollary*, viz.

COROLLARY. One is an aliquot Part of every Number, and the Denominator is that Number itself. So 1 is the 6^th^ part of 6.

3. An *Aliquant Part* is such a lesser Quantity or Number which is not an equal Part; but contains a certain Number of some equal (or *aliquot*) Parts of the Whole. As when any Quantity or Number is not one *aliquot* Part of a greater, yet is equal to 3 4^th^ parts of that greater: So 9 is 3 4^th^ parts of 12; for a 4^th^ of 12 is 3, and 3 times 3 is 9.

4. An *Integral* (or whole Number) is that which represents things absolutely by themselves, without any comparison to other things; *i. e.* they are not considered as Parts of other things; as when we say in general 4 Things, or particularly 4 Words: And they are called *Integral* or whole Numbers only in distinction from Parts.

5. *A Fractional Number* (or a *Fraction*) is that of which each Unit represents a certain *aliquot* Part of another thing, as the Whole to which this Part relates, called hence the *Relative Integer.* For *Ex.* 1 5^th^ or 3 5^th^ parts; or 7 13^th^ parts of any thing. And because the Denomination of the Part, which is also a Number, must be expressed, therefore every Fraction consists of two Members, or requires two Numbers: for there is the Number

Number of things directly and immediately represented (as in the preceding *Examples*, the Numbers 1, 3, 7.) called hence the *Numerator* of the Fraction; and the Number of equal Parts of which the Relative Integer is ſuppoſed to be compoſed, called the *Denominator*, or the Name of the Part, expreſſing the Value of each Unit of the Numerator with reſpect to the Quantity of the Relative Integer. So in this *Ex.* 3 5th parts, 3 is the Numerator, and 5 the Denominator; the common way of placing them being to ſet the Numerator above the other thus, $\frac{1}{5}$ or $\frac{3}{5}$ $\begin{matrix}\text{Numerator.}\\\text{Denominator.}\end{matrix}$

SCHOLIUM. From this Definition of a Fraction it is plain that the Numerator may either be leſs or greater than the Denominator, or equal to it; for we may as reaſonably ſay $\frac{7}{5}$ (or 7 5th parts) as $\frac{4}{5}$; if we underſtand it according to the Definition, *i.e.* as expreſſing 7 things each of which is equal to a 5th of another thing; and not as if 7 were ſuppoſed to be taken out of 5, which is impoſſible. By comparing the Numerator and Denominator, we have this Conſequence; *viz.*

COROLLARY. If the *Numerator* is greater, equal to, or leſſer than the Denominator, the Quantity expreſſed by that fractional Number is greater, equal to, or leſſer than the Integer; becauſe the Denominator repreſents all the Parts of the Integer, and the Numerator ſhews how many are taken.

And this gives riſe to a Diſtinction of *Proper* and *Improper Fractions*, as the Numerator is leſs or not leſs than the Denominator. But a more particular Explication of this, with other Diſtinctions of Fractions, we muſt refer to another place. And here you may *Obſerve*, That inſtead of the Names *Integral* and *Fractional*, we might as properly call them *Abſolute* and *Relative* Numbers; which do very well expreſs their different Natures: For the firſt conſiders things ſimply and abſolutely in themſelves; and the other conſiders things relatively, as Parts of other things.

It is to be alſo *Obſerved*, That Fractions are a more general Kind of *applicate* Numbers: For the Numerator (or the Number of things directly deſigned) is reſtrained; ſo that it does not repreſent a Number of any things indifferently; but is limited to a certain Relation to ſome other thing: nor does it expreſs any Part of that other thing; but ſuch a Part or Parts as the Denominator expreſſes: yet while there is no particular thing named as the relative Whole, it is in this reſpect a general and *abſtract* Fraction, (but not a pure abſolute Number;) ſo $\frac{2}{3}$ is a general and *abſtract* Fraction; but $\frac{2}{3}$ of a Day is *applicate*. Wherefore in every *applicate* Fraction there are two Denominations to be conſidered, which we may call the *Relative* and the *Abſolute*: The firſt is the *Denominator* of the Fraction, and the other is the Name of the Integer. But if the Integer is not one, but a Multitude of Things (as $\frac{2}{3}$ of 6 Pounds) that is to be conceived as an Integer or *one* of its own Kind; or rather we are to conceive this Expreſſion as a mixt Form reducible to a ſimple, wherein the Integer is an Unit of a particular Name; ſo $\frac{2}{3}$ of 6 is equal to $\frac{12}{3}$ of 1. But this muſt be left to its own place.

GENERAL SCHOLIUM.

Of the different Senſes in which Unity *is* Diviſible *and* Indiviſible, *and the Converſion of Numbers from* Integral *to* Fractional, *and contrarily; ſhewing in general wherein their Operations muſt be the ſame, and wherein they differ.*

Here now is the place to explain the different Senſes in which Unity is diviſible or indiviſible. And in the firſt place this is plain, That *Unity* in its own Nature as Number is *indiviſible*; for there can be no Number of Things conceived leſs than *One*: but if we conſider the Subject or Thing to which the Idea of *Unity* is applied; as that is capable of diviſion into real or imaginary Parts; or as it is really a Collection of diſtinct Things united by the Imagination; ſo, what is the Subject of *Unity* in one View and under one Denomination, may be the Subject of Multitude under another. For *Ex.* 1 Pound is the ſame as 20 Shillings; wherefore it is not the Number *One* that is diviſible, but ſome

continued Quantity, as a Yard, a Day, *&c.* or some Number of Things comprehended under a singular Denomination. Hence again we learn to correct a Vulgar Error, *viz.* That a Proper Fraction (*i. e.* whose Numerator is less than its Denominator) is a Number less than *Unity*. It does indeed represent a Quantity less than the relative Integer or Unit; but is not a Number less than *Unity*: For the *Numerator*, which cannot be less than 1, is as properly a Number as if it were applied to things under an absolute Denomination: so $\frac{3}{5}$ of a Pound does as truly express three things, as 3 Pound does; differing only in the Value and Way of denominating the things. Again, because one Quantity or Number cannot be referred to another as a Part or Parts, unless that other be really or conceivably divided into, or composed of such a Number of Parts; therefore, strictly speaking, the relative Integer of every Fraction is what we may call a *collective Unit*, or a real Multitude united together in one Whole under a particular Denomination. Hence a Proper Fraction is in effect some lesser Number compared to a Number greater than it, and always greater than Unity. For *Ex.* $\frac{3}{5}$ is 3 things taken out of 5; or 3 5th Parts of a Whole composed of 5 Parts: and in this View only a Proper Fraction is a lesser Number than its relative Integer; yet not as this is *Unity*, but as it is really a Multitude. To have done; The proper Arithmetical Value of One or any other Number is invariable, *i. e.* One, Two, *&c.* is always the same Number, in whatever things and however denominated; but take the Number with the Application complexly, there may be a difference; so that what is equal to *Unity*, or any other Number, in one denomination, may be a greater Number applied to another; as 1 Shilling is equal to 3 Groats, or to 12 Pence. The mixt Value is the same in all these, yet the Numbers and Denominations differ: Also what is an Integral Number applied to Things under a proper and absolute Denomination, may be converted to a Fraction or relative Number by applying a relative Name or Denomination; so 3 Shillings is the same as $\frac{3}{20}$ Parts of a Pound. In both Expressions the Number and mixt Value is the same, only the Things are differently denominated in the Application; this being indeed all the difference betwixt Numbers *Integral* and *Fractional*; yet this difference is the occasion that these two Kinds must be handled separately: for the Denominator of a Fraction being also a Number, respect must be had to that in every Operation, which occasions more Work than in Integrals. But still the fundamental Operations of Numbers are those performed by Integral or Absolute Numbers; for the Numerator and Denominator of a Fraction, taken by themselves, are of the same general Nature with every other absolute Number, and can have no other Operation applied to them; and the way of making that Application so as to fulfil all that both the Denominations, Relative and Absolute, (where both are considered) do demand, is the only new thing in the Operations of Fractions. Therefore after the Operations in Whole (or Absolute) Numbers are explained, which will employ the remaining Chapters of this Book; the same shall be done for Fractions in the second Book.

CHAP. III.

ADDITION *of Whole and Abstract* NUMBERS.

DEFINITION.

ADDITION *is the finding one Number equal to two or more Numbers taken all together*; that is, finding the most simple Expression of a Number, (according to the established Notation) containing as many Units as are in all the given Numbers taken together; which is hence called their *Sum*. For *Example*, the *Sum* of these Numbers 8, 17, 24, 675, is found by the following Rule to be 724.

SCHO-

SCHOLIUMS.

1. Before we enter upon the particular Rules of *Addition*, it is necessary to make the following Reflection upon the Method of *Notation*, as it is in part the Foundation of all other Rules of Operation; *viz.* That as by any established Notation, whatever it be, we know how to express any Number, or the whole Series of Numbers in a continued Succession from *Unity*, by the adding or joining Unity after Unity for ever; so by Application of this Institution, and the general Axiom, that *The Whole is equal to all its Parts*, we see a possible way of finding and demonstrating the Sum of any two Numbers, *viz.* by beginning with one of the given Numbers, (and it is best to take the greater) and joining to it all the Units of the other one after another, expressing the Sum at every step, according to the Rule of *Notation*, till the last Unit is added; and then you have the Sum sought: because if all the Units in any Number are added successively to another Number, the first Number (which is nothing else but all its Parts together) is certainly added to the second Number. For *Example*, If (according to the present *Notation*) it were proposed to find the Sum of 8 and 7: I take 8, and after it I set down all the Units of 7 separately, and by adding them one by one to 8, and expressing the Sum as it gradually increases, the last and total Sum is 15.

$$\left.\begin{array}{cccccccc} 8, & 1, & 1, & 1, & 1, & 1, & 1, & 1, \\ & 9: & 10: & 11: & 12: & 13: & 14: & 15. \end{array}\right\}$$

In the same manner may we find the Sum of any other two Numbers, or of any Numbers more than two, by first adding any two of them, and then to the Sum adding any other of them; and so on.

Let us next *observe*, What is arbitrary, and what is not so in this Operation. In the first place, As the whole System of the Signs of Numbers, both by *Words* and *Figures*, is a pure arbitrary Institution, so the Addition or Expression of the Sum of Unity, and any Number, is directly and immediately a Part of that Institution; and therefore has no other Reason or Demonstration but that it is so Instituted. Again, tho' the Names of all Numbers are contained in the Institution, yet the finding the Sum of any other two Numbers, or calling it by such a Name in the System, is not immediately a Part of the Institution, but a Consequence, and therefore Demonstrable: For the pure and simple Institution is all comprehended in the System of Signs taken in a gradual Succession from Unity, and proceeding by a continual Increase of Unity; and therefore contains immediately no Question or Case of Addition, but that of adding Unity to Unity, or to any other Number; all other Sums being found by Consequence from this, which therefore have a proper Demonstration, different according as the Consequence is less or more remote. As you'll afterwards learn.

Now the Method of Addition most immediately connected with the Institution is that above explained: But it is easy to perceive how tedious and insupportable this Method of Addition would be upon any System of Notation; and as upon different Systems, the Remedy of this Difficulty would be less or more perfect, so the present admirable Method of *Notation* affords the most easy and perfect Rules for *Addition*, (and all other Operations) whereby such Additions are performed by a few and easy steps, which cannot be done all at once, (as we add Unity to any Number) and would be insupportably tedious to do by so many steps as the preceding Method prescribes: yet this is to be observed, that as the established *Notation* is the Ground-work and Foundation of all, so there are some simple Cases that can be done no other way; as shall be presently explained,

The most simple Cases in any *Problem* are first in the Order of Science; and here the Addition of Unity to any Number is the first and most simple Case: but as it is contained immediately in the Rule of *Notation*, therefore it is supposed in the following *Problem*, as a previous and fundamental Principle.

2. This Sign or Character $+$ set betwixt two Numbers, signifies the Addition of the one to the other; and is a complex or indefinite way of representing the Sum: thus, $3+4$ signifies that 3 and 4 are added together; and we read the Sign by the word *more*. *Exam.* $3+4$ is 3 *more* 4; and thus it expresses the Sum in a complex manner by

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the Parts. And when more Numbers are added, they are joined by the same Sign; thus $3+4+9$ is the Addition or Sum of 3, 4, and 9. The Use of this in particular and determinate Numbers, is chiefly by way of Abbreviation for the neater and shorter Explication of the Work of Addition in particular Examples, as you'll see immediately.

But the principal Use of this Sign of Addition, is for the Expression of the Sum of Numbers, universally or indefinitely represented by Letters in the Algebraick Art. Thus, Any two Numbers being represented by A, B, their Sum is expressed indefinitely thus, $A+B$, and the Sum of A, B and C, is $A+B+C$, and so on; which is the General Rule of the Literal Addition, or Numbers expressed by Letters. Some other particular Considerations relating to this, you'll find in another Place.

Observe again, That as the same Number may be variously represented, either by one simple Expression, or by the simple Expressions of other Numbers variously applied to one another by sundry Operations, whose final Result brings out the same Number; so to express the equality of Value betwixt these different Expressions of the same Number, we use this Sign $=$ set betwixt them; which we use for Abbreviation in explaining the Work of particular Examples in all the common Operations. Thus 8 added to 7 makes 15, which is therefore expressed either complexly $8+7$, or simply 15; and to signify the Equality of these Expressions, we write $8+7=15$, and read it thus, 8 more 7 is equal to 15. Universally, $A+B=D$, signifies that the two Numbers expressed by A and B, are together equal to the Number expressed by D; and so of other Examples of Addition. As $3+4+5=12$, or $A+B+C=D$. Applications of this, in other Operations, you'll find in their Places. I have only this further to say here, That the different Expressions of the same Number, constitute what in the Algebraick Art is called an ***Equation***, that is plainly an Equality of Value betwixt two Expressions of Number: In the finding of which, from the Conditions and Circumstances of any Proposition, with the various Changes and Transformations to be made upon them, by the Application of different Operations, whereby one Equation is deduced from another, consists the Algebraick Art of Reasoning; which, so far as the present Undertaking requires, you'll learn as we proceed.

PROBLEM.

To add two or more Numbers into one Sum, *or simple Expression.*

CASE I. *To add any two Digits or Numbers less than Ten.*

Rule. Take the greater of the two, and to it add all the Units of the other one by one, expressing the Sums gradually according to the Rule of Notation, as explained in *Schol.* 1.

Exam. To add 9 and 6, it is, $9+1+1+1+1+1+1=15$; for adding the 6 Units gradually, and expressing the Sums in the Order of Notation, they make this Series, 9, 10, 11, 12, 13, 14, 15, the last whereof is the Sum sought.

Demon. The Reason of this Rule is already explained; and that there cannot be another Way of adding two Digits, is evident.

SCHOLIUM. Practice by degrees fixes the Sums of all the various Examples of this Case in our Memory; whereby we become capable to pronounce the Answer as readily as the Question is proposed: For upon Reflection it will be found, That we do not always calculate, and add in the manner directed; but know the Sum purely by Memory; which was no doubt acquired by repeated Practice of adding them together Unit by Unit; for it could be done no other way: and therefore it is the only Method we can take to teach young ones, who know nothing but the Names of Numbers in the simple progressive Order from Unity; who may be assisted by their Fingers in this manner, *viz.* Let them tell out the Units of the lesser Number upon their Fingers, then take the greater Number and add to it the Units of the other from their Fingers, expressing the Sums gradually ac-

cording to the progression of Names, and the last is the Name of the Number sought. Again, if thro' any confusion of Thought, the Sum of two Digits should not readily occur, or if one should pretend to deny or doubt of it, this is the natural and certain way of finding or demonstrating it; [which may also be done by dissolving one of the Numbers into two Parts, and adding them successively to the other Number, which is only reducing the Question to a more simple Case, where the Sum may be more easily remembered.] And this brings to my mind, That in Conversation I have met with Persons who would affirm that the Addition of two Digits is a thing not properly demonstrable, but the immediate Effect of an Institution; the contrary of which, I think, I have sufficiently shewn. The Occasion of this Opinion may be the familiarity we have with these simple Cases from our first acquaintance with Numbers; so that remembring the Sums as readily as we do the Sum of Unity and any Number, we are apt to fancy we came by them without any Reasoning or Calculation, because we have them so now. I must also observe, That the Method of our common Books of Arithmetick may have contributed to this; for in these we have but one general Case and Rule of Addition, in which it is supposed that we know already, or can find the Sum of any two Digits; and this perhaps is done upon a Supposition of its being simple and easy: But this would be no reason for omitting it in a Work designed for a just and rational System; which must therefore explain the Connection and Dependence of all the Parts of the Science upon their first Principles, and upon one another.

In the last place, then, since this Case is supposed in all other Cases of Addition, it is necessary the Learner know the Sum of all its Examples as readily as the Question is proposed. In order to which, I shall express them all in the following Table; from which they may be got by heart more easily by those who are Beginners in this Science.

TABLE shewing the Sums of any two Digits.

1	2	3	4	5	6	7	8	9	0
2	3	4	5	6	7	8	9	10	1
	4	5	6	7	8	9	10	11	2
		6	7	8	9	10	11	12	3
			8	9	10	11	12	13	4
				10	11	12	13	14	5
					12	13	14	15	6
						14	15	16	7
							16	17	8
								18	9

The Construction of the *Table* is obvious, and the Manner of using it is this: Take the greater of the two Digits, whose Sum is sought, in the upper Line, and the lesser on the Right-hand Column; in the same Line with this, and under the other stands the Sum. So under 8, on the Head and in the same Line with 6, on the Side stands the Sum 14.

CASE 2. *To add any two or more Numbers into one Sum.*

Rule 1. Place the given Numbers under one another, so that the Figures in like places of each be directly in one Column, [*i. e.* all Figures in the first or Units place in one Column; all in the second place, or Tens, be also in one Column; and so on in the Order of Places.] Then *secondly*, begin at the Units place of the lowest Line or Number; add that Digit to the next above of the same Place and Column; and to this Sum add the next Digit, and so on till all the Digits in that Column are added; then if the Sum is less than 10, set it down under the Figures added; and add up the second Column the same way; and so on thro' all the Columns, taking the Figures in each according to their simple Value. But if the Sum of the first, or any other Column exceeds 10, then it is either a precise Number of Tens (as 20, 30, &c.) or it is such a Number of Tens with some

Number

Number less than 10, (as $24=20+4$; or $68=60+8$.) In the first Case write down 0, and in the other the Number over a precise Number of Tens, (as 4, if it is 34; or 8, if it is 68) and for every 10 in the Sum carry 1 to the next Column; *i. e.* add the Number of Tens in the Sum of every Column to the next Column. Having thus gone thro' all the Columns, the Number of Tens in the Sum of the last Column is to be set down on the left of all the Figures already found; or, the whole Sum of the last Column is set on the left of all the preceding Figures: and all these Figures thus found and placed, express the Sum sought.

$$\left.\begin{array}{r} 654 \\ \underline{243} \\ \underline{897} \text{ Sum.} \end{array}\right\}$$

Ex. 1. $654+243=897$. wrought as in the Margin thus: Beginning at the Units place of the lower Number, I say, $3+4=7$, which I set under the Numbers added; then $4+5=9$; and lastly, $2+6=8$; and the total Sum is 897.

$$\left.\begin{array}{r} 897 \\ \underline{968} \\ \underline{1865} \text{ Sum.} \end{array}\right\}$$

Ex. 2. $897+968=1865$. Thus, $8+7=15=10+5$; therefore I set down 5, and carry 1 to the next place; then $1+6=7$, and $7+9=16=10+6$, for which I set down 6, and carry 1 to the next place; then $1+9=10$, and $10+8=18$, which being set down, the total Sum is 1865.

Let the Learner practise more Examples of this Kind to himself.

Scholiums.

1. If there are more than two Numbers to be added, this Rule plainly supposes, that we can readily, in our Mind, add any Digit to any other Number; which therefore might have been considered as a particular Case: But, rather than make too many Cases, I have made this Supposition; nor has it any thing contrary to good Order, because it is indeed no other than a particular Case comprehended under the general Rule here delivered: so that by practising Examples of this simple Case, we soon acquire the Capacity supposed for more complex Cases. For, if the greater Number has, in the place of Units, such a Digit, as added to the other Digit, makes a Sum less than 10, the Total is plain; thus, $64+2=66$; $22+7=29$. But if the Sum of the two Digits exceeds 10, what's over belongs to the place of Units of the Sum, and 1 for the 10 is to be added to the remaining part of the greater Number. Thus $46+8=54$, *viz.* $6+8=14$; which makes 4 in the place of Units of the total Sum, and then $1+4=5$ in the place of Tens. Also $397+9=406$ for $7+9=16$, then $1+39=40$.

There is also another Supposition in this Rule, *viz.* That we can readily perceive how many Tens are contained in any Number, or in the Sum of any Column of Digits, and also what remains over. Now, tho' these do indeed belong to other Rules, yet the Rule of *Notation* has already taught us them, in the Cases here supposed: For, if we write down any Number, the Figure in the place of Units is the Number over all the Tens contained in it, and the remaining Figures on the Left-hand, taken by themselves, express the Number of Tens; as has been explain'd in the Rule of *Notation* (see *Corol.* 4. after the Problem in Chap. 2.) thus, 40 is 4 Tens and 0 over, 87 is 8 Tens and 7 over, 124 is 12 Tens and 4 over. In small Examples the Number of Tens is easily perceived without writing down the Sum, and in greater Numbers write it down. But there are other Methods of managing Addition, whereby the Number of Tens will be mark'd out in the Operation, which shall be presently explained; but I shall first make some larger Examples, and shew the Application of the preceding Rule to them.

Ex.

Ex. 3d.	Ex. 4th.	Ex. 5th.
868	87869	34
753	26100	596
209	7895	789
546	3746	6897
370	65432	2540
921	789576	4698
689	456789	246
796	324160	86532
5152	1761567	78063
		65
		180460

The Operation of *Ex.* 3. is thus, 6+9=15; +1=16; +6 =22; +9=31; +3=34; +8=42; which is 2 for the first place and 4 to carry to the 2d: thus, 4+9=13; +8=21; +2=23; +7=30; +4=34; +5=39; +6=45; which is 5 for the 2d place, and 4 to carry to the 3d: thus, 4+7=11; +6=17; +9=26; +3=29; +5=34; +2=36; +7=43; +8=51; which being the Sum of the last Column, is all written down; so the total Sum is 5152.——Observe that you are to read the Operation thus, 6+9=15, then 15+1=16, *&c.* But to prevent writing the last Sum twice, I have separated it from the Number to be next added to it by a Semicolon, by which you must understand, that it is added to the next Number. [But not that these Expressions are all equal, *viz.* that 6+9, 15+1, *&c.* are equal.]

After the same manner examine the other two Examples, for your own Practice.

2. If there are many Numbers to be added, so that the Sum of every Column be a great Number, we may save the Memory being too much burdened, and the Work render'd thereby more difficult and uncertain, by various means; as,

First, By making a Point at every 40 or 50, or 100. Thus, Add up the Column till you have a Sum containing any Number of Tens you please, as 50 or 100, making a Point at the last Figure which makes up your Number of Tens, and if there is any excess over Tens, add it to the next Figure, and so go on thro' all the Column: When you have finished it, if the particular Sum after adding the last Figure is less than 10, set it down in the place of the total Sum (or Answer,) and for every Point carry as many Units to the next place as there are Tens in the Number pointed, (*i. e.* 3, 4, 5, or 10, for 30, 40, 50, or 100,) and if the last particular Sum exceed a Number of Tens, set down the Excess, and join that Number of Tens to the Number arising from the Points, and carry the Sum to the next Column; and thus go thro' them all.

So in the annex'd *Example*, which I point at 50, I proceed thus; 9+7 is 16, and 8 is 24, and 7 is 31, and 9 is 40, and 8 is 48, and 9 is 57; here I point at the 9, and take the 7 which is over 50, and say, 7+6 is 13, and 9 is 22, and 4 is 26, and 5 is 31, and 6 is 37, and 7 is 44, and 5 is 49, and 7 is 56, which makes a Point at the 7, and 6 over; with which I proceed, saying, 6+9 is 15, and 8 is 23; therefore I write down 3 in the Sum, and I have 2 (for the 20) to be added to the Number of the two Points, *viz.* to 10 (for every Point is here 5) and so I have 12 to carry in to the next Column; which I add in the same manner.

To prevent blotting of Accounts, which is the Objection made against this Practice, you may make the Points upon a Shred of Paper applied to the Column.

There is also another useful Method of summing up long Columns, *viz.* by distributing the Numbers into Parcels of 9 or 10 Numbers in each, (so that the Sum of each Column in every Parcel shall never exceed 100) adding each Part by itself, and then adding their Sums into one total Sum. As in the following *Example*, which needs no further Explanation.

528
269
5.87.
675
367
986
89.5
764
489
76
29.
38
19
27
38
27
9
5803

DEMONSTRATION of the Rule of CASE 2.

678
957
824
928
653
256
890
348
269 ... 5913
372
908
735
687
943
259
167
896
735 ... 5702
Total 11615

This we deduce from the Nature of Notation, and the common Axiom, *That the Whole is equal to all its Parts*: Thus, every Number consists of as many Parts as there are significant Figures in it; and the Figures standing in the same Places of different Numbers are the similar or like Parts of these Numbers, which being therefore added together according to their simple Values, the Sum has all the same secondary or local Value, *viz.* That of the Place in which these Figures do all stand: But by *Cor.* 4. *Chap.* 2. any Number is equal to as many Tens as the Figures above the place of Units taken by themselves do express, and to that Number of Units more; so 245 is 24 Tens + 5. And again, 1, or any Digit standing in any place is equal to 10 times the Value of the same Digit in the preceding place: wherefore the Sum of the Digits standing all in the like places of different Numbers, is equal to as many Units of the Value of the next higher place as there are Tens in the Sum, and to as many Units of the place added as are over that Number of Tens. Hence we see the Reason of all the Parts of the Rule: for the Parts of Numbers added must be similar, else the Sum is false; so $4 + 7 = 11$, which supposes them to be both of one local Value; but if the 4 be in the place above the 7, the Sum is not 11, but 47: This explains the first Part of the Rule. Then, by carrying forward the Number of Tens found in the Sum of every Column (added according to the simple Values) and adding that Number to the next Column (with which it has a similar Value) we actually add all the Parts of each of the Numbers given; and do add them so, as that all the Units of each particular Digit set down in the total Sum have one local Value, which is the same that the Parts to whose Sum it belongs have in the Numbers added; but the Parts are equal to the Whole, therefore the Numbers set down according to the Rule, express the true Sum.

As to the second Part of the Rule, *viz.* The carrying forwards the Number of Tens, we may see the Reason of it another way; Thus, suppose every Column added by itself, and the Sums set down separately so as the first Figure of the Sum stand in the place of that Column; and let all these Sums be again added in the same manner; and so on till we have the most simple Expression of the Total of the given Numbers. Now this is a very natural way of Operating, which carries its Reason with it; but it is also plain, that the carrying forward the Tens has the very same Effect; for it is only doing that all at once, which is here done at several steps, and is thereby a more compendious Way of finding the Number sought, and therefore preferable.

Example I.

	348
	695
	476
	587
	290
Sum of Units	26
Sum of Tens	370
Sum of Hundreds	2000
Total Sum	2396

Example

Example II.

5789	
6958	
4746	
8984	
4897	
6518	
7968	
9376	
56	Sum of Units
480	— of Tens.
5700	— of Hundreds.
49000	— of Thousands.
06	Sum of Units.
130	— of Tens.
1100	— of Hundreds.
14000	— of Thousands.
40000	— of Ten Thousands.
55236	Total.

§. 2. *Of the Proof of Operations in Arithmetick, and particularly of* ADDITION.

THE Proof of any Operation in Arithmetick is ſome other Operation, by which we are made more certain that the firſt Work is right performed, and the true Anſwer found.

It is different from the Demonſtration of the Rule; for this ſhews that if the Rule is followed, the true Anſwer will be found: But the Proof ſuppoſes the Truth of the Rule, and only ſhews whether we have obſerved it in doing the Work.

The Proof of any Operation ought naturally to be another eaſier than itſelf, in which there is leſs hazard of erring; otherwiſe we are not more certain of the one than of the other: And if we be very ſcrupulous, we may require again a Proof of this ſecond Work, by a third eaſier than the laſt; and ſo on till we have the moſt ſimple Operation. But a little practice puts us beyond the need of all this; and we may be ſatisfied with any one Proof, even tho' it is not an eaſier Work than that to be proved, becauſe there is leſs hazard of Erring in both than in one. However, the eaſieſt Proofs, if they are not tedious, are always preferable.

PROOF of ADDITION.

If we ſhould propoſe a more ſimple Kind of Operation for *Proof of Addition*, we have none but the adding Unit by Unit, already ſpoken of in *Notation*. But the ſimplicity of this Method is infinitely overbalanced by its tedіouſneſs; therefore we muſt be content with another Application of the ſame Rule; which may be made different ways.

1. If the Numbers have been added all together without diſtributing them into Parcels, (as has been explained) then by making ſuch a Diſtribution, and adding them that way, we ſhall prove the Anſwer which was found the other way; for the total Sum muſt be the ſame in both Methods: But if it has been at firſt wrought this way, we may make a Proof by making a different Diſtribution.

2. Whatever way the Work is done at firſt, we may do it again the ſame way, only beginning at the upper Line, and adding downwards.

 3. There

3. There is another way ingenious enough, and very eaſy; but as it ſuppoſes a ſmall Capacity in Subtraction, it may ſeem not agreeable to good Order to propoſe it in this place: For tho' one Operation or Rule may no doubt be made ſerviceable to another, yet every thing ought to be in its Place; therefore you'll ſay, that according to the utmoſt ſtrictneſs of Method, what requires Subtraction, ought to come after Subtraction. But the thing ſuppoſed is no more than that we can readily know the difference betwixt 9 and any Number leſs than 18; and this Capacity cannot but be already acquired by the practice of Addition, (of which Subtraction is but the Reverſe.) There is alſo ſomething in it which produces the ſame Effect as *Diviſion;* but as it does not require the Rule of Diviſion, that furniſhes no Objection: Therefore I ſhall explain this Method here as its proper place.

Rule. Take each of the given Numbers ſeparately, and add all their Figures together as ſimple Units; and in doing ſo, when you have made a Sum equal to 9, or greater than 9, but leſs than 18, neglect the 9, taking what's over and add to the next Figure; and go on ſo till you have gone thro' them all, and mark what is over or under 9 at the laſt Figure; but if the Sum of all the Figures is leſs than 9, mark that Sum. Do the ſame with each of the given Numbers, ſetting all theſe Exceſſes of 9 together in a Line, (in any Order,) then ſum them up the ſame way, marking the exceſs of 9 as before, (or what the Sum is leſs than 9.) Laſtly, do the ſame with the total Sum, and what is under 9, or over any Number of 9's in this, muſt be equal to the Exceſs (or Number leſs than 9) laſt marked; elſe the Work has not been right performed.

I ſhall explain this Practice by one Example.

Example.

	excess of 9's	
2743		7
4678		7
5365		1
12786		6

Beginning with the upper Line, I work thus; $2+7=9$; then $4+3=7$, which I have ſet down on the Right. Then to the ſecond Line, $4+6=10$, which is 1 over 9; then $1+7+8=16$, which is 7 over 9; which I ſet under the Figure laſt found. Again, to the third Line, $5+3+6=14$, which is 5 over 9; then $5+5=10$, which is 1 over 9. Then I do the ſame with the Line of the Figures now found, *viz.* $1+7+7=15$, which is 6 over 9; and finding the ſame Exceſs of 9's in the Sum (12786,) I conclude the Work is right performed.

Obſerve, It will have the ſame Effect if, inſtead of ſetting down the Exceſs of 9's in the ſeveral given Numbers, we carry the Exceſs of one Line into another, and only mark the laſt Exceſs; which ought to be the Exceſs in the total Sum.

Demonſtration. In order to ſhew the Truth of this *Rule,* I muſt premiſe and demonſtrate this *Lemma,* which will be uſeful alſo afterwards.

LEMMA.

The Figure that ſtands in any place of a Number, taken in its ſimple Value, is equal to what will remain after 9 is taken out of the compleat Value as oft as poſſible; i. e. *after all the 9's contained in it are taken away.*

For *Example,* If all the 9's contained in 700 are taken away, there remains the ſimple Number 7.

Demon. Any Figure ſtanding in any place of a Number is equal to ten-times the Value of the ſame Figure in the next lower place, (by what has been ſhewn in Notation;) *i. e.* equal to 9 times $+$ 1 time that Value, (becauſe $9+1=10$.) But 9 times any Number is a preciſe Number of 9's; which being taken away, there remains once the Value of it in that next place, and this again is equal to 9 times $+$ 1 time the Value of the ſame Figure in the next lower place, and the 9 times being taken away, the 1 time remains; and ſo on till you bring it down to the place of Tens, where it is equal to 9 times its ſimple Value $+$ once that Value; and the 9 times taken away, there remains the ſimple Value: But

But thus we have suupofed all the 9's to be taken out of it, and confequently the *Lemma* is true.

Corollary. The Sum of all the Figures in any Number, taken as fimple Units, is the Remainder after as many 9's are taken out of that Number, as are to be found feparately in the compleat Value of each of the faid Figures (becaufe each of thefe Figures taken fimply, is the Excefs of the 9's contained in that Part,) and if that Sum is lefs than 9, it is the Remainder after the 9's contained in the given Number are taken away: but if it is not lefs than 9, the Remainder, after all the 9's are taken out of it, is the Remainder of 9's in the given Number: For it is plain that there can be no more 9's in any Number than what are in the feveral Parts and in the Sum of the Excefs of 9 in the fame Parts.

Now from this *Lemma* and *Corollary*, the Demonftration propofed will be plain: For by adding the Figures of any Number according to the Rule, it is evident we find the Excefs over all the 9's contained in their Sum (taken as fimple Units.) And this is the Excefs of all the 9's contained in the faid Number by the *Corollary*. But again, the Excefs of 9's in each of two or more Numbers being taken feparately, and the Excefs of 9's taken alfo out of the Sum of the former Exceffes, it is plain this laft Excefs muft be equal to the Excefs of all the 9's contained in the Total of all thefe Numbers, (the Parts being equal to the Whole.)

SCHOLIUMS.

1. In the Demonftration of the preceding *Lemma*, I have taken it for a Truth, that 9 times any Number is a precife Number of 9's; *i. e.* that it is equal to that Number of times 9, (without any thing over.) For *Example*, that 9 times 7 is 7 times 9; for the Demonftration of which (if it is required) I refer you to Chap. 5. as I have done already in a like Cafe.

2. To this Proof it is commonly objected, That a wrong Operation may appear to be true; which muft be owned: for if we change the Places of any two fignificant Figures in the Sum, it will ftill appear right. So in the preceding Example, the true Sum is 12786: But fuppofe, thro' miftake, it had been 12768; it is plain this Method of Proof would make it appear right, becaufe there is the fame Excefs of 9's where there are the fame Figures, whatever order they ftand in. But then confider, a true Sum will always appear true by this Proof, (for that is demonftrated) and to make a falfe Sum appear true, there muft be at leaft two Errors, and thefe oppofite to one another; *i. e.* one Figure greater than it ought to be, and another as much lefs; and if there are more than two Errors, they muft always balance among themfelves; *i. e.* the Sum of the Figures that are greater than they ought to be, muft always be equal to the Sum of the Figures that are deficient; elfe it is plain, a falfe Sum will not appear to be right. But now if we confider what an exceeding great Chance there is againft this particular Circumftance of the Errors, and how fimple the Proof-work itfelf is, we may truft to this Proof as fafely as to any other.

CHAP. IV.

SUBTRACTION *of Abstract Whole Numbers.*

DEFINITION.

SUBTRACTION *is the taking one Number out of another; or finding the* Difference *betwixt two Numbers*; *i. e.* the moft fimple Expreffion of that Number whereby the greater of two given Numbers exceeds; or the leffer comes fhort of the other.

Example.

Example. The Difference betwixt 8 and 3 is 5: Betwixt 48 and 19 is 29.

Observe also, That for distinction the greater Number is called the *Subtrahend*, and the lesser the *Subtractor*; and the Number sought, or the Difference, is called also the *Remainder*, *i. e.* what remains after the lesser is taken out of the greater.

SCHOLIUMS.

1. As the Effect of *Subtraction* is plainly the Reverse of *Addition*; so is its Operation: Wherefore we have this first to observe, That by a continual retracting Unity after Unity (by the reverse of what was done in Addition) the difference of any two Numbers may be found; which shews an immediate Connection betwixt this Work and the Rule of Notation. But the insufferable tediousness of this Method is removed by the following Rule, whereby that is done by a few easy steps, which cannot be done all at once, and ought not to be done by more steps than are necessary: and this we owe to the Method of Notation. But we must also observe, That as the subtracting Unity from any Number is the most simple Case, so it is immediately contained in the Rule of Notation, and presupposed in the following *Problem*; and is also the only Method by which one Digit can be subtracted from another; which therefore I make the first *Case* of the following *Problem*, as that upon which all other *Cases* depend.

2. This Sign or Character — set betwixt two Numbers, signifies the *Subtraction* of the one from the other; and is a complex or indefinite way of representing the *Difference*. Thus $7-4$ signifies that 4 is taken from 7; and we read the *Sign* by the word *less*. *Examp.* $7-4$ is read, 7 less 4: and thus it expresses the difference in a complex Manner, by the Whole and Part taken away. And if more Numbers are successively *subtracted*, we prefix the same Sign to each of them; thus, $12-4-3$ signifies that 4 is taken from 12, and then 3 from the first Remainder; or, which is the same thing, that 4 and 3 both (*i. e.* 7) is taken from 12. By this *Sign* we explain in a neat and brief way the Work of particular Examples in *Subtraction*; as you'll presently see.

But the principal Use of this Sign of *Subtraction* is for the Expression of the difference of Numbers represented by Letters in the *Algebraick Art*. Thus, $A-B$ expresses the difference of A and B; $A-B-C$, the difference of A and $B+C$; for by taking all the Parts of one Number successively from another, we take the Whole of that Number from this. And this is the *General Rule of the Literal Subtraction*: other particular Cases you'll earn in another place.

PROBLEM.

To Subtract *one Number from another; or, to find their* Difference.

CASE I. *To find the Difference betwixt two Digits.*

Rule. This is to be done by a continual retracting each *Unit* of the *Subtractor* successively from the *Subtrahend*, and expressing the Differences gradually, according to the *Rule of Notation.*

Example. $9-4=5$. For $9-1-1-1-1=5$. The gradual Series of Differences being 8, 7, 6, 5; because $9-1=8$, $-1=7$, $-1=6$, $-1=5$; the Number sought.

SCHOLIUM. As all other Cases necessarily suppose this one; the Answers of all its Examples ought to be ready in the Memory: and therefore I shall put them all in the following *Table*, for their Use who are mere Novices in *Arithmetick*. But I must here *observe*, That as this is only the *Reverse* of *Addition*, whoever is Master of that, (as they ought to be before they enter on *Subtraction*) will be able at once to pronounce the difference of any two Digits.

TABLE for finding the Differences *of any two Digits.*

1	2	3	4	5	6	7	8	9	0
0	1	2	3	4	5	6	7	8	1
	0	1	2	3	4	5	6	7	2
		0	1	2	3	4	5	6	3
			0	1	2	3	4	5	4
				0	1	2	3	4	5
					0	1	2	3	6
						0	1	2	7
							0	1	8
								0	9

The Use of this Table will, I think, be obvious: Seek the *Subtractor* on the Right-side Column, and against it under the *Subtrahend*, found in the Upper-Line, is the difference sought. *Examp.* Under 7 on the Head, and against 4 on the Side, you find 3, the difference of 7 and 4. Or also thus, Take the *Subtrahend* on the Head, and in the Column under it, seek the *Subtractor*, against which on the Side is the Difference.

CASE 2. *To* Subtract *any Number from another.*

Rule 1. Write the one under the other according to the Order of Places, (and most commonly the *Subtrahend* is set above the other, tho' this is not necessary.) Then, *Secondly*, take the difference betwixt each Figure of the *Subtractor* and its Correspondent in the like place of the *Subtrahend*, beginning at the place of Units, and set the Remainders under them in order; but if the Figure in any place of the *Subtrahend* is less than its Correspondent in the *Subtractor*, add 10 to that Figure, and subtract from the Sum, and set down the Digit which remains; then add 1 to the next Figure of the *Subtractor*, and take the Sum from its Correspondent in the *Subtrahend*; and go on so, adding in the same manner whenever the Figure of the *Subtrahend* is least. All the Figures written down, express the difference sought.

Subtrahend	876
Subtractor	524
Difference	352

Examp. 1. The difference of 876 and 524 is 352, as in the Margin. The Operation is thus, $6-4=2$, which I set down; then $7-2=5$, and $8-5=3$; and the difference sought is 352.

Subtr[d].	74625
Subt[r].	47382
Diff.	27243

Examp. 2. The difference of 74625 and 47382 is 27243. Whose Operation is thus, $5-2=3$; then $2-8$ is impossible, therefore I take 12, and say, $12-8=4$; then (because 10 was added to the last place of the Subtrahend) I add 1 now to the Subtractor, and say, $1+3=4$; then $6-4=2$. Again, $4-7$ cannot be, therefore I take 14, and say, $14-7=7$. Again $1+4=5$, and $7-5=2$. And so the compleat difference is 27243.

You may examine these few more Examples in the same manner.

Examp. 3.	*Examp.* 4.	*Examp.* 5.	*Examp.* 6.
87234	2835	345	58034
53890	925	79	928
33344	1910	266	57106

SCHOLIUM. This Rule ſuppoſes we can readily in our Mind ſubtract any Digit from the Sum of any leſſer Digit and 10; which may be eaſily admitted: but to make the Difficulty leſs, (if there can be any) we may do the Work thus; Subtract from the 10, and add the Remainder to the Figure of the *Subtrahend* to which the 10 ſhould have been added, and ſet this Sum down. So in *Examp.* 5. ſay 9 from 5 cannot, but from 10, and 1 remains; then $1+5=6$; which is ſet down.

DEMONSTRATION of the Second Caſe.

1. Where all the Figures of the *Subtractor* are leſs than their Correſpondents in the *Subtrahend*, the difference of the Figures in the ſeveral like places ſet in the ſame place, muſt all together make the true difference ſought; becauſe as the Parts make up the Whole, ſo muſt the differences of all the ſimilar Parts of any two Numbers make the total Difference of the Wholes, of which theſe are the ſimilar Parts.

2. Where any of the Figures of the *Subtrahend* is leſs than the Correſpondent in the *Subtractor*, the 10 which is added by the Rule you are to ſuppoſe to be the Value of an Unit taken from the next higher place, (which by *Notation* is equal to 10 in this Place,) and then the 1 added to the next place of the *Subtractor* is to diminiſh the next place of the *Subtrahend* by 1 more than is contained in the *Subtractor*, becauſe that 1 was ſuppoſed to be already borrowed from it, and applied to the preceding Place according to its Value there: ſo that inſtead of adding 1 to the *Subtractor*, we may take 1 from the *Subtrahend*; *i. e.* take the *Subtractor* from the *Subtrahend* Figure leſſened by 1. But the Effect is the ſame, ſince either way that 1 is taken from the *Subtrahend*; as it ought to be, ſince it is already applied to the laſt place; which is only taking from one Part, and adding as much to another, whereby the Total is never changed: And by this means the *Subtrahend* is reſolved into ſuch Parts as are each greater than (or equal to) the ſimilar Parts of the *Subtractor*. So in order to ſubtract 26 from 52, reſolve the 52 into theſe Parts, 40 and 12, ſo that the 12 correſpond to the 6 of the *Subtractor*, and the 40 to the 20; which is in effect done by the Rule: for we ſay, 6 from 2 cannot, but from 12, and 6 remains; then it is plain, that for 5 we have but 4 in the next place, and it is the ſame thing to ſay 2 from 4, or 3 from 5.

52
26
—

But there is another thing to be accounted for, which the Rule ſuppoſes, *viz.* That the Difference betwixt any Digit and the Sum of any leſſer Digit and 10, will always be leſs than 10; the Truth of which is plain; for ſince that leſſer Digit wants at leaſt 1 of the greater, (to make a Part equal to the Subtractor) that 1 being taken from the 10 added, there cannot remain above 9.

SCHOLIUM. If it be propoſed to ſubtract two or more Numbers from any one, or one from more; or laſtly, more than one from more than one; the beſt and moſt ſimple way is, firſt to add theſe more together, and then ſubſtract: So let it be propoſed to ſubtract 560 from $467+235$, as in Example 1. below; or, $345+432$ from 978, as in Example 2. or, $3072+5678$ from $2578+9631$, as in Example 3. Alſo when more Numbers are to be ſubtracted out of one, it may be done by taking away firſt one of them, and then out of the Remainder take another, and ſo on till they are all ſubtracted; and this is called *Continual Subtraction.*

Example 1.

	467
	235
Sub^{d}.	702
Sub^{r}.	560
Diff.	142

Example 2.

Sub^{d}.	978
	345
	432
Sub^{r}.	777
Diff.	201

Example 3.

	2578
	9631
Sub^{d}.	12209
	3072
	5678
Sub^{r}.	8750
Diff.	3459

The PROOF *of* Subtraction *may be made either by* Addition *or* Subtraction.

1. By *Addition*, thus; Let the Remainder be added to the *Subtractor*, and the Sum ought to be equal to the *Subtrahend*: For this is restoring back what was before taken away.

2. By *Subtraction*, thus; Subtract the Remainder from the *Subtrahend*; and this Remainder ought to be equal to the former *Subtractor*: Because which soever of the two Parts that make up any Whole is taken away, the other remains. I leave you to apply these Proofs in the preceding Examples.

If we have subtracted more Numbers out of one by continual Subtraction, the Proof of this Work will plainly be, *Adding all the Subtractors and the last Remainder* into one Sum, which must be equal to the *Subtrahend*.

CHAP. V.

MULTIPLICATION *of Whole and Abstract* NUMBERS.

DEFINITIONS.

1. M*ULTIPLICATION is the taking any Number a certain number of times: or, finding a Number which shall contain any given Number a certain proposed number of times*; *i. e.* as oft as that other Number contains *Unity*. For *Ex.* If it is proposed to take 48, 7 times; or find that Number which contains 48, 7 times; the Answer by the following Rule, is, 336.

2. The Number to be multiplied is called the MULTIPLICAND; the Number by which it is multiplied, (or the number of times it is to be taken) is called the MULTIPLIER; and the Number found is called the PRODUCT. The *Multiplicand* and *Multiplier* are also called the FACTORS (of the *Multiplication*) without distinction; because they *make* the *Product*, or Number sought.

SCHOLIUMS.

1. I know there may be made a more general Definition of *Multiplication*, comprehending in it also what is called, *Multiplication by Fractions:* But that being really a mixt Operation of *Multiplication* and *Division*, I thought it more reasonable to make the Definition here to agree only to whole Numbers, which is proper and pure *Multiplication*, according to the more common Sense of the Word; and when you learn the other, it is only joining them both together to make that more general Definition: which I shall do in its proper place.

2. The

2. The Senſe and Effect of *Multiplication* (of whole Numbers) is the ſame with *Addition*; for it's plain, that if we take the *Multiplicand*, and write it down as oft as there are Units in the *Multiplier*, the Sum of all theſe, taken by *Addition*, is the Number ſought.

For *Ex.* to multiply 48 by 7, or to take 48, 7 times; I ſet down 48, 7 times (as in the Margin) and find the Sum 336. But *Multiplication* is a Method of finding the ſame Number more eaſily and expeditiouſly. For *Ex.* to multiply any Number by 468; what a tedious and intolerable thing would it be, to write down the Multiplicand 468 times? But by the following Rules this is prevented, and the Number ſought is found by an eaſy Operation.——*Multiplication* then is only a compendious *Addition*, limited to that particular Caſe wherein all the Numbers to be added are equal to one another, (or the ſame Number:) For it's this Circumſtance that affords us a more eaſy Method of working than by the general Rule of *Addition*. Yet there are ſome ſimple Caſes which admit of no Compend: Theſe are the Multiplication of Numbers under 10, or the Digits, by one another; which are the *primitive Operations* in *Multiplication*, upon which all other Caſes depend. We muſt therefore explain *Multiplication* alſo in two Caſes, as we have done *Addition*. But whereas the *primitive* Caſes of *Addition* depend immediately upon the Rule of *Notation*; the *primitive* Caſes of *Multiplication* depend immediately on *Addition*. For we need not go back to *Notation* for theſe; ſince we have, by an intermediate Step in *Addition*, gained an eaſier way of doing them.

48
48
48
48
48
48
48
——
336

3. This Sign or Character × ſet betwixt two Numbers, ſignifies the *Multiplication* of the one by the other (taking either of them for the *Multiplier* or *Multiplicand*; which does not alter the Product, as will be afterwards demonſtrated) and is a Complex or indefinite way of repreſenting the Product. Thus 7×3 ſignifies that 7 is multiplied by 3, or 3 by 7; which we read 7 times 3, or 3 times 7; whereby the Product is expreſſed in a complex manner by the *Factors*: And if more Numbers are ſucceſſively or continually (as it's called) multiplied together, the ſame Sign is prefix'd to each ſucceſſive Factor. Thus $4 \times 6 \times 3$ expreſſes the Product made by multiplying 4 by 6, and this Product again by 3; and ſo of more Factors, (which will ſtill be the ſame Product in whatever order the Factors are applied, as will be demonſtrated.) By this Sign we explain in a neat and brief way the Multiplication of particular Examples, as you'll preſently ſee.

But the principal Uſe of this Sign of Multiplication is in the Algebraic Art, to expreſs the Product of two or more Numbers in Letters. Thus, $A \times B$ is the Product of A and B; alſo $A \times B \times C$ the Product of A, B and C, and ſo on. But when two or more Numbers are expreſſed, each by one Letter, the Product is alſo expreſſed by theſe Letters ſet together without the Sign. Thus AB is the Product of A and B, ABC that of A, B, and C. This is the general Rule of the *Literal Multiplication*; other particular Caſes you'll learn afterwards.

PROBLEM.

To multiply any Number by another.

CASE I. *To multiply one Digit by another.*

Rule. Add the *Multiplicand* to itſelf as oft as there are Units in the *Multiplier*, and you have the Product ſought.——*Examp.* To multiply 6 by 4, I ſay $6+6$ ($=12$) $+6$ ($=18$) $+6=24$, the Number ſought. But the Products of all the Digits ought to be ready in the Memory; which are eaſily got by the help of the following Table.

TABLE

TABLE of ***MULTIPLICATION*** for the simple Digits, or Numbers less than 10.

1	2	3	4	5	6	7	8	9	1
	4	6	8	10	12	14	16	18	2
		9	12	15	18	21	24	27	3
			16	20	24	28	32	36	4
				25	30	35	40	45	5
					36	42	48	54	6
						49	56	63	7
							64	72	8
								81	9

USE of the TABLE.----Seek the greater of the two Digits in the upper Line, and under it against the lesser, taken in the right-side Column, is the Product sought. *Ex.* To multiply 8 by 4, I take 8 in the upper Line, and under it against 4 I find 32 the Product. Again, to multiply 7 by 9, take 9 in the upper Line, and under it against 7 on the side, you have 63 the Product sought.

DEMON. The Construction of the Table is plainly this: The 9 Digits being set down in the upper Line, each of them is considered as a Multiplicand; and is added to itself successively, as oft as there are Units in every Digit not exceeding itself; and the Multipliers are set in a Column on the side against the respective Sums. For *Example*, the Multiplications of 6 from the upper Line are carried no farther than 6 on the side; and so of the rest. This explains the Reason of the *Rule* when the greater Number is proposed as the Multiplicand, or where the Multiplicand and Multiplier are equal: But, where the lesser is proposed as the Multiplicand, yet we apply the greater to the Table, as if it was the Multiplicand, [which I shall here suppose to be the same in effect, and afterwards it shall be demonstrated, *viz.* that 6 times 8 is the same as 8 times 6, and so of any other two Factors.]

CASE II. *To multiply any two Numbers into one another.*

Rule 1. Make any of the two Factors the *Multiplicand*, or *Multiplier*; but it will be generally most convenient to make that the *Multiplier* which has fewest significant Figures: Then, tho' there's no matter in what order the Factors are set down, yet, when it can be done, 'tis convenient to write the Multiplier under the Multiplicand; so that the first significant Figure on the Right-hand of this be over the first significant Figure of the other, [whether these Figures be in the place of Units or not. See the Examples.] Then, in case there are 0's standing on the Right-hand of either or both Factors, neglect them as if they were not there, and proceed thus:

2. Begin with the first significant Figure of the Multiplier, and by it multiply every Figure of the Multiplicand, one after another, (by *Case* 1.) beginning at the first significant Figure, and proceeding in order to the last Figure on the left; and write down the Products. *Thus*, If the first Product (or that of the first significant Figure of the Multiplicand) is less than 10, write it down; but if it exceeds 10, write down what's over any Number of 10's, and carry that Number (*i. e.* 1 for every 10) to the Product of the next Figure. If this Sum is less than 10, write it down; but if it exceeds 10, write down the Excess of 10's, and carry the Number of 10's to the Product of the next Figure, and so on: setting the Figures to be written down all in a Line after one another, orderly, from the Right to the Left. Having thus gone thro' all the Figures of the *Multiplicand*, (not omitting the 0's that stand mixt with other Figures) write down the compleat Product of the last Figure with the 10's of the preceding added to it.

3. Make

3. Make the ſame Operation upon the *Multiplicand*, with every ſignificant Figure of the *Multiplier*; (paſſing all the o's) taking them in order as they ſtand towards the Left; and ſetting the ſeveral Products under one another: *Thus*, Set the firſt Figure of the Product made by every Figure of the *Multiplier*, at the ſame diſtance from the firſt Figure of the preceding Line, as their reſpective Figures in the *Multiplier* are; and place the reſt of the Figures in order towards the Left, under the preceding.

4. Add all theſe particular Products into one Sum; taking them as they are ſet in the Columns under one another: And if the firſt ſignificant Figure of both *Factors* was in the place of Units, this Sum is the Product ſought (*Ex.* 1, 2, 3, 4, 5) otherwiſe ſet as many o's before it as ſtand before the firſt ſignificant Figure of both *Factors*, (as in *Ex.* 6, 7, 8, 9.)

I ſhall next apply this Rule in *Examples*.

Ex. 1.

```
642  Multiplicand.
  4  Multiplier.
----
2568 Product.
```

To multiply 642 by 4; I ſay $4\times 2=8$, which is ſet down; then $4\times 4=16$, for which I write down 6 and carry 1; then 4×6 $(=24)+1=25$, which being written down, the Product is 2568.

Ex. 2.

```
 85065 Md.
     8 Mr.
------
680520 Pr.
```

To multiply 85065 by 8, I work thus, $8\times 5=40$, for which I write down o, and carry 4; then $8\times 6\,(=48)+4=52$, for which I write down 2 and carry 5; then $8\times o\,(=o)+5=5$, which I write down; then $8\times 5=40$, for which I write down o, and carry 4; then, 8×8 $(=64)+4=68$; which being written down, the Product is 680520.

Ex. 3.

```
   84653  Md.
     469  Mr.
--------
  761877  Prod. by 9.
 507918   ——by 6.
338612    ——by 4.
--------
39702257  Total Prod.
```

To multiply 84653 by 469, I work thus; beginning with 9 (in the Units place of the Multiplier) I multiply by it the whole Multiplicand, in the manner of the preceding *Examples*. Then I take the next Figure of the Multiplier, 6, and by it in the ſame manner multiply the whole Multiplicand; ſetting the firſt Figure of this Product, 8, under the ſecond Figure of the preceding, and the reſt in order; making the ſame Operation with the next (and laſt) Figure of the Multiplier, *viz* 4, and ſetting the firſt Figure of this Product (*viz.* 2) under the ſecond of the preceding, and the reſt in order. All theſe Products ſummed up, as you ſee in the *Ex.* gives the total Product.

Ex. 4.

```
   6452
    806
-------
  38712
51616
-------
5200312
```

To multiply 6452 by 806: After multiplying by 6, I paſs to 8; and becauſe there is one o betwixt them, the firſt Figure of the laſt Line is ſet under the third place of the former, *i. e.* with one place betwixt the firſt Figures of the two Lines, becauſe of the one o betwixt the Multipliers.

Ex. 5.

```
      46007859
        380046
--------------
     276047154
    184031436
  368062872
138023577
--------------
17485102781514
```

To multiply 46007859 by 380046: After making the Product of 6 and 4, I paſs to 8; ſetting its firſt Product two places diſtant from that of the preceding Line, becauſe of the two o's betwixt 4 and 8.

Tri

In the four following Examples, wherein the first significant Figure of each *Factor* is not in the place of *Units*, the Application of the Rule is so plain, that I need make no more words about it.

Examp. 6.	*Examp.* 7.	*Examp.* 8.	*Examp.* 9.
467	584000	46000	376890
2800	93	2700	5004000
3736	1752	322	150756
934	5256	92	188445
1307600	54312000	124200000	1885957560000

In order to the Demonstration of the preceding Rule, and for the sake of some other special Rules following, we must premise these Truths, as belonging to the Theory of *Multiplication*.

LEMMA I.

If one Number is multiplied by another, the Product will be the same as if this other be multiplied by the former; i. e. *Any one of the two* Factors *may be made* Multiplier *or* Multiplicand, *the Product will be the same.*

Examp. 4 times 7 = 7 times 4. A times B = B times A.

Demon. A small Attention to the Idea of Numbers will make this Truth evident; and therefore few Writers think it needs any Demonstration. However, as it is capable of one, and some may require it, I shall satisfy them. Thus;

Any Number B is only a certain Collection of *Units*; wherefore A times B is equal to A times each *Unit* in B, taken separately and added together: but A times 1 is the same thing as A, or 1 time A, (from the Definition of Number:) Therefore A times each *Unit* in B is equal to A, (or 1 time A) taken as oft as there are *Units* in B; *i. e.* B times A. Therefore A times B = B times A.

Or take this other more sensible Demonstration.

Suppose any two Numbers, A, B; let the *Units* of A be represented by a Row of Points, as in the Margin: Repeat this Row as oft as there are *Units* in B, and set them orderly under one another; then it is plain, that there will be as many Columns of Points as there are Points or *Units* in A; each of which Columns has as many Points as there are *Units* in B. Therefore the whole Number of Points which were at first made equal to B times A, (by repeating the Row A, B times) becomes necessarily equal to A times B.

```
       A
  . . . . .
B . . . . .  &c.
  . . . . .
     &c.
```

LEMMA II.

If three or more Numbers are proposed to be continually multiplied, the last Product will still be the same, in whatever Order the Factors *are taken.*

Examp. $3 \times 5 \times 7 = 3 \times 7 \times 5 = 7 \times 3 \times 5 = 7 \times 5 \times 3 = 5 \times 3 \times 7 = 5 \times 7 \times 3$.

Demon. Case I. If there are 3 *Factors* A, B, C; then since the Product of 2 *Factors* is the same whichever of them is *Multiplier*; *i. e.* A times B = B times A: Therefore the thing to be proved here is only, That whichever of the 3 *Factors* is cast in the last place, the Product is still the same, *viz.* that ABC = CBA = ACB. For since AB = BA; therefore ABC = BAC. For the like Reason CBA = BCA, and ACB = CAB.

Now

Now then, to prove that ABC = CBA = ACB: First, taking the *Factors* in the Order ABC, expresses the Product of A by B, or B times A, and this Product AB taken C times; which makes C times B times A, or A taken C times B times; *i. e.* CB times A: So that AB × C = A × CB, or CB × A; *i. e.* ABC = CBA. Again, ACB signifies A taken C times, and this Product AC taken B times; which is B times C times A; *i. e.* A taken B times C times; or, which is the same, C times B times: but this, *viz.* A × CB or CB × A was before shewn to be equal to AB × C, therefore ABC = CBA = ACB.

Case II. If there are more *Factors* than 3, then I prove the Truth proposed thus: I say, if it is true of any particular Number of *Factors* more than 2, it is therefore true if we take in one *Factor* more: but it is true of 3 *Factors*, as shewn above; therefore it is evidently true of 4, and hence again it is true of 5, and so on for ever. What is to be proved then, is the first Part, *viz.* the Connection of the Truth of the Rule for any Number of *Factors* more than 2, with the next Case, or one *Factor* more. Which I prove thus;

Let ABCD, *&c.* be a Product of any Number of *Factors*, in which it is supposed to be no matter in what Order they are taken; therefore I may cast any of its Terms last, the Product will still be equal, *viz.* ABCD, *&c.* = ABC, *&c.* × D = ABD, *&c.* × C = ACD, *&c.* × B = BCD, *&c.* × A. Now, let another Term X be taken into the Question, then it is plain, that whatever *Factor* of the whole A, B, C, D, X, *&c.* we suppose to be last employed, the various Orders in which the preceding *Factors* may be employed, produce the same Effect, by supposition; wherefore all the various Orders of taking the whole *Factors*, wherein any particular *Factor* keeps the last place, produce the same Effect; because the various Orders of the preceding Terms have no different Effect by supposition. And therefore what remains to be proved is only this, *viz.* That the Products are all equal which are made by the several Orders wherein different *Factors* are put in the last place, which will easily appear thus: Since by supposition ABCD, *&c.* = ABC, *&c.* × D = ABD, *&c.* × C, and so on, putting each *Factor* last: Therefore ABCD, *&c.* × X = ABC, *&c.* × D × X = ABD, *&c.* × C × X, and so on. But we may make X and the *Factor* preceding it, change places in each of these Expressions; the Product will still be the same by *Case* 1. because it is a Product of 3 *Factors*, thus; ABC, *&c.* × D × X = ABC, *&c.* × X × D. Also, ABD, *&c.* × C × X = ABD, *&c.* × X × C, and so on; whereby each *Factor* is cast in the last place. Wherefore these Products are all equal, being each equal to the Product of another Order, and all these other Orders equal.

Corol. If two Numbers are proposed to be multiplied together, it is the same thing to do it by the *General Rule* at once; or, if one of the *Factors* is equal to the continual Product of two or more Numbers, then we may multiply the one given *Factor* first by one of these Numbers that produce the other, and then this Product by another of them, and so on thro' them all: Thus, 24 = 4 × 6. Therefore 62 × 24 = 62 × 4 × 6.

Scholium. What an *Aliquot Part* is, has been already explained. And from *Lem.* 1. it is evident, that if two Numbers are multiplied together, each of them is an *Aliquot Part* of the Product, and the other is the Denominator of the Part. So, because 4 × 8 = 32; therefore 8 is $\frac{1}{4}$ of 32, and 4 is $\frac{1}{8}$ of 32. Universally A = $\frac{1}{B}$ of AB, and B = $\frac{1}{A}$ of AB. Whence this follows, that if we multiply any Number by the *Aliquot Part* of another, and then this Product by the Denominator of that Part, the last Product is the same as if the first Number were multiplied by that other Number; which is a Truth manifest also from the nature of an *Aliquot Part*, without regard to this first *Lemma*. Again *observe*, That the Product of two Numbers is very naturally called, the *Multiple* of either of the *Factors*; and it is said to be a *Multiple* of it by the other *Factor*: So 48 is a *Multiple* of 8 by 6, or of 6 by 8. Also if two Numbers are multiplied by the same Number, the Products are called *Like-Multiples* of these Numbers, as 3 × 4 and 3 × 7 are *Like-Multiples* of 4 and 7. Now, as *Multiple* and *Aliquot Part* are directly opposite; so

ſo from the nature of them, this follows; That the Product of one Number by the *Aliquot Part*, or *Multiple* of another, is the like *Aliquot Part* or *Multiple* of the Product of theſe two Numbers. Thus; ſuppoſe $B = CD$; then is AC the $\frac{1}{D}$ Part of AB ($= ACD$,) and AB ($= ACD$) is the *Multiple* of AC by D.

LEMMA III.

If either or both of two Numbers are any how diſtributed into two or more Parts; and if each Part of the one Number is multiplied into the other Number, or into each Part of the other; the Sum of all theſe Products is equal to the Product of the two given Numbers.

Examp. $3+4=7$, and $6 \times 7 = 3 \times 6 + 4 \times 6$. Again, $15 = 9+6$, alſo $8 = 3+5$. Therefore, $8 \times 15 = 3 \times 9 + 3 \times 6 + 5 \times 9 + 5 \times 6$. *Univerſally*, if $A = B + C$, then is $AD = BD + CD$; and if $D = E + F$, hence $AD = BE + BF + CE + CF$.

DEMON. The Reaſon is manifeſt; becauſe the Whole being nothing elſe than the Sum of all the Parts, when one Number or every Part of it is multiplied into every Part of another Number, then is the one Whole multiplied into the other; ſo that the Sum of the Products made by the Parts muſt be equal to the Product of the Whole.

DEMONSTATION of CASE II. *of the preceding PROBLEM.*

I. When the firſt ſignificant Figure of each *Factor* is in the place of Units, (*Examp.* 1, 2, 3, 4, 5.) the Reaſons of the Rule are theſe:

(1.) That either Number may be made *Multiplier* or *Multiplicand*, is demonſtrated in *Lemma* 1. But making that one *Multiplier* which has feweſt ſignificant Figures, is moſt convenient, becauſe it makes feweſt partial Products.

(2.) If the *Multiplier* is a Digit, (*Ex.* 1, 2.) then by multiplying every Figure, *i. e.* every Part, of the *Multiplicand*, we multiply the Whole; and by writing down the Products that are leſs than 10, or the Exceſs of 10's, in the places of the Figures multiplied, and carrying the Number of 10's to the Product of the next place, we do hereby gather together the ſimilar Parts of the reſpective Products; and ſo do the ſame thing in effect as if we wrote down the *Multiplicand* as oft as the *Multiplier* expreſſes, and added them up: For the Sum of every Column is the Product of the Figure in the place of that Column, whereby it appears that the Addition, and the Multiplication according to this Rule, have the ſame Effect: (ſee *Ex.* 1.) Wherefore the Rule is right.

(3.) When there are more ſignificant Figures than one in the *Multiplier*, as in *Ex.* 3, 4, 5. then, by applying them ſeparately, we reſolve the *Multiplier* into Parts; and if the true Products made by each of theſe Parts multiplying the whole *Multiplicand*, are added together, the Sum is the Product ſought, (by *Lem.* 3.) Again; the Product made by each Figure of the *Multiplier* taken in its ſimple Value, is truly found by the Rule; as ſhewn in the preceding Article; and by placing theſe Products with reſpect to one another, ſo as the firſt Figure of each Product ſtands under that Figure of the firſt Product, (or Product of the Figure in the Units place of the *Multiplier*,) which is in the ſame place as the multiplying Figure ſtands in the whole *Multiplier*; theſe Products are added together according to the true Value they ought to have, by conſidering the multiplying Figure in its complete Value. For *Examp.* The Product by the ſecond Figure, or Place of 10's, is ſet and added to the Product of the firſt Figure, as if a Cypher or 0 had been prefix'd to it, whereby it is made 10 times as much as were the multiplying Figure in the place of Units, as it ought to be, ſince that Figure is in the place of Tens, and not of Units; (ſee *Schol.* after *Lem.* 2.) The ſame Reaſon holds in all the other Figures of the *Multiplier*. Therefore the Sum of the Products taken according to the Rule, is the

the true Product sought. And the Reason why we don't actually prefix these o's, is, because it were superfluous as to the Sum. See these Examples.

Ex. 1. $6784 \times 3 = 20352$. For $6784 = 6000 + 700 + 80 + 4$. And the Operation by Parts is thus:

$4 \times 3 =$	12		6784
$80 \times 3 =$	240	or thus,	6784
$700 \times 3 =$	2100		6784
$6000 \times 3 =$	18000		
	20352		20352

Example 2.

Common Way.	Or Thus.	
68749	68749	
853	853	
206247	206247	Prod. by 3.
343745	3437450	Prod. by 50.
549992	54999200	Prod. by 800.
58642897	58642897	Total Prod.

II. When the first significant Figure of either or both *Factors* does not stand in the place of *Units*: Then,

(1) When it is so only in one of the *Factors*, (as *Ex.* 6, 7.) then by taking that *Factor* as if the first significant Figure were in the place of *Units*; *i. e.* neglecting all the o's that stand on the right, we do indeed work only with the 10^{th} or 100^{th}, &c. part of it; therefore the total Product found is accordingly but the 10^{th} or 100^{th} Part of what it ought to be: and therefore to have the true Product, we ought to multiply the last Product by 10, 100, &c. (by *Schol. Lem.* 2.) which is done by setting as many o's before the Product as were before the Multiplier.

(2.) When it is so in both *Factors* (as in *Ex.* 8, 9.) there is the same Reason for setting before the Product the o's that belong to the one *Factor*, as those belonging to the other: For after correcting the Product by the o's of the one *Factor*, it wants to be corrected again by those of the other: Therefore when there are o's belonging to both, they ought all to be set before the Product.

Therefore this *Rule* is true in all possible Cases.

PROOF of MULTIPLICATION.

In the first place, we must observe, That the multiplying of one Digit by another has no other Proof than by Addition: But the Table being examin'd and found true, we are to depend upon that or Memory for these simple Operations; the Proof here designed being for other Cases where either or both *Factors* exceed 9: And this may be done several ways. Thus,

1. By *Multiplication* itself: For if we change the *Factors*, and make that the *Multiplier* which was before the *Multiplicand*, we ought to have the same Product, by *Lem.* 1.

2. By *Division*: But this cannot be applied till that *Rule* is learned.

But both these Methods are too tedious to be of Use; and therefore,

3. The most convenient and easy Proof is by help of the Number 9, like what we have in *Addition*; which is performed thus:

Caſt out the 9's out of the two *Factors* (in the manner taught in the like *Proof of Addition*) and mark the Number that is under or over 9's in each; if either of them is 0, paſs immediately to the Product, and caſt out the 9's of it; the Exceſs ought here to be 0, which makes the Proof. But if the Number under or over 9's is not 0 in either Factor, then multiply theſe two Numbers together, and mark alſo what is under or over 9's in their Product; this Number and what is over 9's in the total Product of the Example ought to be equal.

Ex. 1.

```
  476
   68
 3808
2856
32368
```

In this firſt Example, the Exceſs of 9's in the *Multiplicand* is 8, in the *Multiplier* it is 5; then $8 \times 5 = 40$, and ſo the Exceſs of 9 is 4; exactly equal to the Exceſs of 9's in the Product 32368.

Ex. 2.

```
  87
  36
 522
261
3132
```

In the ſecond Example, the Exceſs of 9's in the *Multiplier* is 0, therefore I paſs immediately to the Product, where I find it is alſo 0.

Demonſtration. The Reaſon of the Practice of caſting out the 9's in the ſeveral Numbers has been already demonſtrated in *Addition*; and the Reaſon of the reſt of the Work will eaſily appear, thus: 1. If either *Factor* is a preciſe Number of 9's, (*i. e.* when there is no Exceſs) as in Example 2. it is plain the Product muſt be ſo too, for it is only that Number of 9's taken a number of times. But, 2. If each of them is equal to a Number of 9's, and ſome leſſer Number over, then let us repreſent them thus; Let one be $A + b$, where A repreſents any Number of 9's, and b the Number over: Let the other be $N + c$, where N is any Number of 9's, and c the Number over. But now, by *Lem.* 4 the Product of theſe *Factors* is equal to the Sum of the Products of all the Parts of the one by all the Parts of the other; and ſo the Product is here $\overline{A \times N}$, $+ \overline{A \times c}$, $+ \overline{N \times b}$, $+ \overline{b \times c}$. But the firſt three Products are each a Number of 9's, becauſe one of their *Factors* is ſo; therefore theſe being caſt away, there remains only $\overline{b \times c}$. And if the 9's are alſo caſt out of this, the Exceſs is the Exceſs of 9's in the total Product; but b, c are the Exceſs in the *Factors*, and $\overline{b \times c}$ their Product: therefore the Rule is true.

Scholium. The Objection made to this *Proof* is the ſame as that mentioned already againſt the like *Proof* for *Addition*; and therefore the ſame Anſwer ſerves here.

§. 2. Containing

Particular Methods *of working* Multiplication *in certain Caſes; whereby it is either contracted into a ſhorter Work than the general Rule, or made eaſier and more certain with as large a Work; and, in ſome Caſes, with a little more Work.*

THE firſt five Cases are ſuch wherein the very ſame Operation is performed as by the *General Rule*; only the Trouble and Time of writing down a great many Figures is ſaved.

Case I. When one of the *Factors* has any Figure whatever in the place of *Units*, and 1 in all the reſt, as 16, 114, 1118; make that the *Multiplier*, and the Product may be got all at once thus;

(1.) Suppoſe the number of Places of the *Multiplier* do not exceed thoſe of the *Multiplicand*, then multiply by the Figure in the Units place of the *Multiplier*, and with every Product, till you come to the Product of that Figure that ſtands over the laſt 1 of the *Multiplier*, (*i. e.* till you have multiplied as many Figures as the Number of Places in the *Multiplier*) add in all the preceding Figures on the right of the *Multiplicand*; to the next Product after this, add in all the preceding except the firſt, (or Units place) and at every

ſuc-

ſucceeding Product exclude always one more in order from the right, [and when you come to theſe excluſions, it will be fit to ſet a Point over the Figure which is to be excluded at the next Operation,] *viz.* the remoteſt on the right of theſe that were taken in at the laſt ſtep. Write down at every ſtep what's under or over 10's, and carry the 10's to the next ſtep; and when you have gone thro' all the Figures of the *Multiplicand*, to the number of 10's carried from the laſt ſtep add in all the Figures of the *Multiplicand* after the laſt Point, pointing alſo the laſt you take in at this ſtep; then take in all from the laſt Point, and ſo on till you take in the very laſt Figure alone; and thus you have the true Product ſought; as in Examples 1, 2, 3.

(2) If the Places of the *Multiplier* exceed in Number thoſe of the *Multiplicand*, do all as in the former *Caſe*; only, when all the Figures of the *Multiplicand* are multiplied, the whole of them muſt be ſumm'd and taken in as oft, and once more, as the difference betwixt the Number of Places of the *Multiplicand*, and the Number of 1's in the *Multiplier*, (*i. e.* plainly till you have got a product Figure under every 1 of the *Multiplier*:) then beginning ſtill at the laſt place, take in one place fewer, (as before) till the laſt place is taken in alone. See *Example* 4.

The Reaſon of this Practice will eaſily appear by comparing a few Examples wrought this way, and alſo the common way.

Example 1.

4658	4658
16	16
74528	27948
	4658
	74528

Example 2.

5276	5276
114	114
601464	21104
	5276
	5276
	601464

Example 3.

263487	263487
1113	1113
293261031	790461
	263487
	263487
	263487
	293261031

Example 4.

847	847
11113	11113
9412711	2541
	847
	847
	847
	847
	9412711

The Operations of theſe Examples are thus; *viz.*

Examp. 1. $6\times8=48$, which is 8 and carry 4; then $6\times5\ (=30)+4$ carried $(=34)$ $+8$ (the next Figure on the right) $=42$; which is 2, and carry 4. Then $6\times6\ (=36)$ $+4$ carried $(=40)+5$ (on the right) $=45$; which is 5, and carry 4. Then $6\times4\ (=24)$ $+4$ carried $(=28)+6$ (on the right) $=34$; which is 4, and carry 3; then 3 (carried) $+4$ (the laſt Figure) $=7$.

Examp. 2. $4\times6=24$; which is 4, and carry 2: then $4\times7\ (=28)+2$ carried $(=30)$ $+6$ (on the right) $=36$; which is 6, and carry 3: then $4\times2\ (=8)+3$ carried $(=11)$ $+7+6$ (the two next on the right) $=24$; which is 4, and carry 2: then $4\times5\ (=20)$ $+2$ carried $(=22)+2+7$ (on the right) $=31$; which is 1 and carry 3: then 3 carried

ried $+5+2$ (on the right) $=10$; which is o, and carry 1: then 1 (carried) $+5$ (on the right) $=6$.

Examp. 3. Here I ſhall ſpare repeating what is ſet down and carried; or the words *carried*, and *on the right*, becauſe you know how to ſupply them. The Work is thus: $3\times7=21$; then $3\times8(=24)+2+7=33$; then $3\times4(=12)+3+8+7=30$; then $3\times3(=9)+3+4+8+7=31$; then $3\times6(=18)+3+3+4+8=36$; then $3\times2(=6)+3+6+3+4=22$; then $2+2+6+3=13$; then $1+2+6=9$; then (o carried $+$) 2.

Examp. 4. Thus $3\times7=21$; then $3\times4(=12)+2+7=21$; then $3\times8(=24)+2+4+7=37$; then $3+8+4+7=22$; then $2+8+4+7=21$; then $2+8+4=14$; then $1+8=9$.

SCHOLIUM. To compare theſe Operations with the ſame Examples at large, *obſerve* what Figures of the *Multiplicand* (as it ſtands written down for every 1 in the *Multiplier*) ſtand under each Figure of the firſt product Line; and theſe ſhew the reaſon of taking in the Figures on the Right-hand of the *Multiplicand*, and how this Rule was invented.

I adviſe a Learner to make himſelf familiar with the Practice of Examples like the firſt; becauſe they occur frequently in common Buſineſs.

Alſo if the *Multiplier* has but two Places, tho' 2 is in the ſecond place, the Product may eaſily be made all at once, by taking in with every Product made by the place of Units, double the Figure on the Right-hand. Practice will make this eaſy, and it will be very uſeful. The like Method may be uſed whatever Figure is in the ſecond Place, (by taking in with every Product as many times the preceding Figure:) but the greater that Figure is, it is the more difficult; and I would only recommend the Practice for 2; unleſs the *Multiplicand* have not above 3 or 4 Places, and theſe alſo ſmall Figures; for then we may uſe this Method with 3, 4, or 5 in the ſecond Place.

$$\begin{array}{r} 546 \\ 28 \\ \hline 15288 \end{array}$$

The *Example* annex'd is thus done; $8\times6=48$; then $8\times4(=32)+4+12=48$; then $8\times5(=40)+4+8=52$; then $5+10=15$.

Note, If 1 is in all the Places, the Practice is ſo much the eaſier, without altering the Rule.

CASE II. If one of the *Factors* has a ſignificant Figure in the Place of Units, 1 in the higheſt Place, and o's betwixt them; make that in Units place the *Multiplier*, and work thus.

1. Suppoſe the places of the *Multiplier* do not exceed in number thoſe of the *Multiplicand*; then multiply by the firſt Figure; and when you are come to the Figure of the *Multiplicand* that ſtands over the 1 in the *Multiplier*, (*i. e.* having made as many Products as there are Places in the *Multiplier*) with that Product take in the firſt Figure of the *Multiplicand*; and with every ſucceeding Product add in the next Figure in order, (and it will be convenient to put a Point over every Figure when taken in, that you may more readily know what is to be next taken in) After the laſt Figure of the *Multiplicand* is multiplied, ſet down all the remaining Figures of the *Multiplicand* that are on the left of the Figure laſt taken in, after having added to them what was carried from the laſt Product.

The Reaſon of this is obvious in the following Examples.

Example 1.

$$\begin{array}{r} \dot{5}\dot{7}\dot{6}\dot{4}8 \\ 103 \\ \hline \underline{5937744} \end{array} \qquad \begin{array}{r} 57648 \\ 103 \\ \hline 172944 \\ 57648 \\ \hline \underline{5937744} \end{array}$$

Example 2.

$$\begin{array}{r} 4653 \\ 1004 \\ \hline \underline{4671612} \end{array} \qquad \begin{array}{r} 4653 \\ 1004 \\ \hline 18612 \\ 4653 \\ \hline \underline{4671612} \end{array}$$

2. Sup-

2. Suppose the Places in the *Multiplier* exceed those in the *Multiplicand*; then if they exceed only by one, there is no place for a Contraction, (*Examp.* 3, 4.) But if there is only one Place more, it may receive a small Contraction thus; Add what's carried at the last Product to the whole *Multiplicand*, and set down the Sum on the left of the Figures already set down; (*Examp.* 5.)

Example 3.

```
     3207
   100005
---------
320716035
```

Example 4.

```
      468
  1000007
---------
468003276
```

Example 5.

```
    8674        8674
   10006       10006
--------    --------
86792044       52044
             8674
            --------
            86792044
```

Observe, If there are any more than one 1 upon the Left-hand of the o's, this will be but a Mixture of this and the preceding Case, to be done thus; *viz.* After you begin to take in the Figures of the *Multiplicand*, take them in gradually, first one, then two, and so on, till you take in as many as the number of 1's in the *Multiplier*, (beginning still at the first Figure of the *Multiplicand*,) and after that (taking in still the same Number of Figures, begin one place nearer the Left-hand; and when you have not as many Figures to take in, take in all you have. The following Examples, without any more Words, will sufficiently explain this.

Example 6.

```
    853467        853467
     11004         11004
----------    ----------
9391550868       3413868
               853467
              853467
              ----------
              9391550868
```

Example 7.

```
     67853945
       111007
-------------
7532262872615
```

Example 8.

```
     4632
   111004
---------
514170528
```

Example 9.

```
    463
  11004
-------
5094852
```

Example 10.

```
     463
  110004
--------
50931852
```

Example 11.

```
      463
  1100004
---------
509301852
```

CASE III. When one of the *Factors* has any Figure greater than 1 in the highest Place, and 1 in all the other Places make that Figure the *Multiplier*: then work thus, *viz.* Take the Figure in Units place of the *Multiplicand*, and set it down; next take the Sum of the first two places, and then the Sum of the first three Places, and so on (beginning still at the first Place) till you have made as many such Operations as the Number of times that 1 is in the *Multiplier*, (and if there are not as many Figures in the *Multiplicand* as to have a Figure more to take in at every Operation, you must continue to take in the whole *Multiplicand* till your number of Operations are compleat) still writing down the Sums as in *Addition*, and carrying the 10's. After this, take the Figure in the highest Place, and by it multiply the whole *Multiplicand* in order; taking into the first Product what was carried from the preceding Operations; and to this and each other Product add the Sum of

of as many of the next following Figures on the left as the Number of 1's in the *Multiplier*; and if there are not as many, add all that are.

Note, It will be convenient to set the highest Place of the *Multiplier* under the first of the *Multiplicand*.

Example 1.		*Example* 2.		*Example* 3.	
4768	4768	7236	7236	86	86
41	41	4111	4111	4111	4111
195488	4768	29747196	7236	353546	86
	19072		7236		86
	195488		7236		86
			28944		344
			29747196		353546

CASE IV. When one of the *Factors* has 1 in the first Place, any Figure in the highest Place, and o's betwixt them, make that Figure the *Multiplier*, and work thus; *viz.* Write down as many of the first places of the *Multiplicand* as are in Number one less than the Places in the *Multiplier*, (*Examp.* 1, 2, 3.) and if there are not as many, make it up with o's set on the left, (*Examp.* 4.) After this, multiply by the last Figure of the *Multiplier*, taking with every Product the Figure of the following Place but one or two, &c. according to the Number of o's in the *Multiplier*, (and set a Point over the Figures taken in, which will be a guide to the next) and you have the Product sought. next

Examp. 1.		*Examp.* 2.		*Examp.* 3.	*Examp.* 4.
678	678	34675	34675	496	7263
301	301	3001	3001	8001	400001
204078	678	104059675	34675	3968496	2905207263
	2034		104025		
	204078		104059675		

CASE V. When one of the *Factors* consists of the same Figure in all its Places (as 66, or 444,) make that the *Multiplier*, and work thus; *viz.* Multiply by that Figure, and out of this Product make up the total Product in this manner: Begin at the Place of Units, and first take one Figure, then two, then three, &c. (setting down the Sums under or over 10's, and carry the 10's;) repeating the Operation still from the first Place, as oft as the Number of Places in the *Multiplier*; and if there is not another Figure to take in at every Operation, you must continue to take in the whole *Multiplicand*, so oft till your Number of Operations be finished. Then begin at the second place, and third place, &c successively; taking in from each as many on the left as the Number of Places of the *Multiplier*, as long as you can find as many; and when they fail, take in all that remains till the last Figure is taken in alone; (minding always to carry the 10's from every Operation to the next.) product

Example 1.		*Example* 2.	
8739	8739	46	46
444	444	33333	33333
34956 Prod. by 4.	34956	138	138
3880116 Total.	34956	1533318	138
	34956		138
	3880116		138
			138
			1533318

SCHOLIUM. If the *Multiplier* consists of 1 in all its places, the Practice is the same; only we have not any previous Multiplication, the second part of the Work being made upon the *Multiplicand* itself.

COROL. When the *Multiplier* consists of any other Figures than 1, we may do the Work thus: First find the Product, as if the *Multiplier* consisted of as many 1's, and then multiply this Product by the Figure of the given *Multiplier*; so the preceding *Example* will stand thus:

8739 × 444
970029 Prod by 111.
4
3880116

The Reason of this you'll find at *Case* 9. joined with this, that $444 = 111 \times 4$.

CASE VI. If the *Multiplier* is a Number which has 9 in all its places, as 9, 99, 999, *&c.* set as many 0's on the Right-hand of the *Multiplicand*, and subtract it from itself so increased: the Remainder is the Product sought, as in these Examples.

	Example 1.	*Example* 2.	*Example* 3.
	$468 \times 9 = 4212$	$3726 \times 99 = 368874$	$7568 \times 999 = 7560432$
Operation.	4680	372600	7568000
	468	3726	7568
	4212	368874	7560432

The *Reason* of this Rule is obvious, for by the 0's prefix'd to the *Multiplicand*, it is multiplied by a Number which exceeds the given *Multiplier* by 1, (so $10 - 1 = 9$, $100 - 1$ Multiplicand $= 99$, and so on.) Wherefore the *Multiplier* being subtracted from this Product, the Remainder must be the true Product sought.

Observe, This Subtraction may easily be perform'd without the trouble of writing the *Multiplicand* oftner than once, by imagining the 0's that ought to be prefix'd: Thus, take the first Figure of the *Multiplicand* from 10, then the next Figure increased by 1, from 10, and so on, till you have made as many Subtractions from 10, as the Number of 9's in the *Multiplier*: then you subtract from the Figures of the *Multiplicand* itself; and when there are no more Figures to subtract, write down all the remaining Figures from which no Subtraction has been made, after subtracting 1 from them when 10 was borrowed in the preceding Subtraction. The preceeding *Examples* sufficiently shew how this is to be done.

Observe again, To multiply any Number consisting all of 9's by itself; the Product has 1 in the place of Units, then after it a Number of 0's fewer by one than there are 9's in the Number multiplied, then 8, and lastly, after it as many 9's as there are 0's before it.

Ex. $999 \times 999 = 998001$, and $99999 \times 99999 = 9999800001$.

The

The *Reason* of this will appear to be universal, by considering one *Example* done in the manner of the preceding Rule; thus,

$$\begin{array}{r} 9999900000 \\ 99999 \\ \hline 9999800001 \end{array}$$

CASE VII. To multiply any Number by 5; multiply it first by 10, *i.e.* set (or suppose) o before it; then take the half of it.

Ex. 1. 6458×5
32290 Prod.

Ex. 2. 7287×5
36435 Prod.

SCHOLIUM. I have here proposed a Division, tho' that Rule is not yet taught: But it was fit to put all together that relates to Multiplication, tho' the Learner should refer this Case till he has Division; yet I think it will be easy to see how any Number is halved, by considering these two Examples. And as to the Use of this Rule, it's certainly easier than multiplying by 5, tho' that is easy itself.

GENERAL COROLLARY *to the preceding 7 Cases.* If the Parts of the *Multiplier* coincide with any of the preceding Cases, we may apply them separately: As in these Examples.

Ex. 1.
$$\begin{array}{r} 7468 \\ 159 \\ \hline 67212 \\ 112020 \\ \hline 1187412 \end{array}$$

Ex. 2.
$$\begin{array}{r} 367845 \\ 4118 \\ \hline 6621210 \\ 15081645 \\ \hline 1514785710 \end{array}$$

For *Ex.* 1. Work by 9 (as in Case 6) then for 15 (as Case 1.)
For *Ex.* 2. Work with 18, and 41 (by Case 1. and 3.)

CASE VIII. If one of the *Factors* consists of two Places, and is equal to the Product of any two Digits, as $28 = 4 \times 7$; multiply by one of these Digits, and the Product found by the other; and the last Product is that sought.

Ex.
$$\begin{array}{r} 7264 \times 28 \\ 4 \\ \hline 29056 \\ 7 \\ \hline 203392 \end{array}$$ Prod.

The Reason of this is obvious, for 7 times 4 times is 28 times. Or may be deduced from *Lemma* 3. *Corol.* For having multiplied by 4, which is only the 7th part of 28, I must multiply again by 7 the Denominator of the Part, to make the true Product by the whole 28.

SCHOLIUM. If the *Multiplier* is equal to the *Product* of any three or more Digits; we may take the same Method, by multiplying continually by all these Digits: But this will not in every Case afford any Compend; nor will it always appear easily what Digits will produce the given Number: However the Reason of the Practice is evident from *Lem.* 3. *Cor.* For $7486 \times 8 \times 9 \times 6 = 8 \times 9 \times 6 \times 7486. = 432 \times 7486$, because $8 \times 9 \times 6 = 432$.

Ex. 7486×432 $(= 8 \times 9 \times 6)$
$$\begin{array}{r} 8 \\ \hline 61488 \\ 9 \\ \hline 553392 \\ 6 \\ \hline 3320352 \end{array}$$ Product.

CASE IX. In Multiplication of great Numbers, one part of the hazard of erring proceeds from the too great Distance betwixt the Figures of the several Product Lines and the corresponding Figures of the *Multiplicand*; and also their not standing directly under one another, so that we must always look athwart from the one to the other.

In many Cases, tho' we cannot contract the Work, either as to the Operation or Number of Figures written, yet we may make it more simple and easy, by help of Addition or Subtraction, or more simple Multiplication; and sometimes with these Methods we may

mix

mix ſome of the former. The following Examples will ſufficiently inſtruct you how to do the like in other Caſes: For the different Circumſtances of the *Multiplier* makes the Variety unlimited, and therefore there can be no general Rule.

```
Ex. 1.     648
            76
       -------
          3888
         4536
       -------
         49248
```

```
Ex. 2.     467          Or thus.     467
            78                        78
        ------                    ------
          3736                      3269
         3269                        3736
        ------                    ------
         36426                     36426
```

```
Ex. 3.   57684
            63
       -------
        173052
       346104
       -------
       3634092
```

```
Ex. 4.     974
           862
       -------
          1948
         5844
        7792
       -------
        839588
```

```
Ex. 5.    5689
           286
       -------
         11378
        45512
          34134
       -------
       1627054
```

```
Ex. 6.    4956          Or thus.    4956
           168                       168
        ------                    ------
         39648                     79296
        79296                        39648
        ------                    ------
        832608                    832608
```

```
Ex. 7.    5648
           459
       -------
         50832
       254160
       -------
       2592432
```

```
Ex. 8.  724689
           545
      ---------
       3623445
      26611005
      ---------
     398955505
```

The Operation of theſe Examples *is thus:*

Ex. 1. After multiplying by 6, I add that Product to the *Multiplicand*, inſtead of multiplying by 7.

Ex. 2. After multiplying by 8, I ſubtract the *Multiplicand* out of this Product, inſtead of multiplying by 7. Or, according to the other Method, I firſt multiply by 7, and to this Product add the *Multiplicand*. The reaſon of placing the two Lines as you ſee them ſtand here, is obvious.

Ex. 3. After multiplying by 3, I double this Product for 6.

Ex. 4. I multiply by 2, then multiply this Product by 3 (for 6) then add theſe two Lines (the firſt Figure of the one to the firſt of the other, and ſo on) for 8.

Ex. 5. I multiply by 2, this Product by 4, (for the 8) and then ſubtract the firſt Line from the ſecond (for the 6) or multiply the firſt Line by 3.

Ex. 6. I multiply by 8, and double this Product (for the 16.) Or multiply by 16 and halve this Product (for the 8.)

Ex. 7. I multiply by 9, and becauſe $9 \times 5 = 45$, I multiply the laſt Product by 5, either the common way, or by *Caſe* 7.

Ex. 8. I multiply by 5 (*Caſe* 7.) for the laſt 5; and this Product by 9 (*Caſe* 6.) for the 45.

An Universal Method *for all Caſes; whereby, tho' there is no Contraction, and even ſome more to do, yet it makes the Work ſo eaſy, that there is no time loſt, at leaſt in large Examples, and more Certainty in the Operation. Thus:*

Write down the *Multiplicand*, then double it; add this Sum to the *Multiplicand*, and this again, and ſo on, every Sum to the *Multiplicand*, till you have nine Numbers. And it's plain that thus you have a Table of the Products of the *Multiplicand* by all the Digits; made up by a very ſimple and eaſy Operation: And then you have no more to do, but transfer your ſeveral Products out of this Table, and ſum them up.

TABLE.

TABLE.

1	467853798
2	935707596
3	1403561394
4	1871415192
5	2339268990
6	2807122788
7	3274976586
8	3742830384
9	4210684182

Example.

```
          467853798
            6839754
   ----------------
         1871415192
        2339268990
       3274976586
      4210684182
     1403561394
    3742830384
   2807122788
   ----------------
   3200004886285692
```

SCHOLIUMS.

1. This Method is univerſal; but we need not apply it in every Caſe, for that would not always be beſt: But in ſuch Examples as this, I think the Eaſe and Readineſs with which it's done, does more than ſave the time ſpent in making the Table; with this Advantage, that the Work is perform'd with much more Certainty, becauſe it's more ſimple.

2. Again this may be contracted in many Caſes: for there is no neceſſity always to make the Table for all the nine Digits. And it may happen, that by help of ſome of the preceding Methods, we can as eaſily make a Table for few, or no more than we have uſe for in the *Multiplier*; nor is it any great matter in what order they ſtand in the Table.

Ex. 1. 78659 by 6897. In making this Table, I firſt take 3; then double its Product for 6; and do the reſt by the common way.

Table for *Ex.* 1.

1	78659
3	227977
6	455954
7	534613
8	613272
9	691932

Ex. 2. 783596 by 3856. Firſt I multiply by 3, then by 5, (as in *Caſe* 7.) then add theſe Products for 8, then ſubtract the firſt from this for 7.

Table for *Ex.* 2.

1	783596
3	2350788
5	3917980
8	6268768
7	5485172

MULTIPLICATION by *NEPER*'s RODS.

The great and everlaſting Honour of our Country, the Lord *Neper*, conſidering the vaſt Advantage of the preceding Method of *Tabulating* the *Multiplicand*, to make it yet more eaſy, contrived the following Machine, for the more certain and ready way of making the Table: To underſtand which, we muſt ſet before us what they call,

PYTHAGORAS's Table of *Multiplication*.

1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

The Conſtruction of this Table is the ſame as that ſhewn in *Problem* 1. with this difference, that here there is a compleat Column of Products from every Digit on the head, to 9 times that Digit; ſo that either of the Factors may be found on the head, or on the left ſide.

Now, ſuppoſe this Table to be made upon a Plate of Metal, Ivory, Wood, or Paſt-board; and then conceive the ſeveral Columns (ſtanding downwards from the Digits on the head) to be cut aſunder, and theſe are what we call *Neper's Rods*, for *Multiplication*. But then there muſt be a good number of each; for as many

times

times as any Figure is in the *Multiplicand*, so many Rods of that Species, (*i. e.* with that Figure on the Top of it) must we have; tho' 6 Rods of each Species will be sufficient for any Example in common Affairs. There must also be as many Rods of o's. But before we explain the way of using these Rods, there is another thing to be known, *viz.* That the Figures on every Rod are written in an Order different from that in the Table; *Thus*, The little square Space or Division in which the several Products of every Column are written, is divided into two Parts by a Line a-cross from the upper Angle on the Right to the lower on the Left; and if the Product is a Digit, it is set in the lower Division; if it has two Places, the first is set in the lower, and the second in the upper Division; but the Space on the Top is not divided. Also there is a Rod of Digits not divided, which is called the *Index-Rod*; and of this we need but one single Rod. See here the Figure of all the different Rods, and the Index, separate from one another.

NEPER's Rods.

Index Rod.										
1	1	2	3	4	5	6	7	8	9	0
2	/2	/4	/6	/8	1/0	1/2	1/4	1/6	1/8	/0
3	/3	/6	/9	1/2	1/5	1/8	2/1	2/4	2/7	/0
4	/4	/8	1/2	1/6	2/0	2/4	2/8	3/2	3/6	/0
5	/5	1/0	1/5	2/0	2/5	3/0	3/5	4/0	4/5	/0
6	/6	1/2	1/8	2/4	3/0	3/6	4/2	4/8	5/4	/0
7	/7	1/4	2/1	2/8	3/5	4/2	4/9	5/6	6/3	/0
8	/8	1/6	2/4	3/2	4/0	4/8	5/6	6/4	7/2	/0
9	/9	1/8	2/7	3/6	4/5	5/4	6/3	7/2	8/1	/0

USE of the RODS.

Lay down first the Index-Rod; on the Right of it, set a Rod on whose Top is the Figure in the highest Place of the *Multiplicand*; next to this again set the Rod on whose Top is the next Figure of the *Multiplicand*; and so on in order to the first Figure: Then is your *Multiplicand* tabulated for all the 9 Digits; for in the same Line of Squares standing against every Figure of the Index-Rod, you have the Product of that Figure; and therefore you have no more to do but transfer the Products, and sum them.

But in taking out these Products from the Rods, the Order in which the Figures stand obliges you to a very easy and small Addition, thus; Begin to take out the Figure in the lower Part (or *Units* place) of the Square of the first Rod on the Right; add the Figure in the upper Part of this Rod to that in the lower Part of the next, and so on; which may be done as fast as you can look upon them. To make this Practice as clear as possible, take this Example.

Examp.

The Rods set together for the Number 4768.

1	4	7	6	8
2	/8	1/4	1/2	1/6
3	1/2	2/1	1/8	2/4
4	1/6	2/8	2/4	3/2
5	2/0	3/5	3/0	4/0
6	2/4	4/2	3/6	4/8
7	2/8	4/9	4/2	5/6
8	3/2	5/6	4/8	6/4
9	3/6	6/3	5/4	7/2

Examp. Multiply 4768 by 385.

Against 5 in the Index I find this Number, according to the Rule,	23840
Against 8 this Number, — —	38144
Against 3 this Number, — —	14304
Total Product,	1835680

To make the Use of the *Rods* yet more regular and easy, they are kept in a flat square Box, whose Breadth is that of ten Rods, and the Length that of one Rod; as Thick as to hold six, (or as many Rods as you please;) the Capacity of the Box being divided into ten Cells, for the different Species of Rods. When the Rods are put up in the Box, (each Species in its own Cell, distinguished by the first Figure of the Rod set before it on the Face of the Box near the Top,) as much of every Rod stands without the Box as shews the first Figure of that Rod: Also upon one of the flat Sides, without and near the Edge upon the Left-hand, the Index-Rod is fixed; and along the Foot there is a small Ledget; so that the Rods, when applied, are laid upon this Side, and supported by the Ledget, which makes the Practice very easy. But in case the *Multiplicand* should have more than 9 Places, that upper Face of the Box may be made broader.

Some make the *Rods* with four different Faces, and Figures on each, for different purposes. But I have explained what is necessary for *Common Multiplication*, and shall leave it.

A NEW METHOD *by a small Moveable* TABLE.

To Those who want *Rods*, I propose the following Methods, which, with the help of *Case* VI. and VII. and some Hints given in *Case* X. may, I believe, be near as easy and expeditious as the *Rods*. Thus:

1. Make a Table of the *Multiplicand* only for the Numbers 1, 2, 5; (using *Case* VII. for 5.) and make it upon a bit of loose Paper, that it may be always applied directly and immediately over the Place where every particular Product is to be written down, (for much of the difficulty lies, as I have already observed, in the Distance and cross Position of the *Multiplicand* to the several Products,) and out of this small Table find your Product thus:

Suppose for a Multiplicand 685497.

TABLE.

1	685497
2	1370994
5	3427485

When the Figure of the Multiplier is 2 or 5, here you have the Products; then for 3 add 2 to 1, (*i.e.* the Numbers against 2 and 1;) for 4, double the Number against 2. For 6, add 5 and 1; or multiply 2 by 3. For 7 add 5 and 2. For 8 add 5, 2 and 1; for 9, use the Method of *Case* VI.

2. We

2. We may alſo make the Table for 3 and 5, as here: And then for 2, double 1; for 4, add 3 and 1; for 6, double 3, or add 5 and 1; for 7, double 3 and add 1; that is, to every Product add the Figure of the *Multiplicand.* For 8, add 5 and 3; for 9, uſe *Caſe* VI.

1	685497
3	2056491
5	3427485

3. Or alſo make the Table for 3 and 7: And to get the 7, I multiply 3 by 2, and add 1. Then in uſing the Table, to get the Product by 4, add 2 to 1. For 5, uſe *Caſe* VII. For 6, double 2; for 8, add 1 and 7; for 9, uſe *Caſe* VI. or multiply 3 by 3.

1	685497
3	2056491
7	4798479

4. Or we may make the Table for 1, 2, 4. Then for 3, add 1 and 2; for 5, uſe *Caſe* VII. or add 1 and 4. For 6, add 2 and 4; for 7, add 1, 2, and 4; for 8, double 4; for 9, uſe *Caſe* VI.

1	685497
2	1370994
4	2741988

5. Or laſtly, make the Table for 1, 3, 4. Then for 2, double 1; for 5, uſe *Caſe* VII. for 6, double 3; for 7, add 4 and 3; for 8, double 4; for 9, uſe *Caſe* VI.

1	685497
3	2056491
4	2741988

Uſe any of theſe Tables you pleaſe; and for different Multipliers, one of them may, perhaps, be preferable to another. But if all the 9 Digits are in any Multiplier, it is indifferent which of them you chuſe; tho' I think the third Method has the Advantage.

CHAP. VI.

DIVISION *of Abſtract Whole* NUMBERS.

DEFINITION.

DIVISION *findeth how oft one* Number *is contained in another.*

The Number divided, (or which is conſider'd as the containing Number) is called the *Dividend*; the Number dividing, (or which is conſidered as contained) is called the *Diviſor*; and the Number ſought, is called the *Quotient*, or *Quote*, (from *Quoties, How oft*;) becauſe it ſhews the *how oft* ſought. *Examp.* If we enquire how oft 3 is contained in 12, the Anſwer is 4 times: And 12 is the *Dividend*; 3 the *Diviſor*; 4 the *Quote.*

SCHOLIUMS.

1. Every greater Number is not a *Multiple* of every leſſer: therefore when a greater Number is propoſed to be divided by a leſſer, and is not a Multiple of it, this Operation finds how oft the Diviſor is contained in the Dividend; and alſo what remains after the Diviſor is taken out of the Dividend as oft as poſſible: So that in ſome Caſes there are four Numbers concerned in *Diviſion*; *viz.* the *Diviſor*, *Dividend*, *Quote*, and *Remainder.* *Examp.* 3 is contained in 14, 4 times, and 2 *remains.*

2. As *Multiplication* is only a compendious *Addition*; ſo is *Diviſion* only a compendious *Subtraction* of one Number out of another as oft as poſſible: For it is plain, that as oft as any Number is contained in another, ſo oft preciſely it can be taken out of it; ſo

ſo that if we find how oft the leſſer can be taken out of the greater, we thereby find how oft it is contained in it: but to find how oft it can be taken out, is plainly the work of Subtraction; for by taking the leſſer out of the greater, and the ſame leſſer out of the firſt Remainder, and out of every Remainder ſucceſſively till the Remainder be o, or leſs than the Subtractor, we have found what is required is *Diviſion*; the Number of Subtractions being the Quote. So in theſe Examples, we find that 3 is contained in 12, 4 times; becauſe it can be taken out of it 4 times, and o remains. And 3 is contained in 14, 4 times, and 2 remains; becauſe being 4 times ſubtracted, there remains 2.

Ex. 1.	*Ex.* 2.
12	14
3	3
9	11
3	3
6	8
3	3
3	5
3	3
o	2

But tho' the thing ſought in *Diviſion* may be found by Subtraction, yet it would be intolerable Labour in moſt Caſes; and therefore the following Rules of *Diviſion* are contrived, which do that by a few eaſy ſteps which cannot be done all at once, and would be too tedious to do by Subtraction.

Again *obſerve*, That tho' there are here alſo, as in the preceding Operations, ſome more ſimple Caſes, the ready performance of which is uſeful in more complex Caſes: yet in moſt Caſes we are left to gueſs at the Anſwer in the ſeveral ſteps of the Work; with this help only, that we have a certain Rule for proving the Number gueſſed to be right, and when it is wrong, how to come nearer to it at the next gueſs, till at laſt we find it; as you will preſently learn.

3. In all Caſes where there is a Remainder in the *Diviſion*, or when the *Dividend* is not a Multiple of the *Diviſor*, the Number called the *Quote* is the direct and proper Anſwer to that Queſtion, How oft is the *Diviſor* contained in the *Dividend*? yet the Remainder may be brought in fractionally as a Part of the *Quote*, making the Remainder the Numerator, and the *Diviſor* the Denominator of a Fraction: And then we may ſay, That the *Dividend* contains the *Diviſor* ſo many times as the *Integral Quote* expreſſes, and ſuch a Part or Parts of a time, (*i. e.* ſuch a Part or Parts of the Diviſor,) as that Fraction expreſſes. Thus, for Example, 3 is contained in 14, 4 times, and 2 remains; which being $\frac{2}{3}$ Parts of 3, we may ſay, that 14 contains 3, 4 times and $\frac{2}{3}$ Parts of a time; *i. e.* that it contains 4 times 3, and $\frac{2}{3}$ Parts of 3: And this mix'd Expreſſion, $4+\frac{2}{3}$; or thus, $4\frac{2}{3}$, may be called *The Complete Quote*, in diſtinction from the Integral Quote. There will be the ſame Reaſon in all Caſes for completing the *Quote* by a Fraction made of the Remainder and Diviſor, and underſtanding it as we have done in this Example. Thus, univerſally, if any Number A is contained in another B, a number of times expreſſed by D, with a Remainder R; then is the complete Quote $D\frac{R}{A}$: For any Number R being, the ſame as R times 1, and 1 being ſuch a Part of any Number A, as that Number denominates; *i. e.* the $\frac{1}{A}$ Part; therefore R is $\frac{R}{A}$ Parts of A. Wherefore, if by the words, *How oft*, in the Definition, we mean how many times and parts of a time, (as now explained,) the Integral Quote muſt always have this Fraction, made of the Remainder and Diviſor, annex'd to it as a part of the complete Anſwer of the Queſtion.

Hence again *obſerve*, That in this ſenſe, the Dividend may be a leſſer Number than the Diviſor; for tho' a leſſer Number does not contain a greater, yet it contains a certain Fraction of it, which is what we call containing it a certain part or parts of a time; and the Quote is a Fraction whoſe Numerator is the Dividend, and its Denominator the Diviſor. So 3 divided by 5, the *Quote* is $\frac{3}{5}$, ſignifying that 3 contains 5, $\frac{3}{5}$ parts of a time; or that it contains $\frac{3}{5}$ parts of 5.

4. But again, for the ſame *Reaſon*, (*viz.* that any Number A is equal to or contains $\frac{A}{B}$ Parts of any Number B) the complete Quote of any Number divided by any other Number

Number may be indefinitely expressed by a Fraction whose Numerator is the Dividend, and its Denominator the Divisor. Thus 5 divided by 3, the *Quote* is $\frac{5}{3}$; and 3 divided by 5, the *Quote* is $\frac{3}{5}$. *Universally*, A divided by B, the *Quote* is $\frac{A}{B}$.

Now, as Fractions arise from *imperfect Division* when the Dividend is not a Multiple of the Divisor; so the Consideration of Fractions does necessarily begin with *Division of Whole Numbers:* For which Reason there are some things relating to Fractions must be explained in this Part; particularly I must here shew you, That the Fraction $\frac{A}{B}$ is in all Cases equal in Value to the complete Quote of A divided by B; and this being demonstrated, it will follow, that if we use such an Expression of a Quote in any Reasoning or Operation, instead of the more direct and immediate Expression of the Quote, which it is often very convenient to do, the Effect will be the same. And this I demonstrate thus: 1. If A is lesser than B, there is no other Quote. 2. If A is greater than B, then is $\frac{A}{B}$ an improper Fraction; and from the nature of such a Fraction, (as it is defined and explained in *Chap.* 1.) this is manifest, That as oft as B is contained in A, so many Units (of that kind to which the Fraction refers) is the Fraction equal to, and to such a proper Fraction more, whose Numerator is the Remainder, (after B is taken out of A as oft as possible) and its Denominator B. For when the Numerator and Denominator are equal, the Value of the Fraction is 1; so $\frac{3}{3}$ Parts, or $\frac{B}{B}$ Parts of any thing, is equal to that Thing or Unit: Therefore, if B is contained in A, R times without a Remainder, $\frac{A}{B}$ is equal to R times 1, or R. And if there is a Remainder n, the Value of it is $R + \frac{n}{B}$, or $R\frac{n}{B}$.

The *Division* or *Quote* of two Numbers is also expressed by this Sign $\div$ set betwixt them; the *Dividend* being set first, thus $5 \div 3$, signifies 5 divided by 3. $A \div B$ signifies A divided by B. It is also sometimes expressed by this Sign), with the *Divisor* before the *Dividend*, thus 3) 5. A) B.

Here then you have the *General Rule* of the Literal Division.

Corollaries.

I. If the Integral Quote (or Number of times the whole Divisor is contained in the Dividend,) is multiplied into the Divisor, and to the Product be added the Remainder, (after the Divisor is taken out of the Dividend as oft as possible) the Sum is equal to the Dividend. For Example, 3 is contained 4 times in 12, and nothing remains; therefore 4 times 3 is 12. Again, 3 is contained 4 times in 14, and 2 remains; therefore 4 times 3, and 2 added, is 14: For $4 \times 3 = 12$, and $12 + 2 = 14$. *Universally*, if $A \div B = q$, and nothing remaining, then is $Bq = A$. But if r remains, then is $Bq + r = A$.

II. The Remainder in *Division* must always be less than the Divisor; for else the Divisor is not taken out of the Dividend as oft as possible: and this therefore is a Mark that the *Quote* is taken too small: As again, if the Product of the Divisor and Quote exceed the Dividend, it is a sign the Quote is taken too great. And hence, lastly, the Product of the Divisor and integral Quote is the greatest Multiple of the Divisor contained in the Dividend.

Scholiums.

I. We have here learnt a mutual *Proof* of *Multiplication* and *Division*, as these Operations are in their Effects directly opposite to one another. Thus, in *Multiplication*, if the Product is divided by any one of the Factors, the Quote is the other. And in *Division*, if the Divisor is multiplied by the integral Quote, and to the Product be added the Remainder, the Sum is the Dividend. Other *Proofs of Division* you will find afterwards.

II. If we compleat the Quote by a Fraction made of the Remainder and Divisor, then it is a general Truth, that the Divisor multiplied by the compleat Quote, produces the Dividend: For being multiplied by the integral Quote, the Product is the greatest Multiple of the Divisor contained in the Dividend; and multiplied by the Fraction, (*i. e.* such a Fraction of it being taken,) it produces the Remainder; thus, $14 \div 3 = 4\frac{2}{3}$; then $4 \times 3 = 12$, and $\frac{2}{3}$ of $3 = 2$. And $12 + 2 = 14$. *Universally*, if $A \div B = q\frac{r}{B}$, then $A = Bq + \frac{r}{B}$ of $B = Bq + r$, as in *Corol.* 1. *Observe* also, That as in Whole Numbers, it is no matter which of two Numbers is the Multiplier; so, to multiply a Whole Number by a Fraction, or this by that, has the same Effect, as will be explained in *Book* 2. But in the Case now before us, the Reason is obvious. Thus to multiply B by $\frac{A}{B}$ is only taking $\frac{A}{B}$ Parts of B, which is A. Again, to multiply $\frac{A}{B}$ by B, is taking B times, A times $\frac{1}{B}$; (for $\frac{A}{B}$ is A times $\frac{1}{B}$.) But B times $A =$ A times B, therefore B times A times $\frac{1}{B} =$ A times B times $\frac{1}{B} =$ A times 1; (for B times $\frac{1}{B} = 1$) or A. Wherefore to multiply the two Parts of the complete Quote by the Divisor, or to multiply the Divisor by these, the Sum of the Products is the Dividend. For $\frac{r}{B}$ of $B = B$ times $\frac{r}{B} = r$; and $Bq + r = A$. Or taking the complete Quote fractionally, then $\frac{A}{B}$ of $B = B$ times $\frac{A}{B} = A$.

PROBLEM.

To Divide *one Number by another.*

CASE I. *When the* Divisor *is a Digit, or single Figure, and the* Dividend *either a Digit, or a Number of two Figures, whereof that in the place of Tens is less than the* Divisor.

Rule. Take such a Digit as, multiplied into the Divisor, will exactly produce the Dividend; but if there is not such a Digit, take the greatest, which multiplied into the Divisor, makes a Product less than the Dividend; that Digit is the integral Quote, and the Remainder (which must be less than the Divisor) set fractionally over the Divisor, compleats the Quote. The Reason of this Rule is in *Schol.* 1. preceding.

Examp. 1. $12 \div 3 = 4$; because $4 \times 3 = 12$. *Examp.* 2. $26 \div 4 = 6$, and 2 remains; so the complete Quote is $6\frac{2}{4}$; for $6 \times 4 = 24$: then $24 + 2 = 26$. *Examp.* 3. $8 \div 5 = 1$, and 3 remains; so the complete Quote is $1\frac{3}{5}$.

Whoever is familiar with the Table of *Multiplication*, can find at first hearing the Answer to any Example of this Case. Or we may take help of that Table, thus: Seek the Dividend in the Table; and if it is not there, seek the greatest Number which is less than it; the Digit in the same Line on the Side of the Table, or in the same Column on the head of the Table, is the Divisor, and the other is the integral Quote.

CASE II. *To* Divide *any Number by another.*

Rule. Set the Divisor on the left of the Dividend, as in the following Examples; then take as many Figures from the Left-hand of the Dividend as are in Number equal to the Places of the Divisor; and if these Figures, considered by themselves, make a Number less than the Divisor, take in one Figure more, [which will necessarily make a Number greater than the Divisor,] this we call the first *Dividual*, (or partial *Dividend.*) We are then to find how oft the Divisor is contained in this Dividual; and here it is that we are left in a great measure to guess at the Figure sought: But this we know, that the Quote cannot exceed 9, as shall be afterwards demonstrated. And then also (by *Corol.* 2. preceding,) it must

must be such, that the Product of the Divisor by it do not exceed the Dividual, (for then it is too great a Figure;) and the Remainder (or Difference of the Dividual and Product) be less than the Divisor, (else the Quote-Figure is too little;) and thus we must find the Quote by trials. But to prevent too many useless trials, we have this help, *viz.* Find, by *Case* I. how oft the first Figure on the left is contained in the first on the left of the Dividual, when this and the Divisor have equal number of Places; But in the first two Figures on the left of the Dividual, when this has one place more than the Divisor; and this Number limits the guessing, so that you cannot take a greater; and if this happens to exceed 9, [which it will in no Case but the last, and that where the first of the two Figures in the Dividual is equal to the Divisor-Figure; for it is certain this will be found in these at least 10 times,] your guess begins at 9. But then it will often happen that this Number is too great, and we have no other general Rule, or Help here, but to begin at this Figure and make trials, by multiplying the Divisor; for if the Product does not exceed the Dividual, that is the Figure sought: if it does exceed, take the next lesser Figure, and with it make the like trial; and go on so gradually till you find a Figure whose Product does not exceed the Dividual; for then the Remainder will certainly be less than the Divisor, which is the true Proof of the Quote-Figure's being right. Having thus found the first Figure of the Quote, set it down (on the Right-hand of the Dividend) and write the Product of it by the Divisor under the Dividual, and subtract that out of this; and then before the Remainder (on the right) set the next Figure of the Dividend, (or the Figure on the right of the first Dividual,) and this Number is the second Dividual. Then in the same manner as before, find how oft the Divisor is contained in this Dividual; set the Figure found on the Right-hand of the Quote Figure last found; then multiply the Divisor by it: write down the Product under the Dividual, and subtract as before; then to the Remainder prefix the next Figure of the Dividend, and this is the next Dividual to be divided as before. In this manner proceed till every Figure of the Dividend is employed step by step: And *observe*, that if any Remainder with one Figure of the Dividend prefix'd, makes a Number less than the Divisor, set 0 in the Quote, and prefix also the next Figure; and so on.

All the Quote Figures thus found, taken in order as they are placed as one Number, is the true Quote sought; and the last Remainder is what the Dividend contains over so many times the Divisor as the Quote expresses.

I shall next illustrate this Rule by Examples.

Examp. I.

```
Divr. Divd. Quote.
  4 ) 6392 ( 1598.
      4
      --
      23
      20
      ---
       39
       36
       ---
        32
        32
        ---
        00
```

To divide 6392 by 4, I proceed thus: I seek how often the Divisor 4 is contained in 6, (the first Figure of the Dividend) which is but once; therefore I set 1 in the Quote; then $1 \times 4 = 4$ (or the Divisor multiplied by the Quote is 4,) which I write under the Dividual 6, and subtracting, the Remainder is 2, to which I prefix 3, the next Figure of the Dividend, and then I take 23 for my next Dividual; and examining how oft 4 is contained in it, I find 5 times, (for had I taken 6, it were too great; for $6 \times 4 = 24$, which is greater than the Dividual 23; and had I taken 4, it were too little, for $4 \times 4 = 16$, and $23 - 16$, is 7, which is greater than the Divisor 4;) therefore I place 5 in the Quote on the Right of the former; and under 23 set $20 = 5 \times 4$; then subtracting, the Remainder is 3, to which I prefix the next Figure of the Dividend, *viz.* 9; then is 39 my next Dividual: and in this I find the Divisor 4 contained 9 times, which I write in the Quote on the right of the former; then $9 \times 4 = 36$, which I write under the Dividual 39, and the Remainder is 3, to which prefixing the next (and last) Figure of the Dividend 2. This Number, *viz.*

32

32 is my next (and last) Dividual, in which the Divisor is contained 8 times, which I set in the Quote on the right of the preceding Figures; then setting down the Product 32 ($=8\times4$,) there is no Remainder; and the Quote sought is 1598, that is, 4 is contained in 6392, 1598 times.

Examp. 2.

```
36) 85609 (2378.
    72
    136
    108
     280
     252
      289
      288
        1 Remr.
```

To divide 85609 by 36, I proceed thus: There are two Places in the Divisor, therefore I take the first two Places of the Dividend, *viz.* 85, which making a Number greater than the Divisor, I take them for the first Dividual, and seeking how oft 3 (the first Figure of the Divisor) is contained in 8 (the first Figure of the Dividual, because they have equal Places) I find it 2 times, which I find also by trial to be the true Quote of 85 by 36; so I place 2 in the Quote, and subscribing the Product 72 ($=2\times36$) I subtract it from 85, the Remainder is 13, to which prefixing the next Figure of the Dividend 6, my next Dividual is 136. Then I seek how oft 3 (the first Figure of the Divisor) is contained in 13, the first two of the Dividual, (because it has one place more than the Divisor,) and I find it 4 times: but this is too great for the whole Divisor, (because $4\times36=144$) therefore I try the next Figure 3, and find it right; therefore I set 3 in the Quote, and subscribing the Product 108 ($=3\times36$,) I subtract it from 136, and the Remainder is 28; to which prefixing 0, the next Figure of the Dividend, my next Dividual is 280. Then I seek how oft 3 is contained in 28, and I find 9 times; but this is too great, (for $9\times36=324$.) I take again 8, and find it also too great, (for $8\times36=288$) and at last I find 7 to be right; therefore I set 7 in the Quote, and subscribing the Product 252 ($=7\times36$) the Remainder is 28; to which prefixing 9, the next and last Figure of the Dividend, I have for my next and last Dividual 289, in which I find as before, that the Divisor is contained 8 times, and 1 remains. So the true Quote is 2378, and 1 over; which is $2378\frac{1}{36}$.

Examp. 3.

```
465) 2744897 (5903
     2325
      4198
      4185
       1397
       1395
          2 Remr.
```

To divide 2744895 by 465, I proceed thus: my first Dividual is 2744; I seek how oft 4 (the first Figure of the Divisor) is contained in 27 (the first two of the Dividual.) I find it 6; but this is too great, and I take 5, and find it right. Then multiplying and subtracting, and prefixing to the Remainder the next Figure of the Dividend, my next Dividual is 4198; and here I find the Divisor contained 9 times. Then proceeding as before, my next Dividual is 139; which being less than the Divisor, I set 0 in the Quote, and then prefix another Figure; so that my next Dividual is 1397, in which the Divisor is contained 3 times, and the Remainder is 2; so the true Quote is 5903; or, taking in the Remainder, it is $5903\frac{2}{465}$.

Examp. 4.

```
462) 3235386 (7003
     3234
        1386
        1386
        0000
```

To divide 3235386 by 462. The first Dividual is 3235, in which the Divisor is contained 7 times, and the Remainder is 1; the next Dividual is 13, in which the Divisor is not contained; therefore I set 0 in the Quote, and bringing down the next Figure, I have 138 for a new Dividual, which is also less than the Divisor, therefore I set another 0 in the Quote; and bringing down another Figure, the next Dividual is 1386, in which the Divisor is contained 3 times, and 0 remains.

Examp. 5.

```
372) 149100 (400
     1488
      300 Remr.
```

To divide 149100 by 372. The first Dividual is 1491, and the Quote of this is 4, then the Remainder is 3, and the rest of the Dividuals are 30 and 300, which are each less than the Divisor, and therefore the Quote is 400.

Examp. 6. To Divide 210800 by 62, the Quote is 3400.

```
62) 210800 (3400
    186
    ---
     248
     248
```

In Cases like this, when there is no Remainder in any step, and that the Remaining Figures of the Dividend are all o's, we have no more to do than to join as many o's to the preceding Figure, found in the Quote.

These Examples will be sufficient to make a diligent Learner understand this Rule; and I shall only set down a few more Examples, with their Answers, leaving the Operation for an Exercise to the Student.

Examp. 7. To divide 5713070046 by 678, the Quote is 8426357.
Examp. 8. To divide 6069944827 by 8376, the Quote is 724683 $\frac{19}{8376}$
Examp. 9. To divide 293682135936 by 8405, the Quote is 34960078 $\frac{346}{8405}$.

In order to the *Demonstration* of the preceding Rule, and for the sake of some other special Rules to follow, we must premise the following Truths in the Theory of *Division*.

LEMMA I.

1. If two Numbers consist of an equal Number of Places, the lesser is not contained in the greater above 9 times.

2. Again, If the greater of two Numbers has but one Place more than the lesser; and supposing also that, excluding the first Figure on the Right of that greater Number, the remaining Figures on the Left make a Number less than the lesser given Number, then this lesser Number is not contained in the greater above 9 times.

DEMON. *Part* 1. If a Cypher is prefix'd to the lesser of two Numbers, (which have both the same Number of Places,) it is thereby multiplied by 10; and consequently that is the least Number which contains it 10 times: but the other given Number having one Place fewer than this Product, is a lesser Number, and consequently does not contain the lesser given Number 10 times, or does not contain it above 9 times. *Example*, 11 is less than 99; but is not contained in it 10 times, for 10 times 11 is 110, which is greater than 99. *Universally*, Let A be the greater, and B the lesser of two Numbers having an equal Number of Places; B × 10 contains B precisely 10 times, and it is a Number that has one Place more than B or A, and consequently is a greater Number than A; wherefore B is not contained 10 times in A.

Part 2. The lesser given Number (is 476) is greater, by supposition, than as many Places (475) on the left of the greater given Number (4759;) and must exceed it by at least 1: therefore 10 times this lesser Number, (*viz.* 4750,) must want at least 10 of 10 times the given lesser Number, (*viz.* 4760.) But whatever Digit we add to this deficient Product, or put in the Place of the o, (making in the present *Examp.* 4759,) it cannot make up the defect of 10; and therefore the given lesser Number (476) is not contained in the given greater Number (4759) 10 times, or not above 9 times.

SCHOL. The second Part of this *Lemma* is but a particular Case (accommodated to our present purpose) of a more general *Theorem*; which is this, *viz.* If any Number A, is greater than another, B; and if B is multiplied by any Number R, then A is not contained R times in R B, nor yet in the Sum of R B, and any Number N which is less than R; *i. e.* in R B + N.

The *Reason* is; since A is not once contained in B, neither is R A once contained in R B, which must want at least as many Units as R to make it equal to R A; and since N is less than R, R B + N cannot be equal to R A; *i. e.* A, which is contained precisely R times in R A, is not contained R times in R B + N, which is less than R A.

L E M M A II.

If any Number N is resolved into any Parts A, B, C, *&c. i. e.* if N = A + B + C, *&c.* then,

1. If all these Parts A, B, C, *&c.* are severally Multiples of any Number D, or all except one; then dividing A, B, C, *&c.* severally by D, the Sum of the Quotes is equal to the Quote of N ÷ D. (*Examp.* 1, 2, 3.) And the Remainder in the Division of that Part which is not a Multiple of D, is the Remainder in the Division of N ÷ D.

2. If there are more than one of the Parts of N, that are not Multiples of D; and if the Sum of the Remainders, in the Division of these Parts that are not Multiples of D, is less than D, then the Sum of the Integral Quotes, is the Integral Quote of N ÷ D; and the Sum of the Remainders, is the Remainder in the Division of N ÷ D. (*Examp.* 4.)

3. If the Sum of the Remainders is equal to, or greater than D; then, being divided by D, and the Integral Quote added to the Sum of the Integral Quotes of the Parts of N, this last Sum is the Integral Quote of N ÷ D; and the Remainder in the Division of the Sum of the Remainders, is equal to the Remainder in the Division of N ÷ D (*Examp.* 5.)

Demon. The Reason of all these Articles is easily seen from the Equality of the Whole and all its Parts. In the first and second Article it is obvious: for as oft as D is contained in A, B, C, *&c.* severally, so oft at least as the Sum of these times, it must be contained in the whole N; and if the Sum of the Remainders in the Division of the Parts A, B, C, *&c.* is less than D, then it is plain that D is contained no oftner in N than the Sum of the times it is contained in all its Parts A, B, C, *&c.* and the Remainder in N ÷ D, must be the Sum of the Remainders in the Division of the Parts, when this Sum is less than D; but if this Sum is equal to, or exceeds D, (as supposed in Article 3.) then since these Remainders are Parts of the Dividend, it is evident, that as oft as D is contained in their Sum, that must be added to the Sum of the times it is contained in A, B, C, *&c.* and this last Sum is the times it is contained in N; and the Remainder on the Division of the Sum of the Remainders, is the Remainder in dividing N by D.

Examp. 1.

D N = A + B + C.
Divisor 4) 36 = 16 + 12 + 8. Dividends.
Quotes 9 = 4 + 3 + 2.

Examp. 2.

Divisor 5) 48 = 25 + 15 + 8. Dividends.
Quotes 9 = 5 + 3 + 1.
Remainders 3 = 3.

Examp. 3.

Divisor 6) 46 = 24 + 18 + 4. Dividends.
Quotes 7 = 4 + 3 + 0.
Remainders 4 = 4.

Examp. 4.

Divisor 6) 53 = 20 + 27 + 6. Dividends.
Quotes 8 = 3 + 4 + 1.
Remainders 5 = 2 + 3.

Schol. If we take the complete Quotes by Fractions made of the Remainders and Divisor; then it is an *Universal Truth*, That the Sum of the Quotes of the Parts of N divided severally by any Number D, is equal to the Quote of N divided by D. For the fractional Parts of the Quotes have all the same Denominator D, and their Numerators are the several Remainders. But from the nature of Fractions it is obvious, that several Fractions having the same Denominator, and being referred to the same Integer, their Sum is the Sum of the Numerators, applied as a Numerator to the same Denominator. *Examp.* $\frac{2}{7}$ of any thing, and $\frac{3}{7}$ of the same thing, make $\frac{5}{7}$. So that if the Sum is an improper Fraction, we find its Value in a whole Number, or with a proper Fraction annex'd, by dividing the Nu-

Examp. 5.

Divisor 6) $88=46+29+13$. Divid.
Quotes 14 , $7+4+2$.
Remainders 4 , $4+5+1$.

In this last Example, the Sum of the Remainders is $4+5+1=10$, which is greater than the Divisor 6; and being divided by it, the Quote is $10\div6=1$, and 4 remains. Then this Quote 1 added to the Sum of the former Quotes, the Sum is $7+4+2+1=14$, the Quote of $88\div6$, and the Remainder of $88\div6$, is the same as that of $10\div6$, which is 4.

merator by the Denominator, (as before explained.) It is plain then, that if the Fractions belonging to the complete Quotes of the Parts of N divided by D, are added together, and the Value of the Sum added to the Integral Quotes, the Operation is the same as expressed in the *Lemma:* for it is adding the Remainders; and if their Sum exceeds the Divisor, taking the Number of times the Divisor is contained in it, and adding this Quote to the Sum of the Integral Quotes; which makes the Universal Truth here proposed evident.

COROLLARIES.

I. If a Number N is resolved into any Number of Parts, and these Parts be divided severally by any Number D, in this manner, *viz.* First divide one Part, and if there is a Remainder, add it to another Part, and divide the Sum; and so on, adding the Remainder of every Division to the next Part; and if any Part with the preceding Remainder is less than the Divisor, then we add another. Having thus gone through all the Parts, the Sum of the Quotes is the Quote of the first Number N divided by the same Divisor D; and the last Remainder in the Division of the Parts, is the Remainder in the Division of N by D.

That this is in Effect the same Case as the first Article of the preceding *Lemma*, or a plain Consequence of it, will be obvious by considering, that if the first Remainder is taken out of the first Dividend, it leaves a Multiple of the Divisor, *viz.* the Product of the Divisor and Quote; and the same being true in all the rest of the Steps, it follows that the Case is the same as if N were resolved into Parts equal to these Multiples of the several Quotes by the Divisor; all which Multiples with the last Remainder make up the Dividend N. For Example, $50=17+8+25$, then $17\div3=5$, and 2 remains, which added to 8, makes 10; then $10\div3=3$, and 1 remains, which added to 25, makes 26; and $26\div3=8$, and 2 remains; lastly, the Sum of the Quotes is $5+3+8=16$, the Integral Quote of $50\div3$; and the last Remainder 2, is the Remainder of $50\div3$. And this Work is the same in effect as if we resolve 50 into these Parts, $15(=3\times5)+9(=3\times3)+24(=3\times8.)$

Or this Truth is plain independently of the *Lemma*, because the Divisor is taken out of every Part of the Dividend as oft as possible, by carrying the Remainder of every Part forward to the next.

II. If the same Divisor D is applied to two different Dividends, whereof the greater is a Multiple of the lesser, as N and Nm; then if N contains D, a number of times q, without a Remainder; Nm will contain D, m times as oft as N does, or mq times; *i. e.* if $N\div D=q$, then $Nm\div D=mq$. Again, if $N\div D$ has a Remainder r, then Nm will contain D at least mq times with a Remainder equal to mr. And if mr is equal to, or greater than D, then, as oft as D is contained in mr, so many times oftner than mq is it contained in Nm. The Deduction of this from the *Lemma* is plain; because Nm is resolvable into $N+N+N$, *&c.* taking N as oft as m expresses; so that if $N\div D=q$, and r remains; then D is contained in Nm at least mq times, with a Remainder equal to rm. See these Examples, wherein $m=100$.

Examp.

Examp. 1.
D N q.
3) 18 (6.

Nm mq.
3) 1800 (600.

Examp. 2.
D N q.
3) 23 (7.
$21 = Dq.$
$2 = r.$

then
D Nm mq.
3) 2300 (700.
$2100 = D \times mq.$
$200 = rm.$

SCHOL. These Examples are of the Kind which we have particular Use for in Demonstrating the *Rule of Division:* And we have this further to be carefully remarked in all Examples like the second, *viz.* That tho' the Remainder (mr) in the second Part of the Example, is greater than the Divisor, whereby the Integral Quote is not so great as the number of times that D may be got in Nm, yet of the Number it wants to be added to it, (which is the Quote of mr by D,) all the Figures will fall in the Places of the o's standing on the right of that Part of the Quote which is already found, and can never be of the same local Value with any of the other Figures. So 3 is found in 2300 as many times oftner than 700 times, as the Quote of 200 by 3; yet no Figure of this Quote can rise to the Place of 100; the Reason of which, and of all such Cases, is explained in *Schol.* to *Lemma* I. For 3 being greater than 2, is not contained 100 times in 200. And which will also be true, tho' we set any other Figure in the Place of the o's that stand on the right of the Remainder, since the Remainder without these o's is less than the Divisor. See the *Scholium* referred to.

III. If N($=$A$+$B) is a Multiple of C; and if A is also a Multiple of C, then so must B be. Again, if $N = A + B + C + D$, *&c.* and if N is a Multiple of R; and also if each of the Parts of N to the last, are Multiples of R, so must that last be.

LEMMA III.

If one Number is divided by another, and the Quote again divided by the same, or any other, and every succeeding Quote again divided as long as you please or can; the last Quote will be equal to the Quote of the first Dividend by the continual Product of all these Divisors.

DEMON. 1. We shall first suppose the several Dividends are Multiples of the Divisors; and in this Case it will easily appear, as in the annex'd Example.

2) 168 (84.
3) 84 (28.
7) 28 (4.
then because
$2 \times 3 \times 7 = 42$;
therefore
42) 168 (4.

The Reason is this. If we take the last Quote and all the Divisors in a reverse Order, and multiply them continually, they must produce the first Dividend, (by what is already shewn of the mutual Proof of *Multiplication* and *Division.*) Thus, $4 \times 7 = 28$. $28 \times 3 = 84$. $84 \times 2 = 168$. But we may take these Factors in any Order, they will produce at last the same Number, (by *Lemma* II. in *Multiplication.*) And if we order them so as the last Quote in Division be the last Factor in multiplying, the Truth proposed will be manifest, thus; Because $4 \times 7 \times 3 \times 2 = 168$. Therefore also $2 \times 3 \times 7 \times 4 = 168$. But $2 \times 3 \times 7 = 42$; therefore $42 \times 4 = 168$; and $168 \div 42 = 4$. The same Reasoning will hold in all Cases, which we may represent by Universal Characters. Thus; if $A \div b = M$; and $M \div c = N$; and $N \div d = q$: then $A \div bcd = q$. For, $qd = N$; $Nc = M$; and $Mb = A$; that is, $qdcb = A$; or $bcdq = A$: But $bcdq \div bcd = q$; *i.e.* $A \div bcd = q$.

2. Suppose there is a Remainder in each Division, yet the last Quote will still be equal to the Quote of the first Dividend by the Product of all the single Divisors, tho' the Remainder will not be the same as the Remainder of the last Division. That we may see the Truth proposed in this Case, and also how to find by the several Divisors and Remainders

ders, what the Remainder would be upon dividing by the Product of the Diviſors, we ſhall conſider the *Example* annex'd; wherein the thing propoſed is proved. But to ſee the Reaſon of it, we muſt take the *reverſe Multiplication*, as in the Margin on the Right.

3) 479 (159
rem. 2

5) 159 (31
rem. 4

4) 31 (7
rem. 3

then becauſe
$3 \times 5 \times 4 = 60$
therefore
60) 479 (7
rem. 59

Thus; Take the laſt Quote and the ſeveral Diviſors, as ſo many ſimple *Factors*, and multiply them continually, taking in the correſpondent Remainders with the Product, (to make up the ſeveral Dividends.) Now the firſt thing to be ſhewn from this Multiplication is the reaſon why, That tho' the laſt Product (479) exceeds the continual Product of the *Factors* $7 \times 4 \times 5 \times 3 = 420$, (as it muſt do, becauſe of the Numbers taken in;) yet it cannot exceed it by a Number as great as the continual Product of the ſeveral *Factors* excluding the firſt 7, (*i. e.* the ſeveral Diviſors, *viz.* $4 \times 5 \times 3 = 60$;) which is thus ſhewn.

laſt quote	7	:	3, laſt rem.
laſt div[r].	4		
2[d] quote	31	:	4, 2[d] rem.
2[d] div[r].	5		
1ſt quote	159	:	2, 1ſt rem.
1ſt div[r].	3		
	479		

The Number by which the laſt Product, or Sum (479) exceeds the continual Product of all the ſimple *Factors* (420,) is plainly to be found thus; *viz.* Take the Product of the laſt Remainder (3) by the laſt Diviſor but one, (5;) then to this Product (15) add the next preceding Remainder (4,) and multiply the Sum (19) by the next preceding Diviſor (3,) and to this Product (57) add the next preceding Remainder (2,) and ſo on: for the laſt Remainder 3 is taken in with the firſt Multiplication; then it is multiply'd in the ſecond Multiplication by 5, with which the ſecond Remainder is taken in, (which makes $15 + 4 = 19$.) Then is all this multiply'd in the third Multiplication by 3, and the firſt Remainder 2 is taken in, and the whole is 59; ſo the true Remainder ſought is 59. But this muſt be always leſs than the Product of the Diviſors, becauſe in the ſeveral Multiplications the Numbers taken in are leſs than the correſpondent Multipliers, (for they are Remainders of a Diviſion wherein that Multiplier was the Diviſor;) wherefore the Product of any of the Diviſors by the following Remainder, (or Remainder of the next Diviſion,) with the preſent Remainder added, is leſs than the Product of the ſame Diviſor by the following Diviſor, and conſequently the continual Product of all the Diviſors is greater than the Product of all the Diviſors (excluding the laſt,) with the Remainders taken as directed. This being once clear, the Truth concerning the laſt Quote is manifeſt; for nothing elſe can make it different from the laſt Caſe but theſe Remainders, or Numbers added in the Multiplication, which can never make the Quote leſs: and, by what is now ſhewn, they cannot make it greater, becauſe all the Increaſe upon the laſt Dividend is leſs than the total Diviſor, (or Product of the particular Diviſors.)

As to the fractional Part of the Quote, it will be of the ſame Value, tho' not of the ſame Expreſſion. But the Truth of this will not appear ſo eaſily till we have learnt the *Doctrine of Fractions*, where you'll find it particularly explained.

DEMON-

DEMONSTRATION *of the preceding Rule of* CASE II. *of* DIVISION.

As to finding every one of the Figures of the Quote singly, as the true Quotes of the Divisor into the several Dividuals considered by themselves, we need no further Demonstration; because each of them is found by trial, and proved to be true by a certain and infallible Mark, (discovered in *Corol.* 2. to the *Definition*) before it is admitted. All that I have more to add, is to observe, That you have in *Lemma* 1. the *Reason* that the *Divisor* can never be found in the *Dividual*, (or *Partial Dividend*, as it is to be taken by the Rule) above 9 times.

The only thing then that remains to be proved is, That the several Figures of the Quote taken as one Number, according to the Order in which they are placed, is the true Quote of the total *Dividend* by the *Divisor*; and this will be easily shewn, thus: It is plain, that in this Operation we have resolved the *Dividend* into Parts, and divided them severally in the manner proposed in *Corol.* 1. *Lem.* 2. For we have first taken one Dividual, then added the Remainder of that to another Part of the *Dividend*, and after dividing this, we have added to the Remainder another Part of the *Dividend*, and so on. But we have considered the several Dividuals without regard to the Places they possess in the *Whole Dividend*, and thereby taken them in a less than their true Value; but if that Defect is supplied by placing the several Quotes, (or Parts of the total Quote) so as they have the true Value they ought to have from the complete Value of their respective Dividuals, (or Parts of the *Dividend*,) and that by so placing them they make one Number equal to the Sum of their complete Values; then is that Number the true Quote sought, (by the said *Corol.* 1. *Lem.* 2.) But thus it actually is: for the complete Value of the first Dividual is 10, or 100, *&c.* times the Value in which it is taken in the Operation, according as there is one, two, *&c.* Figures standing before it: Also its quote Figure standing first on the Left-hand, there are as many Figures of other Quotes set before it, as the Number of remaining Figures in the *Dividend*; because for each of these there is, by the Rule, a Figure put in the Quote: therefore this first quote Figure receives by its Place, a Value 10, or 100, or 1000, *&c.* times its simple Value, according as there are one, or two, or three, *&c.* Figures before it, corresponding to the true Value of the Dividuals, (as ought to be done by *Corol.* 2. *Lem.* 2.) Therefore this first quote Figure taken in its complete Value from the place it stands in, is the true Quote of the Divisor in the complete Value of the first Dividual. For the same reason, all the rest of the Figures in the Quote taken according to their Places, are each the true Quote of the Divisor in the complete Value of their Dividuals; because as the first Figure on the right of each succeeding Dividual, is one Place more to the right of the preceding, (or has one Figure fewer standing before it,) so ought their Quotes to have; and so are they also ordered: Consequently taking all the quote Figures in order as they are placed by the Rule, they make one Number, which is equal to the Sum of the true Quotes of the several Parts of the *Dividend*; which is therefore the true Quote of that whole *Dividend*.

To leave no Obscurity in this Demonstration, I shall illustrate it by two Examples. In which I shall set down the Dividuals, Quotes, and Remainders according to their true Values.

Examp

Examp. I.

Diviſor. Dividend.

36)	85609	Quotes.
1ſt dividual	85000 (	2000
36 × 2000 =	72000	
1ſt remain.	13000	
add	600	
2ᵈ dividual	13600 (	300
36 × 300 =	10800	
2ᵈ remain.	2800	
add	00	
3ᵈ dividual	2800 (	70
36 × 70 =	2520	
3ᵈ remain.	280	
add	9	
4ᵗʰ dividual	289 (	8
36 × 8 =	288	2378 Sum
laſt remain.	1	of the Quotes.

In the firſt Example, the Dividend 85609 is reſolved into theſe Parts, *viz.* 8500 + 600 + 00 + 9. For tho' the firſt Dividual is conſidered as 85, yet it is truly 85000; and therefore its Quote inſtead of 2, is 2000, and the Remainder 13000; and ſo of the reſt, as you ſee in the Operation. But if we take the Multiples of the Diviſor by the ſeveral Quotes, with the laſt Remainder, and conſider the Dividend as diſtributed into theſe Parts, (which are here 72000 + 10800 + 2520 + 288 + 1,) then the Work is reduced to the Conditions of ***Lemma*** **2.** *Article* 2.

Examp. 2.

Diviſor. Dividend.

465)	2744897	Quotes.
1ſt dividual	2744000 (	5000
465 × 5000 =	2325000	
1ſt rem.	419000	
add	800	
2ᵈ dividual	419800 (	900
465 × 900 =	418500	
2ᵈ rem.	1300	
add	97	
3ᵈ dividual	1397 (	3
465 × 7 =	1395	5903
laſt rem.	2	

In this ſecond Example, when we have got the ſecond Quote, the Remainder is 1300; then we add the two next Figures of the Dividend, becauſe the Figure of the Quote muſt be of the ſame local Value as the laſt of theſe Figures: For ſince 465 is not contained in 139, it is not contained 10 times in 1390; and ſo the next Figure in the Quote after 9 muſt be 0, and the ſignificant Figure of the Quote of 1390 ÷ 465, muſt be in the ſecond Place after 9, *i. e.* in the preſent Example, in the Place of Units: and therefore we take in alſo the Figure 7, which is in the Place of Units, to find at once all that Part of the Quote which belongs to the Place of Units; for had we divided 13900 by 465, the Quote is 2, and the Remainder 460; to which adding the laſt Figure of the Dividend 7, the Sum is 467, in which the Diviſor is contained once, and 2 remains; and ſo theſe two Quotes both in the place of Units, *viz.* 2 + 1 make 3, which is more conveniently found, as in the Operation in the Margin. The like reaſon you'll find in all Caſes where there are 0's in the Quote. And for the laſt Example, take 113764 ÷ 28 = 4063, the firſt Dividual is 113000, the Quote 4000, and the Remainder 1000; to which if we add 700, the Sum 1700 does not contain the Diviſor 28 ſuch a number of times as can fall in the Place of Hundreds; therefore we take in another Figure, which makes 1760; and the Quote 6 falls in the ſecond Place after the preceding Quote 4. The laſt Figure of the Quote is 3.

§. II. Particular Rules *for contracting the Work of Division in certain Cases: And, for managing it with more Certainty, tho' with more Work, in all Cases.*

CASE 1. When the Divisor is a Digit, the multiplying of the Divisor and Quotes, and also the Subtraction of the Products from the Dividuals, may be easily performed without writing down any thing but the Remainders; and these, with the Quotes, set more conveniently, as in this *Example*; wherein 37546 is divided by 4. Thus, 4 in 37 is 9 times, and 1 remains; the Quote 9 I set under the Dividend, and the remainder 1 above; then this Remainder, with the next Figure of the Dividend prefix'd, makes 15; the next Dividual, in which 4 is contained 3 times, and 3 remains; the Quote 3 is set after the preceding Quote, and the Remainder 3 over the 5 of the Dividend. Then is the next Dividual 34, whose Quote is 8, and 2 remains; then the last Dividual is 26, and the corresponding Quote is 6, and 2 remains. *Again*, in this *Case* it will be very easy to do the Work without writing down the Remainders, only conceiving them, in the places where they ought to be. And the Conveniency of doing it this way, you'll see in *Case* 3. Observe also, that you may easily use the same Practice, if the Divisor is 11 or 12.

$$\begin{array}{r} 1322\phantom{\tfrac{2}{4}} \\ 4)37546\phantom{\tfrac{2}{4}} \\ 9386\tfrac{2}{4} \end{array}$$

CASE 2. If the Divisor has o's in the first Places next the Right-hand, take no notice of them in the Operation, making the Divisor only the remaining Figures on the Left; and exclude as many Figures, whatever they are, from the Right of the Dividend, as those o's of the Divisor; making the remaining Figures on the Left the Dividend. Having finished this Division, you have found the integral Quote sought: And for the fractional Part, to the Remainder of the Division prefix all the Figures excluded from the Dividend; that is the true Remainder that would happen if the Division were done by the common Method: This Remainder, with the given Divisor, makes the Fraction. But observe, that if there are any o's standing clear on the Right-hand of this Remainder, you may omit them all (in making your Fraction) if they do not exceed the Number of o's excluded from the Divisor; or as many of them, as are equal in Number to those in the Divisor; and omitting the same Number of o's in the Divisor, of the remaining Figures on the Left make your Fraction.

Example 1. To divide 84700 by 4600, I divide 847 by 46, the Quote is 18, and 19 remains; but the true Remainder is 1900: Making this Fraction $\frac{1900}{4600}$, which is equivalent to this $\frac{19}{46}$.

Ex. 2. To divide 3640 by 800, I divide 364 by 8, the Quote is 45, and 4 remains; but the true Remainder is 420: Making this Fraction $\frac{420}{800}$, equal to this $\frac{42}{80}$.

Examp. 3. To divide 68704 by 2400, I divide 687 by 24; the Quote is 28, and 15 remains; but the true Remainder in the Question is, 1504, and the Fraction is $\frac{1504}{2400}$.

Examp. 4. To divide 367854 by 800, I divide 3678 by 8; the Quote is 459, and the Remainder 6; but the true Remainder is 654, and the Fraction $\frac{654}{800}$.

The *Reason* of this Rule is contained in *Lem.* 4. Thus, if an equal Number of o's are excluded from Divisor and Dividend, the remaining Figures on the left express like aliquot Parts of them, *viz*, a tenth Part if one o; a hundredth, if two o's were excluded, and so on. But like aliquot Parts of two Numbers make the same Quote as their Wholes, (by *Lem.* 4.) *Ex.* 46 and 847 are the hundredth Parts of 4600 and 84700, and so have the same Quote 18, with a Remainder 19, which is the hundredth Part of the Remainder in dividing 84700 by 4600; and therefore the two o's cut off from the Dividend are to be prefix'd to it, to give it the true Value. *Again*, if the Figures excluded the Dividend are not all o's, yet if we suppose them so, the Quote is right: And that it cannot be increased by the Value of the Figures cut off, whatever they are, is plain; be-

cause being prefix'd to the Remainder found by the Rule, they must make a less Number than the given Divisor; since that Remainder is less than the Divisor without its o's, which are as many in number as these Figures prefix'd to the Remainder, which therefore can never make up the Defect. So in *Ex.* 4. when 3678 is divided by 8, the Remainder is 6, which being less than 8, no Figures prefix'd to it can make a Number equal to 8, with as many o's prefix'd. But these Figures being a part of the Dividend, belong to the Remainder; which, instead of 600, as it would have been had the Figures cut off been o's, is 654.

Tho' this is the proper Demonstration of this Rule, yet you may be sufficiently satisfied of the Justness of it, by considering any Example wrought at length; wherein it will easily appear, that by excluding the o's in the Divisor, and as many Figures in the Dividend, we only save the Trouble of writing many superfluous Figures, and yet bring out the same Quote. As in these *Examples.*

```
4600)84700(18        800)367854(459
     4600                3200
     ----                ----
     38700                4785
     36800                4000
     -----                ----
      1900                 7854
                           7200
                           ----
                            654
```

As to that part of the Rule for contracting the Fraction, you'll find the Reason of it explain'd in Book II.

CASE 3. If the Divisor is the Product of two or more Digits, and that you can easily discover these Digits; then divide first by any one of these, and the Quote by any other, and so on: the last Quote is that sought. And for the Fraction, multiply the Divisor by the last Quote, and take the Product from the Dividend, you have the Remainder, which would happen by dividing after the common way. Or find it thus; Multiply the last Remainder (of the Work) by the preceding Divisor (or the last but one) and to the Product add the preceding Remainder; this Sum multiply by the next preceding Divisor; and to the Product add the next preceding Remainder, and so on, till you have gone thro' all the Divisors and Remainders to the first. But when there are no Remainders in any of the particular Divisions, the Dividend is a Multiple of the Divisor.

Ex. 1. To divide 9048 by 24, I divide by 4 and 6, because $4 \times 6 = 24$. Thus; $9048 \div 4 = 2262$, then $2262 \div 6 = 377$.

Ex. 2. To divide 754683 by 42, I divide by 6 and 7, because $6 \times 7 = 42$, as in the Margin; wherein the first Quote is 125780, and 3 remains, which I have set over a Line after the Quote; then the second Quote is 17968, and 4 remains, which multiplied into the preceding Divisor 6, produces 24, to which the first Remainder 3 being added, makes 27; so that the true Remainder is 27, and the fractional part of the Quote $\frac{27}{42}$.

$$\begin{array}{c|l} 6 & 754683 \\ 7 & 125780\frac{3}{} \\ & 17968\frac{27}{42} \end{array}$$

Ex. 3. To divide 18472 by 32, I divide by 4 and 8, because $4 \times 8 = 32$; the first Quote is 4618, and nothing remains; the second Quote is 577, and 2 remains, which multiplied into 4, produces 8, the true Remainder. But in this Case, where there is no Remainder in the first Step, the Fraction may be made of the Remainder, and Divisor of the second Step. So here it may be $\frac{2}{8}$.

$$\begin{array}{c|l} 4 & 18472 \\ 8 & 4618 \\ & 577\frac{8}{32} \end{array}$$

Ex. 4. To divide 48767 by 15, I divide by 3 and 5, because $3 \times 5 = 15$: the first Quote is 16255, and 2 remains; the second Quote is 3251, and nothing remains; wherefore there is no Product to be added to the first Remainder: and so that is the true Remainder, and the Fraction is $\frac{2}{15}$.

$$\begin{array}{c|l} 3 & 48767 \\ 5 & 16255\frac{2}{} \\ & 3251\frac{2}{15} \end{array}$$

Ex.

Ex. 5. To divide 3428689 by 126, I divide by 3, 6, 7, because $3\times6\times7=126$: the first Quote is 1142896, and the Remainder 1; the second Quote is 190482, and the Remainder 4; the last Quote is 27211, and the Remainder 5, which multiplied into the preceding Divisor 6, produces 30; to which the preceding Remainder 4 added, makes 34; which multiplied by the preceding and first Divisor 3, produces 102; and the Remainder 1 added, makes 103 the true Remainder, and the Fraction is $\frac{103}{126}$.

$$\begin{array}{r|r} 3 & 3428689 \\ 6 & 1142896\,\frac{1}{} \\ 7 & 190482\,\frac{4}{} \\ & 27211\frac{103}{126} \end{array}$$

The *Reason* of this Rule is explained in *Lem* 3. and as to the Contraction of the Fraction in *Ex.* 3. you'll learn the reason of it in Book II.

This Practice is of very good use, especially where the Divisor is the Product of two Digits; because when it is so, they are easily discovered: and the use of it you'll find more remarkably in the next Chap. §. 5. Observe also, that if the Factors of the Divisor are 11 or 12, it's easy to divide by them as by a Digit. Thus to divide by 144, chuse 12, 12, because $12\times12=144$. For 33 take 3, 11. For 84, take 7, 12.

CASE 4. One who is tolerably acquainted with the Practice of Division, according to the preceding general Rule, may contract the Work, by omiting to write down the Product of every Figure of the Quote by the Divisor; doing it in mind, and gradually as the Product is made, subtracting it from the corresponding Figures of the Dividual; setting the Remainder either above or below the Dividend, in the manner of the following Examples: For it's no matter whether the Figures of any Remainder or Dividual stand all in one Line, if they are duly situated with respect to one another, from Left to Right-hand. Also, instead of setting the Quote on the Right-hand of the Dividend, it may stand as conveniently under or over the Dividend.

Ex. 1. $72849\div46=1583$, and 31 remains.

That you may perceive the manner of working without Confusion, I shall represent it as it appears separately at every Step.

```
  26             3            31           313
46)72849        268          2686         26861
    1         46)72849     46)72849     46)72849
                 15           158          1583 31/46 Quote.
```

The first Dividual is 72, the Quote 1, and Remainder 26. The second Dividual is 268, the Quote 5, and Remainder 38; which is found gradually; Thus, $5\times6=30$; then 0 (the first Figure of the Product) from 8 (the first Figure of the Dividual) remains 8. Again $5\times4=20$, and 3 (the Number of 10's carried from the last Product) is 23: then 23 from 26 (of the Dividual) remains 3; whence the next Dividual is 384; the Quote 8, and Remainder 16; Found thus, $8\times6=48$, then 8 (of the Product) from 4 (of the Dividual) cannot be taken, but from 14, and 6 remains: Again $8\times4=32$, and 4 (from the last Product) is 36; then 36 from 37 (instead of 38 of the Dividual, because 1 was taken from the 8 in the last Step to make 14) leaves 1. Or it comes to the same thing, if to the Product we add 1, for the 10 that was borrowed in the last step; so the Product 36 and 1 (borrowed) is 37; which taken from 38, there remains 1. The next Dividual is 169, the Quote 3, and Remainder 31.

By this Example you may understand how to do, or examine others. See the following.

```
               33                          3                        23
              4304     Or it may          304                      2527
Ex. 2.  68)24786       stand thus.  68)24786                      445545
     Quote.   364 34/68                   364 34/68     Ex. 3. 467)3247893
                                                          Quote. 6954 375/467
```


Observe,

Obſerve, When in any Step there is 10 borrowed in the Subtraction, then the Remainder will be always greater than the Subtrahend, (becauſe if any Digit is taken from the Sum of 10 and a leſſer Digit, the Remainder muſt be greater than that leſſer Digit.) And when you come to the next Step, look back upon the laſt Remainder, its being greater than the Subtractor over which it ſtands, is a certain ſign that 10 was borrowed in the laſt Step, and conſequently that 1 for that 10 is to be taken from the Subtractor, or added to the Product in this Step: And this Obſervation is very uſeful, becauſe we are apteſt to forget this 1. So in *Ex.* 3. the laſt Dividual is 2243, the Quote 4, and Remainder 375: Thus found, $4\times7=28$; then 8 from 13 leaves 5. Again, $4\times6=24$, and 2 (carried from the laſt Product) is 26, and 1 (borrowed, becauſe 5 is greater than 3 over which it ſtands) makes 27; then 7 from 14 leaves 7. Again $4\times4=16$, and 2 (carried from the laſt Product) and 1 (borrowed) makes 19, which taken from 22, leaves 3.

In the common way of practiſing this Method, they daſh a Line thro' every Figure of the Dividual, gradually as the Figures of the Remainder are ſet over them, in order to prevent Confuſion; becauſe the next Dividual appears the more diſtinctly from the Figures of the preceding Dividuals that are thus cancelled. As in this *Example*, repreſented in all its Steps ſeparately.

```
   9          2           21
24)5726       90          904
   2      24)5726    24)5726
              23          238 14/24
```

Obſerve, As it is to the ſame Purpoſe whether we write the Remainders over or under the Dividend, ſo the Method of placing them has a little Confuſion in it, which is helped by daſhing the Figures. But I think it a more convenient way to ſet the Remainders under the Dividuals, and ſo as the Figures of the ſame Remainder be in a Line; leaving the next Figure of the Dividend, which makes up the Dividual, where it ſtands, and ſetting the Quote over the Dividend, as in theſe *Examples*.

```
   238  Quote.
24)5726 Dividend.
   9  }
   20 } Remainders.
    14}
```

```
    364  Quote.
68)24786 Dividend.
   43  }
    30 } Remainders.
     34}
```

I ſhall recommend this Method of Diviſion only to ſuch as, by Practice, have acquired a Habit of cloſe and careful Attention; otherwiſe 'tis too difficult. But there are ſome particular Circumſtances wherein 'tis very eaſy and convenient, as in the two following *Caſes*.

CASE 5. If the Diviſor has the Figure 9 in all its Places, as 9, 99, 999, *&c.* then the Quote is either 1, when the Dividual is equal to the Diviſor; or if the Dividual has one place more than the Diviſor, the Quote is either the firſt Figure on the Left of the Dividual, or the next greater Figure. Thus, if the firſt Figure added to the remaining Figures (taken as one Number) makes a Sum leſs than the Diviſor, that firſt Figure is the Quote, and that Sum is the Remainder, after the Product of the Diviſor and Quote is taken out of the Dividual. But if that Sum is equal to, or greater than the Diviſor, the Difference is the Remainder, and the Quote is the Figure next greater than the firſt Figure of the Dividual.

Thus the Quote and Remainder being eaſily found, (without the Product) we may chuſe the Method of the preceding *Caſe* for placing them; and we may alſo make it ſhorter, by not writing down ſuch Figures of the Remainder which are the ſame with their Correſpondents in the Dividual, but leaving them where they ſtand.

Ex.

Ex. 1.
$$\begin{array}{r} 35 \\ 72428 \\ 99)468793 \\ \text{Quote. } 4735\tfrac{28}{99} \end{array}$$

Here the firſt Dividual is 468, the Quote 4, and the Remainder 72 (=68+4.) The next Dividual is 727, the Quote 7, and Remainder 34 (=27+7.) The next Dividual is 349, the Quote 3, and Remainder 52 (=49+3.) The next Dividual is 523, the Quote 5, and Remainder 28.

Ex. 2.
$$\begin{array}{r} 0 \\ 92\ \ 4 \\ 999)67934582 \\ \text{Quote. } 68002\tfrac{584}{999} \end{array}$$

Here the firſt Dividual is 6793, the Quote 6, and Remainder 799 (=793+6.) The next Dividual is 7994, the Quote is 8 (=7+1,) and the Remainder 2, (becauſe 994+7=999+2.) The next Dividual 25, and the next again 258, being each leſs than the Diviſor, the Quote Figures are 0; the laſt Dividual is 2582, the Quote 2, and Remainder 584 (=582+2.)

By theſe Examples you may eaſily do or examine any other; ſee the following.

DEMONST. The thing to be demonſtrated in this *Rule*, is the way of finding the Quote and Remainder when the Dividual has one place more than the Diviſor, which, upon a little Attention, will be very obvious. For 99 wants 1 of 100, (and any Number expreſſed thus by 9's, wants 1 of a Number expreſſed by 1, and as many 0's.) And however oft 100 is contained in any Number of the ſame Number of places, (which is always as oft as the firſt Figure on the Left expreſſes, the remaining Figures on the Right being the Remainder; ſo $462 \div 100 = 4$, and 62 remains) ſo oft at leaſt muſt 99 be contained in it: and the Remainder will be the Sum of the Remainder when divided by 100, and of the Quote or firſt Figure; which is the Remainder when 99 is taken out of any Number of Hundreds. Thus, $468 \div 99 = 4$, and 72 remains, *viz.* $68+4$; for in $400 \div 99$, the Remainder is 4. And therefore, ſince $4+68$ is leſs than 99, it muſt be the true Remainder in dividing 468 by 99. *Again*, $697 \div 99 = 7$ and 4 remains; for 99 is contained in 600 ſix times, and 6 remains; then $6+97=99+4$, in which 99 is contained once, and 4 remains: Wherefore 99 is contained in 697, 7 times, and 4 remains. The ſame Reaſoning holds in all Caſes.

Examp. 3.
$$\begin{array}{r} 73 \\ 7622 \\ 999)746985 \\ \text{Quote. } 747\tfrac{732}{999} \end{array}$$

Examp. 4.
$$\begin{array}{r} 3\ \ 9 \\ 99)39976 \\ \text{Quote. } 403\tfrac{79}{99} \end{array}$$

Examp. 5.
$$\begin{array}{r} 92\ \ 4 \\ 999)67934582 \\ \text{Quote. } 68002\tfrac{584}{999} \end{array}$$

Examp. 6.
$$\begin{array}{r} 71 \\ 99)99467 \\ \text{Quote. } 1004\tfrac{71}{99} \end{array}$$

Examp. 7.
$$\begin{array}{r} 9\ \ \ 61 \\ 99)6039457 \\ \text{Quote. } 61004\tfrac{61}{99} \end{array}$$

Examp. 8.
$$\begin{array}{r} 9 \\ 99)40596 \\ \text{Quote. } 410\tfrac{6}{99} \end{array}$$

Examp. 9.
$$\begin{array}{r} 809 \\ 99)67419 \\ \text{Quote. } 681 \end{array}$$

The Uſe and Conveniency of this *Caſe*, you'll find chiefly in *Chap.* 4. *Book* 5. where ſuch Diviſors frequently occur.

CASE 6. If the Diviſor has any other Figure than 9 in all its Places, as 44, 666, *&c.* doing the Work in the manner of *Caſe* 4, will be eaſier than if the Figures were all different: But it will be ſtill eaſier with a little more Work. Thus, divide by 11 or 111, *&c.* with as many Places as the Diviſor has; and here the Quote is eaſily ſeen, for it is either the firſt Figure on the Left of the Dividual, or the next leſſer Number; or 0, if the Dividual has the ſame Number of Places as the Diviſor. But if the Dividual has one Place more than the Diviſor, the Quote is 9. Then the Product of the Diviſor and Quote having the ſame Quote Figure in all its Places, the Remainder is eaſily found without writing down the Product, and ſo may be diſpoſed in the manner of *Caſe* 4. Then, laſtly, divide this Quote by that Digit of which the Diviſor conſiſts, by *Caſe* 1.

Ex.

Examp. 1. 44)562387.

Thus.

11261
11)562387
4) 51126 $\frac{1}{\ }$
12781 $\frac{23}{44}$

Examp. 2. 777)4672869

Thus,

1 10
230872
111)4672869
4) 42097 $\frac{102}{\ }$
10524 $\frac{213}{777}$

A General Method *to make* DIVISION *certain and easy.*

The Work of *Division* may be made more certain and easy with the writing a few more Figures; by making a Table of the Products of the Divisor by all the Digits; as was done in §. 2. of the preceding *Chap.* for Multiplication. To be used thus; Seek the Dividual, or the next lesser Number, in the Table against it is the Quote Figure, and that Number itself is the Product of the Divisor and Quote: By this means the Work may be done as fast as Figures can be written. Nor need the Products be copied out of the Table, but taken where they stand. The Remainders may be found and written under the Dividual: And here also we may chuse to write the Dividuals and Remainders under the Dividend, and without bringing down the Figures of it to the Remainders, taking them where they stand, as has been shewn in *Case* 4.

Example.

1	483	)35268749(72560 $\frac{69}{483}$
2	986	3401
3	1469	1258
4	1952	986
5	2435	2727
6	2918	2435
7	3401	2924
8	3884	2918
9	4367	69

Here the first Dividual is 3526; the nearest Number to it in the Table is 3401, against 7, which is therefore the Quote, and 3401 is the Product. The next Dividual is 1258, whose next Number in the Table is 986, *&c.*

But the Work may be made without writing the Products out of the Table; and it will stand thus.

1	483	)35268749(72560 $\frac{69}{483}$
2	986	125
3	1469	272
4	1952	292
5	2435	6
6	2918	
7	3401	
8	3884	
9	4367	

Scholium. As to this Method of Practice, the Readiness with which the Quote and Product is found, may be reckoned a sufficient Balance for the Time and Trouble of making the Table, in all Cases where the Divisor exceeds two Places, and the Dividend is so large a Number that the Quote will consist of several Figures; and especially too, if we subtract the Products as they stand in the Table, without writing them under the Dividual: And this part will still be more easy, if we write the Table upon a separate bit of Paper, so as we can place each Product under the Dividual.

And, lastly, in such Questions and Calculations where the same Number must be frequently used as a Divisor, this Method of a Table, especially a moveable one, is most convenient, as you'll find afterwards.

NEPER's Rods may also be used for making the Table of the Divisor.

Of the PROOFS *of* DIVISION.

1. We have already explained one *Proof* of *Division* by *Multiplication*, which is this; *viz.* The Product of the Divisor and Integral Quote added to the Remainder, is equal to the Dividend.

2. But it may be also proved by *Division*. For the *Dividend* being divided by the Integral Quote, the *Quote* of this *Division* will be equal to the former *Divisor*, with the same *Remainder*. Thus, 3 is contained 4 times in 14, and 2 remains: But 4 times $3 = 3$ times 4; therefore 4 must be contained 3 times in 14, with the same Remainder 2; as it actually is. The same Reason is good in all Cases.

3. Lastly, *Division* may be proved by casting out the 9's. Thus; Subtract the *Remainder* out of the *Dividend*, what remains here ought to be the Product of the *Divisor* and *Quote*; which you may prove by casting out the 9's, as was done in *Multiplication*.

CHAP. VII.

Of APPLICATE NUMBERS.

Explaining the preceding Fundamental Operations, as they are Applicable to Questions *about* Particular Things, *with their Circumstances in Human Affairs.*

§. 1. *Of* Applicate Numbers; *and their Distinction of* Simple *and* Mix'd.

With TABLES *of the Variety of* COINS, WEIGHTS, *and* MEASURES *of* GREAT BRITAIN.

WHY Numbers are called *Applicate*, we have learned already. But now we are to consider, That for the Use of Society, it was necessary that certain greater Quantities should be subdivided into other lesser ones; and these again into others lesser; each having its distinct and proper Denomination; but all considered as subordinate Species of the greater; in order to the giving and receiving more or less of any Goods or Commodity, as occasion should require. As for Example, One Pound (of Money) is divided into 20 Shillings, one Shilling into 12 Pence, and one Penny into 4 Farthings. These several lesser Quantities, as they have distinct Denominations, are as really *Integers* of their own kind as the greater, of which they are a Part; and a Number of each Species considered by itself, is called a *simple Number*; as 48 Pounds, or 56 Shillings, *&c.* But when we take together a Number of several Species, taking still less of each inferiour Species than what makes an *Unit* of the higher, and considering it as a Part of that *Unit*, this makes a *mix'd Number*. For Example; 48 *l.* 14 *s.* 9 *d.* 2 *f.* is one mix'd Number. Again consider, that as the Numbers of the inferiour Species that make a mix'd Number are less than an Unit of the superiour, and have always a known and cer-

certain Relation to them, (as 1 Shilling is a 20th Part of a Pound;) so they are in effect *Fractions*, and being summed up with a regard to that relation (as we shall learn) they are truly considered as *Fractions*, (or Numbers related to one another) in the Operation. But the relative Denomination (which in every Fraction is a Number) being suppress'd and understood, [tho' considered in the Operation] and each Species distinguished by proper Denominations, which of themselves express no such Relation; they are all considered as *Whole Numbers:* as indeed each Species is in the most strict and proper sense, considered as an Integer with respect to the lower. So that each of them, except the highest and lowest, is considered both as a *Whole Number* and a *Fraction*.

This Account of the Nature of *mix'd Numbers* might perhaps be sufficient; yet there are several Reflections, useful to such as would have complete Notions of Things, concerning the Nature of the various Kinds of *mix'd Numbers*, that may be very proper in this Place: for tho' it may be thought a Digression from the business of *Arithmetical Operations*, yet it can never be impertinent to make useful Reflections upon the Subject of these Operations.

The Nature and Design of Society has introduced among Men a Necessity of exchanging such things as are the Product of their different Applications and Labour; for every Country does not produce, nor every Man apply himself to every thing: Now, whatever way this Exchange is made; whether things are valued by their real Use, or by Fancy, there is always some Equality supposed, or made by agreement of Parties, betwixt certain Quantities of one thing and another. And that Commerce might be regular and certain, it was necessary to constitute some fixed and standard Quantities under certain and constant Names; otherwise Men could never be able to treat about these Exchanges unless they were together, and the Subjects were immediately before them; and even then not without great Inconveniency. Again, because Men need less and more on different occasions, it was necessary there should be various Quantitys of every Kind, which differing only as *less* or *more*, it was convenient that each (or several) of the greater should contain a precise Number of the lesser as distinct and certain Parts of them, whereby subordinate Quantitys coming under one general Name, constitute one kind of *mix'd Number*; the several Parts or Denominations of which we call the several Species of that Kind.

The more common Subjects of these *mix'd Numbers* are the external sensible Objects which we see and feel, and which in general we call Bodies.

Now the Quantity of Bodies can only be considered in two respects, either as to their *Bulk*, *i. e.* the external Measure of *Length*, *Breadth*, and *Thickness*; or their *Weight*: And to compare different Bodies in these respects, there must be certain common standard *Measures* and *Weights* to which all others are compared. In some Bodies only one Dimension, which we call *Length*, is considered; because the other two are either inconsiderable in themselves; or rather, because in a comparison of more and less of these things, the other Dimensions are equal, (or supposed to be so;) and the Dimension chosen is that which admits of most Degrees: so that the more and less are here according to the *Length*. Hence proceed what we call the *Measures of Length*. *See the following Tables of mix'd Numbers.* Again, in some, Length and Breadth are both considered; and from this proceed the *Superficial* or *Square* Measures. Others are measured in all their three Dimensions; hence the *Solid* and *Cubical* Measures: under which may be comprehended what we call the *Measures* of *Capacity*, by which are measured the Quantity of *Liquids*, and of all such things as consisting of small Particles either altogether distinct, or cohering very loosely, cannot be measured singly, nor given out by Number; but are considered according to the *Bulk* they make, when being laid together they appear as one continuous Body; for Example, Corns, and Meal, or any solid Body reduced to Powder.

Again, other things are more conveniently measured by their *Weight*.

In the next place we must *observe*, That some things are exchanged by Number, the Individuals (which must all be of one Species of things) being really separated and distinct;

ſtinct; and which having neither ſuperiour nor inferiour Species, are not valued by Weight or Meaſure; or one of them being ſo valued, the reſt are ſuppoſed to be equal; or the things are ſuch, as cannot be weighed or meaſured. In this manner are Cattle, and innumerable other things bought and ſold by *Tale*, (in the common Phraſe;) and for theſe there is no diſtinct Order of *mix'd Numbers*. But there is alſo a kind of *mix'd Numbers* conſtituted for ſome things that are exchanged by Tale; the Species of which are called *Groſs* and *Dozen*, *&c.* Now here the Species are not any real continued Quantitys, but certain Numbers diſtinguiſhed by particular Denominations, which therefore require no Standards, but to have a true Idea of the Number they are affixed to. Whereas in other things, the ſeveral Denominations give us immediately the Idea of ſome continued Quantity; and we apply Number to them only by an arbitrary Subdiviſion into Parts: So that we may conclude, that *mix'd Numbers* ariſe more generally and properly from the imagined Parts of continued Quantitys, either *Solids*, *Superficies*, or *Lines*.

As to COINS, *obſerve*, That they are properly meaſured by Weight: But the Weight being aſcertained by publick legal Marks, their Species have not the proper Denomination of Weight; and therefore we don't ordinarily talk of them as things *weighed*; yet when there is any ſuſpicion of falſe Weights, they are compared to ſome ſtandard Weights.

As Numbers are not only applied to Bodies, and their imagined Parts, but alſo to every thing that is capable of *more* and *leſs*; as to the conceivable Parts of Time; ſo we have alſo from this laſt a particular kind of *mix'd Number*.

Again *obſerve*, That for different things we have different kinds of Weights and Meaſures; ſo we have *Troy Weight* and *Averdupoiſe Weight*, *&c.* We have different Meaſures for *Corn*, *Beer*, *Wine*, *&c.* Wherein theſe different *Weights* and *Meaſures* coincide and agree, or what the Relation betwixt them is, and by what means their Standards were firſt ſettled, is not ſo ſtrictly the buſineſs of this Work to conſider. The Statutes explain and determine theſe things; and perhaps Cuſtom only is the Foundation of ſome of them.

The laſt thing I ſhall obſerve upon this Subject, is, That of the Denominations of *Coins*, *Weights*, and *Meaſures*, ſome are merely imaginary, *i. e.* are not Names of any one real diſtinct Quantity, but of ſome poſſible Quantity ſuppoſed equal to a certain Number, or a certain Part of ſome real ſtandard Quantity. So for Example, a *Pound of Money* is an imaginary Quantity equal to 20 Shillings. A *Laſt* is an imaginary Quantity equal to 12 Barrels. And again, you muſt *obſerve*, that there are many more Denominations known, and uſed upon different Occaſions in treating or ſpeaking of theſe things, than are convenient or ordinarily uſed in keeping Accounts. In the following *Tables* I ſhall give you a full account of all the Kinds and Denominations that are commonly known; and diſtinguiſh thoſe that are uſed in keeping Accounts.

TABLES *of the moſt common* COINS, WEIGHTS, *and* MEASURES, [*Real and Imaginary*] *of* GREAT BRITAIN.

Engliſh Money.

4 Farthings	=	1 Penny.
4 Pence	=	1 Groat.
6 Pence	=	1 Teſter.
12 Pence	=	1 Shilling.
5 Shillings	=	1 Crown.
6 Shillings + 8 Pence	=	1 Noble.
10 Shillings	=	1 Angel.
13 Shillings + 4 Pence	=	1 Mark.
20 Shillings	=	1 Pound.

The *Real Coins* now Current and commonly known, are theſe:

1. Of Copper-Money; a Farthing, and a Halfpenny.

2. Of Silver-Money; a Penny, Two-pence, Four-pence, Six-pence, a Shilling, Half a Crown, a Crown.

3. Of Gold-Money; Half a Guinea = 10 Shillings + 6 Pence; a Guinea = 21 Shillings, or 1 Pound + 1 Shilling.

There are many other Gold-Coins, and ſome Silver, but not very common.

Accounts are kept in Pounds, Shillings, Pence, and Farthings; which are marked by these Characters, $l : s : d : q$ or f. Whose Relations I mark over them, thus; $\overset{20}{l} : \overset{12}{s} : \overset{4}{d} : f$. *i. e.* $4f = 1d$. $12d = 1s$. $20s = 1l$.

Observe, In *Scotland* they use the same Denominations, except Farthings, and 1 Pound *English* = 12 Pound *Scotch*. But they begin now to use *English* Money in their Accounts.

English Weights.

Troy Weight.

24 Grains		1 Penny-weight.
20 Penny-weight	=	1 Ounce.
12 Ounces		1 Pound.

Accounts are kept in the same Denominations, marked thus:

$\overset{12}{\text{lb}} : \overset{20}{\text{oz}} : \overset{24}{\text{dw}} : \text{gr.}$

Apothecary's Weight.

20 Grains		1 Scruple.
3 Scruples	=	1 Dram.
8 Drams		1 Ounce.
12 Ounces		1 Pound.

Marked thus.

$\overset{12}{\text{lb}} : \overset{8}{℥} : \overset{3}{ʒ} : \overset{20}{℈} : \text{gr.}$

Averdupoise Weight.

4 Quarters of a Dram		1 Dram.
16 Drams		1 Ounce.
16 Ounces		1 Pound.
14 Pound	=	1 Stone.
28 Pound		1 Quart. of a Hund.
4 Quarters of a Hundred		1 Hundred Weight.
20 Hundred		1 Tun.

In keeping Accounts, this Weight is subdivided into two Kinds, called *Averdupoise Weight, the Greater*, and *the Lesser*.

The Greater comprehends these Denominations, Tun. Hundred Weight. Quarter. Pound. Marked thus; $\text{T.} \overset{20}{\text{C.}} \overset{4}{\text{Qr.}} \overset{28}{\text{lb.}}$ But the last 3, *viz.* C. Qr. lb. are sufficient.

The lesser comprehends these; Stone. Pound. Ounce. Dram. Quarter. Marked thus: $\overset{14}{\text{St.}} \overset{16}{\text{lb.}} \overset{16}{\text{oz.}} \overset{4}{\text{dr.}} \text{qr.}$

Observe, In *Scotland* the Stone is commonly reckoned 16 Pounds.

The *Original* of all Weights in *England* was a *Corn* of Wheat taken out of the Middle of the Ear, and well dried; of which 32 made one Penny-weight; instead of which, they made afterwards another Division of the Penny-weight into 24 Grains. Mr. *Ward* (in his *Young Mathematician's Guide*,) cites a Statute of *Edward* III. by which there ought to be no Weight used but *Troy*. But Custom, says he, afterwards prevailed in giving larger Weight to coarse and drossy Commodities, and thereby introduced the Weight called *Averdupoise*, And as to the Proportion betwixt *Troy* and *Averdupoise* Weight, he says, That by a very nice Experiment he found that 1 Pound *Averdupoise* is equal to 14 Ounces, 11 Penny-weight, 15 and $\frac{1}{2}$ Grains *Troy*. So that neither the Ounce nor Pound are the same.

By *Troy* Weight are weighed *Jewels*, *Gold*, *Silver*, and *Bread*.

By *Averdupoise* Weight are weighed all *Grocery* Wares.

The Apothecary's Pound is *Troy* Weight: but instead of subdividing the Ounce into *dw.* they divide it into Drams and Scruples.

Sheeps Wool Weight has these Denominations; 7 Pounds = 1 Clove: 2 Cloves = 1 Stone: 2 Stones = 1 Tod: $6\frac{1}{2}$ Tods = 1 Wey: 2 Weys = 1 Sack: 12 Sacks = 1 Last.

Liquid

LIQUID MEASURE.

Wine Measure.

2 Pints	=	1 Quart.
4 Quarts	=	1 Gallon.
42 Gallons	=	1 Tierce.
$1\frac{1}{2}$ Tierce	=	1 Hogshead.
$1\frac{1}{3}$ Hogshead	=	1 Punchion.
$1\frac{1}{2}$ Punchion or 2 Hogsheads	=	1 Butt or Pipe.
2 Butts or Pipes	=	1 Tun.

Which may for Accounts be reduced to these; Tun. Hogshead. Gallon. Quart. Pint. Thus Marked:

$$\overset{4}{T} : \overset{63}{hd} : \overset{4}{gal} : \overset{2}{qt} : pt.$$

Or it may be sufficient to use $\overset{63}{hd} : \overset{8}{gal} : pt.$

Observe also, That all Spirits, Mead, Perry, Cyder, Vinegar, Oil, and Honey are measured as Wine. Again, 18 Gallons make 1 Runlet; and $31\frac{1}{2}$ Gallons make a Wine or Vinegar Barrel.

Ale and Beer Measure.

2 Pints	=	1 Quart.	A Firkin of Soap and Herrings are the same with that of Ale.
4 Quarts	=	1 Gallon.	
8 Gallons Ale / 9 Gallons Beer	=	1 Firkin	
2 Firkins	=	1 Kilderkin.	
2 Kilderkins	=	1 Barrel.	
$1\frac{1}{2}$ Barrels	=	1 Hogshead.	

Mr. *Ward* says, This Distinction of the Ale and Beer Measures are now used only in *London*; but in all other Places of *England* it is by a Statute of Excise made in the Year 1689, without distinction $8\frac{1}{2}$ Gallons to 1 Firkin. And this Measure may be reduced to these Denominations, *viz.*

$$\overset{48}{hd} : \overset{4}{gall} : \overset{2}{qt} : pt. \text{ for Ale.}$$

$$\overset{54}{hd} : \overset{4}{gall} : \overset{2}{qt} : pt. \text{ for Beer.}$$

Or, according to the last Account for both,

$$\overset{51}{hd} : \overset{4}{gall} : \overset{2}{qt} : pt.$$

There is also another way of keeping Accounts, especially in the Affairs of the *Revenue*, as the *Excise*; where they make the lowest Denomination a Cubical Inch, (*i. e.* a Measure 1 Inch long, 1 Inch broad, and 1 Inch deep:) And then of Wine Measure, 231 Cubick Inches make 1 Gallon. In Ale and Beer Measure, 282 Cubick Inches make 1 Gallon; and you may chuse as many of the superiour Denominations in your Accounts as you please.

As to the Original of Liquid Measure, it is from *Troy* Weight. Thus; 8 ℔ *Troy* Weight of Wheat gathered out of the Middle of the Ear, and well dried, is, by the old Statutes of *Henry* III. *&c.* ordained to be a Gallon of Wine Measure; neither were any other Measures allowed, tho' Time and Custom has introduced others. Mr. *Ward* mentions an Experiment he was witness to at *Guildhall*, before the Lord-Mayor of *London* and others; whereby it was found that the old Standard Wine Gallon contained exactly 224 Cubical Inches; tho', says he, for several Reasons, the supposed Content of 231 Inches was continued.

Observe, In *Scotland*, the common Denominations of Liquid Measure are these: Hogshead. Gallon. Pint. Mutchkin. Gill. and 4 Gill = 1 Mutchkin; 4 Mutchkins = 1 Pint; 8 Pints = 1 Gallon; and 16 Gallons = 1 Hogshead. They also call 2 Mutchkins, 1 Chopin; and 2 Pints 1 Quart. The *English* Pint is a very little larger than a *Scotch* Mutchkin. But the Excise in *Scotland*, since the Union of the two Nations, is calculated upon *English* Measure.

Dry Measure, *called also* Corn Measure.

2	Pints		1 Quart.
2	Quarts		1 Pottle.
2	Pottles		1 Gallon.
2	Gallons		1 Peck.
4	Pecks		1 Bushel, Corn.
5	Pecks	=	1 Bushel, Water.
4	Bushels		1 Coomb.
2	Coombs		1 Quarter.
4	Quarters		1 Chalder.
5	Quarters		1 Tun or Wey.
2	Weys		1 Last.

For Accounts, these Denominations are sufficient;

$$\overset{4}{\text{Ch}} : \overset{8}{\text{qr}} : \overset{4}{\text{bush}} : \overset{16}{\text{pk}} : \text{pt}.$$

I have taken this Table as it is in *Wingate* and others; but *Ward* says, 10 Quarters = 1 Wey, and 12 Weys = 1 Last of Corn.

As in Liquid Measure, so in dry, the lowest Denomination used in the Calculations of the Revenue is a Cubic Inch, whereof $268 + \frac{4}{5}$ make 1 Gallon: For the *Winchester* Bushel with a plain round Bottom and equally wide from Top to Bottom, is $18\frac{1}{2}$ Inches wide, and 8 Inches deep; whence follows by Calculation, that $268\frac{4}{5}$ Cubical Inches make 1 Gallon.

The common Denominations of Corn Measure in *Scotland*, are $\overset{16}{\text{Chaldron}}.\ \overset{4}{\text{Boll}}.\ \overset{4}{\text{Bushel}}.\ \overset{4}{\text{Peck}}.\ \text{Quarter}.$ But they are different Measures from the *English* of the same Name.

Measures *of* Length.

1st.

12	Inches		1 Foot.
3	Feet		1 Yard.
45	Inches		1 Eln.
2	Yards	=	1 Fathom.
$5\frac{1}{2}$	Yards		1 Pole or Perch.
40	Poles		1 Furlong.
8	Furlongs		1 Mile.

For Accounts use these Denominations;

$$\overset{8}{\text{Mile}} : \overset{220}{\text{furl}} : \overset{3}{\text{yd}} : \overset{12}{\text{feet}} : \text{Inch}.$$

2d.

4 Nails = 1 Quarter
4 Quar^r = 1 Yard.

Marked

$$\overset{4}{\text{yd}} : \overset{4}{\text{qr}} : \text{na}.$$

The Original of Long Measures is from a Corn of Barley, whereof 3 taken out of the Middle of the Ear, and well dried, make 1 Inch; and therefore 1 Barley-Corn is the least Measure, but not used in Accounts.

Time.

60	Seconds		1 Minute.
60	Minutes		1 Hour.
24	Hours	=	1 Day.
365	Da. + 5 ho. + 48 min. + 57 sec.		1 Year.

Because 1 Day is not an *aliquot* Part of a Year, therefore all these Denominations cannot conveniently be mixed in Accounts. And we may chuse to make Days the greatest Denomination in Accounts; and then any Number of Days may be again reduced to Years, by dividing them by 365 Days, 5 Hours, 48 Minutes, 57 Seconds, as will be afterwards taught. In Astronomical Calculations there is a necessity to be thus exact: But for common Uses we may neglect the 5 Hours, 48 Minutes, 57 Seconds, and make 365 Days = 1 Year. Or also casting away 1 Day + 5 Hours, *&c.* we may call 364 Days = 1 Year: And make this Division of the Year, *viz.*

60

60 Seconds		1 Minute.
60 Minutes		1 Hour.
24 Hours		1 Day.
7 Days	=	1 Week.
4 Weeks, or 28 Days		1 Month.
13 Months		1 Year.

But to make 365 Days = 1 Year, and use no Denomination betwixt Year and Day, is the better way in Calculations of Interest, where it occurs most; as you'll find afterwards: yet Months and Weeks make a convenient Division of Time.

Of SUPERFICIAL *or* SQUARE MEASURE.

16 Square Quarters of an Inch		1 Square Inch.
144 Square Inches		1 Square Foot.
9 Square Feet	=	1 Square Yard.
30¼ Square Yards		1 Square Pole.
40 Square Poles		1 Rod of Land.
4 Rods		1 Acre.

Observe, Square Measure is that which is as long as broad; and therefore as 4 Quarters make 1 Inch in Length, so a Surface 1 Inch long and 1 Inch broad is divisible into 16 Parts, each ¼ Inch long and broad; and so of the rest. The Reason of which will be understood after you know what Multiplication is.

For small Surfaces the Denominations of inch. foot. yd. are enough. And for Land these of Acre. Rod. Pole. Also when we say so many square Feet or Yards, *&c.* it were the same thing to say so many Feet or Yards long, and one broad. And thus a Rod is 40 Poles (or 220 Yards) long, and 1 Pole (or 5½ Yards) broad; which is also 1210 Yards long, and 1 broad.

SCHOLIUM, *relating to the following* Applications.

Those who would be very nice and scrupulous as to the Method and Order of bringing in the following *Applications*, would first explain all the four fundamental Operations of *Addition*, *Subtraction*, *Multiplication*, and *Division*, as applied to simple Numbers, before they say a word of mixed Numbers; and they have this Reason for it, *viz.* Because the Addition of mixed Numbers is indeed not the simple Effect of Addition, but of that and Division too: and therefore, according to the strictest Method, ought to be brought as a mixed Application of both. It is true, that according to the Order we have hitherto followed, (the general Rule of Division being already taught) we may propose any Rule with a Division to be performed; for this cannot be called the proposing to one the doing of a thing which he has not yet learned: yet still it will have thus much of *Disorder* in it, that we shall anticipate something that does more immediately and properly belong to the Application of *Division*. But that being simple, and the Reason of it very obvious, I have chosen rather to explain the Practice of *Simple* and *Mixed Numbers* one immediately after the other, in each of the four Operations. And to serve those who incline to learn the Addition of Applicate Numbers before they learn Division of Abstract Numbers, I have also explained the Method necessary to be used in that Case.

§. 2. ADDITION *of* APPLICATE NUMBERS.

CASE I. *To add* Simple Numbers *all of one Denomination.*

RULE. This is done in all reſpects as *Abſtract Numbers*, theſe being no other than *Numbers applicable to any kind of thing.* So that a particular Application can never alter the Rule and Reaſon of Operation. And that they ought to be all of one Denomination, is alſo plain. But ſee the following *Scholiums*.

SCHOLIUMS.

L. Sterling.		Years.
2468		347
7890		256
5678		789
2345		563
L. 18381	Sum	Y. 1955

1. That is called a *Proper Addition*, when Numbers are ſo added together as that their Sum is a new Number diſtinct from either of them; and that *Applicate Numbers* may be added in this manner together, the *Units* of each that are to be added muſt be of the ſame Value and Denomination, or applied to the ſame Species of things, elſe the Sum can have no particular Denomination, and ſo be of no uſe in Practice. So 4 Men and 3 Books make 7 things; but are not either 7 Men or 7 Books.

Again, Numbers of different Denominations are ſaid to be added *improperly* when each Number is diſtinctly repreſented by itſelf with ſome Mark or Word for Addition; as for Example, 3 *l.* and 4 *s.* or 3 *l.* + 4 *s.* or more ſimply 3 *l:* 4 *s.* and ſuch Additions conſtitute *mixed Numbers*. But ſtill it is to be minded, that *applicate Numbers* cannot be added, even improperly, *i. e.* to make one *mixed Number*, unleſs they have certain Relations to one another, ſo that a certain Number of one kind is equal to one of another. And for the ſame Reaſon when there are ſeveral *mixed Numbers* to be added, the Numbers of each Species muſt be ſeparately and diſtinctly added together, as in the following Caſe: Concerning which, careſully obſerve the following Article.

2. The Sum of ſeveral *mixed Numbers* may be found and expreſſed after two very different manners. (1.) We may add the Numbers of every Species by itſelf into one complete Sum; and expreſs theſe ſeveral Sums diſtinctly and ſeparately. Or, (2.) Regarding the mutual Relations of the ſeveral Species, the total Sum may be found and expreſſed in a more ſimple manner; ſo that there ſhall be no Number in it of any inferiour Species but what is leſs than an *Unit* of the next above. [The Value of the Sums of each inferiour Species being expreſſed in Numbers of the next higher Species, gradually to the higheſt.]

Now the *Firſt* is the only Method natural and proper to *Addition*; for the Anſwer it finds is the pure Effect of *Addition*; and is indeed only ſo many diſtinct Queſtions of ſimple Numbers added, without any dependence or regard to one another. As in the annex'd *Example*.

	l.	s.	d.
	24	14	10
	68	18	09
	352	06	11
	467	10	08
1ſt Method	911	48	38
2d Method	913	11	02

But the Anſwer found by the ſecond Method, (which ſhall be taught immediately,) is the moſt Simple and Uſeful in Buſineſs; becauſe it expreſſes the Whole in one Species (and that the higheſt) as near as poſſible; and ſo makes the Compariſon of different Quantities more ſimple and eaſy: yet the Operation is more complex; for it is the mixed Effect of *Addition* and *Diviſion*. I have in this *Example* expreſſed the Sum both ways. How the ſecond is performed, you learn in the following *Rule*.

I *Laſtly*,

Lastly, Observe, That tho' the Numbers to be added are all of one Denomination, and so far belong to the first *Case*; yet if they are of such kind of things as have superior Species, the Answer may, in some Cases, be found and express'd two ways; either in their own Species by simple Addition, as in *Case* 1. or regard may be had to the superior Species: And then, if their Sum taken in their own Species is greater than an Unit of the Superiour, the total Value of the given Numbers may be express'd in the superiour Species, as far as it reaches. And so it will be either a simple Number of some one superiour Species, or a mix'd Number of several Species, (the Rule for the performing of which, is contain'd in that of the next *Case.*) For *Example*, several Numbers of *d.* or *sh.* being proposed to be added, we may find the Sum all in *d.* or *sh.* or in *l. s. d.* as far as the Value will reach. So if we take the Shillings of the preceding *Examp.* for the given Numbers of a Question, the Answer is either 48 *s.* or 2 *l.* 8 *s.* which you'll find equal to it by the Rule given in the following *Case.*

CASE II. *To add* MIX'D NUMBERS.

Rule 1. In every Line of mix'd Numbers let the Species be distinctly separated, and in order to this, write first down the Names (or Characters) of the Species, [the highest on the Left-hand, and the rest in order toward the Right] and then write every Line of Numbers in order under these, the Number of each Species under its own Name; and in every Species observe duly the Order of places of each Figure.

2. Beginning at the lowest Denomination, add all the Numbers of that Column together (as simple Numbers) and when you have found the Sum, you must find how many Units of the next superiour Denomination it's equal to: Thus; Divide it by that Number of the Species added, which is equal to an Unit of the next above; what remains in the Division write down under the Numbers added, as a Part of the total Sum which belongs to that Species, and the Number of the Quote take and add to the Numbers of the next Species. But if the Sum is less than an Unit of the next, set down what it is, and there is nothing to be carried to the next. Go thus thro' every Denomination, till you come to the last or highest, and write down the total Sum of that as it is, because it has relation to no higher; and all these Numbers, set down under every Denomination, make the total Sum.

Examp. of Money.

l.	*sh.*	*d.*	*f.*
4760 :	14 :	11 :	2
6854 :	16 :	10 :	3
5923 :	08 :	06 :	2
640 :	10 :	09 :	0
832 :	00 :	11 :	3
7925 :	18 :	08 :	1
8894 :	19 :	10 :	3
35832 :	10 :	08 :	2

Thus, in the annex'd Example, the total Sum of the Farthings you'll find to be 14; then, because 4 Farthings = 1 Penny, I divide 14 by 4, the Quote is 3, and 2 remains, which is written in the Sum under Farthings, and the Quote 3 I carry to the Pence; and the Sum is 68*d.* which I divide by 12 (because 12*d.* = 1 *sh.*) the Quote is 5, and 8 remains, which is written down under *d.* and the 5 carried to the Shillings, whose Sum is 90, which I divide by 20 (because 20 *sh.* = 1 *l.*) the Quote is 4, and 10 remains, which is written under *sh.* and the 4 carried to the Pounds, whose Sum is 35832, making the total Sum 35832 *l.* 10 *s.* 08 *d.* 2 *f.*

The *Reason* of making the Divisions directed (which is the only new thing we have to account for) is plainly this, *viz.* Because 4 *f.* = 1 *d.* therefore as oft as 4 Farthings is contained in any other Number of Farthings, so many Pence is that Number of Farthings equal to: and the like Reasoning is good in all other Cases.

But for those who do not yet understand the Rule of Division (and even tho' they do) the following is a convenient Method.

To add MIX'D NUMBERS *without the Rule of Division.*

Rule. Begin at the lowest Species, and add the Numbers thereof together; not by single Columns, as you do simple Numbers, but take the whole Number that is in every Line together, and add them to one another, pointing when you come to such a Sum as is equal to, or greater than an Unit of the next higher Species, (but less than two such Units) and carry on the Excess, adding it to the next Number; and so thro' all that Species; setting down in the total Sum what Excess there happens to be after the last Point. Then for every Point carry 1 to the next Species, and go thro' all the Species in the same manner; but the highest you are to add by single Figures, as simple Numbers.

Examp.

l.	*s.*	*d.*
46	14·	6
68	18	11·
72	10·	10·
94	9	8
282	13	11

The Operation of this Example is thus: Beginning at the Pence of the lowest Line, I say 8 + 10 = 18, which is 12 + 6; therefore I make a Point at the 10, and carry forward the 6; thus 6 + 11 = 17, which is 12 + 5, this makes another Point at 11, and 5 to carry forward; then 5 + 6 = 11, which being less than 12, I write it down: Then for the two Points I carry 2 to the Shillings; thus, 2 + 9 (= 11) + 10 = 21, for which I make a Point (for the 20) at 10, and carry forwards the 1 over 20; thus 1 + 18 (= 19) + 14 = 33, for which I make another Point at 14, and the Remainder 13 (over 20) is written down: then for these two Points I carry 2 to the Pounds, and add them as in *Case* 1.

Observe, We need not point the Shillings, but take this easy Method, *viz.* Add the first Column and write down what's over 10's, (as simple Numbers) then carrying the Number of 10's to the second Column, (or Place of 10's) sum it up, and if the Sum is an even Number, set down 0 and carry the half of the Sum to the Pounds; but if it's an odd Number, set down the odd 1, and carry the half of the Remainder.

The Reason of this Practice is plain, for two 10's make 20; and we may easily suppose any body, the least acquainted with Addition, can take the half of an even Number.

ANOTHER METHOD.

Some propose to make Tables, that may serve instead of Division; whereby, when the Sum of any Species is taken by itself (as simple Numbers) you may, by Inspection, find how many Units are to be carried to the next higher Species: The Method of which will be very obvious, by considering the following *Examp.* for Money, which is made by simple Addition. Thus; Beginning at 4 *f.* write against it 1 *sh.* then 4 + 4 = 8, and against it write 2; then 8 + 4 = 12, and so on, still adding the last Sum to the first Number. The same way proceed in all other Species. And for the Length of the Table, you carry it on as far as you please; which, for long Pages or Columns of Numbers may require 40 Lines: But if you'll subdivide the Column into Parcels of about 12 or 20 Numbers, taking their Sums separately, and adding them together, then a Table carried so far will be sufficient.

TABLE *for the Addition of Money.*

f.	*d.*	*d.*	*sh.*	*sh.*	*l.*
4 =	1	12 =	1	20 =	1
8 =	2	24 =	2	40 =	2
12 =	3	36 =	3	60 =	3
16 =	4	48 =	4	80 =	4
20 =	5	60 =	5	100 =	5
24 =	6	72 =	6	120 =	6
28 =	7	84 =	7	140 =	7
32 =	8	96 =	8	160 =	8
36 =	9	108 =	9	180 =	9
40 =	10	120 =	10	200 =	10

The

The *Use* of this Table (and all others of the same kind) is obvious; for, having summed any Species, seek that Number in this Table (in its proper Column, if it does not exceed the greatest Number in the Table) and if that precise Number is not there, take the next lesser, and against it you have the Number of the next superiour Species contained in the Sum: which Number you are to carry to the next superiour, and take the Difference betwixt the Sum and that Number next less, which you are to write down under the Numbers added. I shall leave you to examine the preceding *Examples* by this Table, or make others for your Exercise.

Observe, Those who propose such Tables do it to prevent blotting of Accounts by pointing; for they design them for the more Ignorant, who can't do Division. But as I have said enough already to the Objection against pointing, I shall only observe, that any Accounts such Persons can be intrusted with, can't require so great Nicety as to make pointing a Fault. And I think 'tis plain, there is less Trouble with it in the Practice, and is even more convenient than to do the Work by Division (when one can do it so) because more simple.

EXAMPLES *for the Exercise of* ADDITION *in* MIX'D NUMBERS.

Money.

	20	12	4
l.	*s.*	*d.*	*f.*
346	14	08	3
268	16	10	2
4689	09	11	2
7846	10	06	3
6320	00	04	0
25683	18	00	2
64	12	11	1
45219	17	05	1

Troy Weight.

	12	20	24
℔.	℥.	*dw.*	*gr.*
4768	11	18	20
2345	10	15	23
3689	08	10	18
875	06	12	10
762	04	19	06
86	07	08	09
67	11	13	22
12597	01	19	12

Averdupoise Weight, the Greater.

	4	28
Ct.	*qr.*	℔.
372	3	27
468	2	20
593	0	10
678	2	18
976	3	19
678	1	24
789	2	06
4558	1	12

Wine Measure.

	4	63	4	2
Ton.	*hd.*	*gal.*	*qt.*	*pt.*
436	3	62	2	1
678	2	60	3	0
569	1	48	1	1
456	0	29	3	1
789	1	36	2	0
987	2	54	1	1
672	3	46	3	1
4591	1	24	1	1

Long Measure.

	8	220	3	12
Mile.	*furl.*	*yd.*	*f.*	*in.*
3467	5	219	2	10
4567	7	184	1	11
5678	6	062	2	08
78967	4	009	0	09
56789	3	084	2	11
24608	2	147	1	10
35791	1	210	2	06
209871	0	80	0	05

In this *Example* of *Long Measure*, because 220 Yards = 1 Furlong, therefore in adding up the Yards, the easiest way is to add up the Column of Units, writing down what's over 10's, and carrying the Number of 10's to the other two Columns, sum them both together, pointing at every 22, or dividing the Sum by 22.

GENERAL SCHOLIUM, *concerning the more special Application of the Rules of Arithmetick.*

A good *Arithmetician* must be capable of something more than barely to perform any Operations with given Numbers, when the Question is simply proposed to Add, Subtract, *&c. i. e.* when he knows what Operation is to be applied, and to what Numbers. For, the great Art of Application lies in the Solution of such Questions as, to distinguish them from the other, I call *Mix'd* or *Circumstantiate* Questions, *i. e.* wherein no Operation is named, but we are left to find the proper Work from the Nature and Circumstances of the Question. Now, for this there are not any determinate and general Rules: it depends upon the good Sense and Judgment of the Arithmetician, whereby he can distinctly and perfectly comprehend the Nature and Circumstances of a Question. It supposes him to understand the Nature of the Subject about which the Question is; and lastly, to understand perfectly well the true general Import and Effect of the several simple Operations of Arithmetick. By which means he may know when the Reason of the Question requires such an Operation.

The more simple the Circumstances of a Question are, it will be the more easy; and where there is but one Operation to be applied, it will be always obvious: But, where a Variety of Circumstances occur, and several Operations become thereby necessary, the Difficulty increases; which Experience only can make easy. And therefore, as a proper Introduction to that Experience, I shall give you, after each of the Rules, some practical Questions, whose Solutions being considered, may help to guide the Judgment in like Applications.

The Effect of *Addition* being the Discovery of a Number, which is equal to certain given Numbers, taken all together; whenever the Sense and Reason of a Question shews that any given Numbers must be collected, or that the Number sought is equal in Value to several Numbers given, then Addition is the Rule; as in the following Examples.

MIX'D PRACTICAL QUESTIONS *for* ADDITION.

Quest. 1. A Father was 18 Years 4 Months old, (reckoning 13 Months to one Year, and 28 Days to one Month) when his eldest Child was born. Betwixt the eldest and second were 11 Months 10 Days. Betwixt the second and third were 3 Years 8 Months. When the third is 12 Years, 6 Months, 20 Days, how old is the Father? Answer 35 Years, 4 Months, 2 Days. For that all these Numbers ought to be added together, is manifest.

	13	28.
ye.	*mo.*	*da.*
18 :	04 :	00
:	11 :	10
3 :	08 :	00
12 :	06 :	20
35 :	04 :	02

Quest. 2. I bought a Parcel of Goods, whereof the first Cost was 40*l.* 10*s.* paid for packing them 13*s.* for Carriage 1*l.* 6*s.* 8*d.* and spent about the Bargain making, 15*s.* 6*d.* What do these Goods stand me in all? Answer, 43*l.* 5*s.* 2*d.*

l.	*s.*	*d.*
40 :	10 :	00
00 :	13 :	00
01 :	06 :	08
00 :	15 :	06
43	05 :	02

Quest.

Quest. 3. There is owing me by the following Debtors, *viz.* *A.* owes 20 *l.* 15 *s.* *B.* owes 100 *l.* *C.* owes 56 *l.* 10 *s.* 8. *d.* *D.* owes 82 *l.* 18 *s.* 4 *d.* What is the Amount of the whole? Answer, 260 *l.* 4 *s.*

l.	*s.*	*d.*
20	15	00
100	00	00
56	10	08
82	18	04
260	04	00

Quest. 4. The Distance betwixt two Places is such, that if 3 Miles and 5 Furlongs is taken from it, what remains is equal to 8 Miles, 4 Furlongs and 100 Yards. What is the Distance of these two Places? Answer, 12 Miles, 1 Furl. and 100 Yards.

m.	*furl.*	*yd.*
3	05	000
8	04	100
12	01	100

§. 3. SUBTRACTION *of* APPLICATE NUMBERS.

CASE 1. To subtract simple Numbers of one Denomination.

Rule. This is done as *Abstract Numbers.*

	l.
Sub^d^.	468
Sub^r^.	253
Diff.	215 *l.*

	Gallons.
Sub^d^.	72306
Sub^r^.	9462
Diff.	62844 *Gallons.*

CASE 2. To subtract *Mix'd Numbers.*

Rule 1. Having set the Subtractor orderly under the Subtrahend, with a due regard to the Places and Species; 2. Begin at the lowest Species of the Subtractor, and take the Number of that from its correspondent in the Subtrahend, and write down the Difference. Do the same with all the Species, and you have the Difference sought.

But if the Number of any inferiour Species in the Subtractor, is greater than its Correspondent in the Subtrahend, then to this add a Number equal to an Unit of the next Species [for *Examp.* to *sh.* add 20, and to *d.* add 12] and subtract from that Sum; and then to the Number of the next Species in the Subtractor add 1, (or take 1 from the Subtrahend) and then subtract: And when you come to the highest Species, do as in simple Numbers, *i. e.* add 10 where you can't subtract.

Examp. 1.

	l.	*sh.*	*d.*
Sub^d^.	72	18	10
Sub^r^.	29	10	04
Diff.	43	08	06

Examp. 2.

l.	*sh.*	*d.*
48	14	6
26	17	10
21	16	08

The first *Example* is simple and easy. For the second, I do it thus, 10 (in the *d.*) from 6 cannot, but from $12+6=18$, and 8 remains, to be set down in the *d.* Then $1+17$ (in *sh.*) $=18$, which from 14 cannot, but from $20+14$ $(=34)$ and 16 remains, to be set down in *sh.* Then $1+6$ (in *l.*) is 7, which taken from 8, 1 remains: Then 2 from 4, and 2 remains. So the total Difference is 21 *l.* 16 *s.* 8 *d.*

The *Reason* of this Practice is sufficiently plain from the like Reasoning used for the abstract Rule.

SCHOLIUMS.

1. What was obferved in Addition, is true here alfo, *viz.* That Number ought to be fet down in any inferiour Species, but what's lefs than an Unit of the next; otherwife the Species are confounded: Yet ftill it's arbitrary to make any Species the higheft, and then in it we write down any Number.

2. *Obferve,* That this *Rule* fuppofes we can readily (or in our Mind) difcover the Difference betwixt any Number of any Species lefs than an Unit of the next higher, and any other Number greater, but lefs alfo than fuch an Unit; or betwixt any fuch Number and a leffer increafed by a Number equal to fuch an Unit. And to do this readily, requires a little Practice; but then the Operation for each Species may be performed feparately, Figure by Figure, which will remove this Suppofition; yet it will be more tedious Work; and we muft by Practice acquire the Capacity which the Rule fuppofes. And to make it fomewhat eafier, take this Method: When the Subtrahend Figure is leaft, take the Subtractor Figure out of the Number to be added, and the Remainder add to the Subtrahend Figure; the Sum is the Number to be fet down in the Difference. As in *Examp.* 2. above, I fay 10 *d.* from 6 can't, but from 12, and 2 remains; which added to 6, makes 8 to be fet down.

Other *Examples* in Subtraction.

Troy Weight.

	12	20	24
℔.	*oz.*	*dw.*	*gr.*
342	08	10	06
84	03	15	20
258	04	14	10

Wine Measure.

	33	4	2
hd.	*gal.*	*qt.*	*pt.*
24	42	2	1
15	56	3	0
8	48	3	1

Long Measure.

	4	4
yd.	*qr.*	*na.*
18	2	0
9	3	1
8	2	3

MIX'D PRACTICAL QUESTIONS *for* SUBTRACTION.

Queft. 1. Having borrowed 20*l.* 13*s.* 4*d.* and paid thereof 8*l.* 16*s.* 8*d.* What's yet due? Anfwer, 11*l.* 16*s.* 8*d.*

l.	*s.*	*d.*
20	13	4
8	16	8
11	16	8

Queft. 2. Having bought 2 hund. Weight, and 3 qr. of Sugar; and fold thereof 1 hund. 2 qr. and 14 ℔. what is yet unfold? Anfwer, 1 *C.* 14 ℔.

C.	*qr.*	℔.
2	3	00
1	2	14
1	0	14

Queft. 3. A Father was 24 Years, 9 Months, 10 Days old when his eldeft Son was born; and is now 56 Years, 3 Months, and and 22 Days. How old is the Son? Anfwer, 31 *ye.* 7 *mo.* 12. *da.*

	13	28
Ye.	*mo.*	*da.*
56	3	22
24	9	10
31	7	12

Queft. 4. What is that Sum of Money, which being added to 36*l.* 6*s.* 8*d.* will make 50*l*? Anfwer, 13*l.* 13*s.* 4*d.*

l.	*s.*	*d.*
50	00	00
36	06	08
13	13	04

The

The following two Questions are mixed in *Addition* and *Subtraction*.

Quest. 5. Having borrowed 100 *l.* and paid at one time 20 *l*: 13 *s* : 4 *d.* at another time 33 *l*: 6 *s* : 2 *d.* How much is yet due? *Answ.* 46 *l* : 6 *d.*

	l : s : d.
Borrd	100 : 00 : 00
	20 : 13 : 04
	33 : 06 : 02
Paid	53 : 19 : 06
	46 : 00 : 06

Quest. 6. There are 2 Casks of Sugar; the Weight of the one full Cask is 1 *C* : 2 *qr.* of the other 1 *C* : 18 ℔. The Weight of one empty Cask is 25 ℔. of the other, 1 *qr* : 7 ℔. What is the Weight of Sugar in both Casks? *Answ.* 2 *C* : 14 ℔.

	C : qr : ℔.
	1 : 2 : 00
	1 : 0 : 18
Full Casks	2 : 2 : 18
	0 : 0 : 25
	0 : 1 : 07
Empty Casks	0 : 2 : 04
	2 : 0 : 14

§. 4. MULTIPLICATION of APPLICATE NUMBERS.

INTRODUCTION.

IN order to know what Variety there is in *Multiplication of Applicate Numbers*, we must first consider, What is a *Simple* and *Proper* Question of Multiplication; and I think it is plain, That this is a Question proposed with no other Circumstances, or in no other Form, but barely *To multiply one Number by another*; and then it is certain that the given Numbers must be such as are agreeable to the general Nature and Definition of that *Operation*; which being no other than the repeating or taking the one Number as oft as the other expresses, or contains *Unity*; it is manifest that the Multiplier must always be an abstract Number, expressing simply the number of times that the Multiplicand is to be taken. Therefore the whole Variety in Multiplication of Applicate Numbers depends upon the *Multiplicand*; which may be either a *simple* or *mixed* Number, making only two different Cases.

Hence we see, That the proposing simply to multiply one Number by another, both Applicate, is absurd. For Example, to multiply 8 *l.* by 3 *l.* For what's that to say, *To multiply by* 3 *l?* To multiply by 3 is intelligible; but what has the Name *l.* to do here? Now, because some have thought fit to propose such Questions without explaining the true Sense and Meaning of them, and given us very perplex'd Rules for solving them, which may be made more general and much easier, if the Sense of the Question is once rightly conceived; I shall here therefore make it appear more evidently, that Questions proposed in that simple manner, are Nonsense in Terms; by which means I shall lead you to the Sense that must be put upon all such Questions: and in its proper place you'll find the Rules for solving them.

Suppose then, that it is proposed to multiply 8 *l.* by 3 *l.* I should ask the Proposer, What he means by multiplying 8 *l?* He can make no other Answer, But that it is the repeating 8 *l.* a certain number of times, (that being the simple and proper Definition of *Multiplication.*) Then I ask him, How many times he would have it repeated? And if he

he anfwers 3 *l.* I hope the Abfurdity is manifeft. He may indeed fay, that he means to have it taken as oft as 3 *l.* contains 1 *l.* But then the direct fimple Anfwer to my Queftion is 3 times, and not 3 *l.* And fo the Denomination of *l.* applied to 3, does not belong to it as a Multiplier, and is only a certain circumftantial way of fignifying how oft the *Multiplicand* is to be repeated: which is going round about to no purpofe, when your Meaning can be expreffed more fimply; for you fee it muft ftill end in this, that the Multiplier is an abftract Number expreffing only how oft the other Number is to be repeated.

Again obferve, That the Authors of fuch Queftions give us this for a Principle, That the two Terms muft be both applicate to one kind of thing, as *Money*, *Weight*, or *Time*; fo that they would very readily pronounce this Queftion, Nonfenfe, *viz.* To multiply 8 *l.* by 3 Days. But here their confufed Notion of this matter will appear yet clearer; for this Propofition is every whit as reafonable as the other: Becaufe if 3 *l.* cannot be a direct and reafonable Anfwer to that Queftion, *How oft?* till you explain it by faying, as oft as 3 *l.* contains 1 *l.* it is plain, that by the fame Method, I may propofe to multiply 8 *l.* by 3 Days; meaning to take 8 *l.* as oft as 3 Days contain 1 Day; which is equally good Senfe as the other.

But *further:* Since an Explication is neceffary; and fomething muft be underftood which is not directly expreffed in propofing to multiply 8 *l.* by 3 *l.* it may as well fignify the taking of 8 *l.* 60 times, [*i. e.* as oft as 3 *l.* contains 1 *fh.* which is 60 times;] for fo I may explain it. *Again*, to multiply 8 *l.* by 3 *fh.* fignifies, according to their meaning, taking $\frac{3}{20}$ Parts of 8 *l.* (becaufe 1 *fh* $= \frac{1}{20}$ of a *l.*) which is not purely a Queftion of *Multiplication*, but of that and *Divifion* too. But why may it not as well fignify the taking 8 *l.* 3 times (*i. e.* as oft as 3 *fh.* contains 1 *fh.*) or 36 times, (*i. e.* as oft as 3 *fh.* contains 1 *d.*) All thefe meanings are equally reafonable; fince the fimple propofing to multiply 8 *l.* by 3 *fh.* limits it to none of them: for it has no meaning till it be explained; or, it has any one of thefe meanings indifferently. And this fhews how ambiguous fuch Propofitions are; or rather, no Propofitions at all, till their meaning is thus cleared up and determined.

But now, after all this, you are to know that Queftions may occur which are folved by *Multiplication*; yet the mix'd Circumftances of the Queftion be fuch, that all the given Numbers may be particularly applied; whereby the Reafonablenefs of fuch Propofitions as I have here cenfured, may feem to be fairly accounted for. To which I anfwer, That in all fuch Cafes, before we know what is to be done, we are obliged to reafon upon the Nature and Circumftances of the Queftion, and by that means we difcover, that fome one given Number of things is to be taken as oft (*i. e.* multiplied) as fome other contains an *Unit* of a certain Denomination. But then, as the *Multiplication* imports only the taking a number of things a certain number of times, fo in the Operation, the *Multiplier* fignifies only *how oft* the other is taken, (this being the proper and formal Notion of a *Multiplier*) tho' it is difcovered by the Circumftance of a Number applied to fome particular thing in the Queftion.

Again, the very fame Number of things in different Queftions will be confidered in different Views, when it expreffes the *Multiplier*, and fo make different *Multiplications*; which fhews that a fimple Propofition of multiplying 8 *l.* by 3 *l.* or 3 *fh.* (and all Queftions of this kind) have no determinate Meaning, or are plainly Nonfenfe. Nor is the Propofition lefs or more reafonable, tho' the two Terms are applied to different things which can be explained to the fame meaning as the other, and occur as often in the Circumftances of mixed Queftions.

I fhall illuftrate all this by particular Examples immediately; but fhall firft explain the Practice in fimple Queftions: which, according to the Diftinction above-mentioned, confifts of two Cafes.

CASE I. *To multiply a* Simple Number.

RULE. This is to be done by the *General Rule for Abstract Numbers*; and the Product is an Applicate Number of the fame Denomination with the *Multiplicand.*

Example 1.	*Example* 2.
7464 *L.*	467 Years.
8	5
59712 *L.*	2335 Y.

SCHOLIUMS.

I. Since any one of two Numbers may be made the Multiplier, and that it is fometimes more convenient to make it the one than the other, tho' that one happens to be the Applicate Number, (*i. e.* the Number to be multiplied) there is no matter; becaufe it is but fuppofing the Denomination to be fhifted and applied to the other Number, and then the Effect is the very fame. For Example: To multiply 3 *l.* by 48, I apply 3 as the Multiplier, as if it were 48 *l.* by 3; whereby I make the Multiplier according to its proper Notion, Abftract; which gives the fame Product and of the fame Denomination: For fince 48 times 3 = 3 times 48; if the Denomination of the Multiplicand is the fame, the Product is in all refpects the fame.

II. *Multiplication* being in effect only a compendious Addition of Numbers equal among themfelves, or the Repetition of the fame Number, the Obfervations made in *Schol.* 2. after *Cafe* 1. in *Addition*, are applicable here alfo: *viz.* That the Product of a mixed Number may be found and expreffed two ways; either, 1. By taking the Product of each Number by itfelf as fo many diftinct Queftions and Applications of *Cafe* 1. which make a confufed Anfwer, tho' the Work is fimple Multiplication. Or, 2. Regarding the Relations of the Species, and expreffing it in the higheft as far as poffible; which is confidering it properly as a mixed Number, whereby we make a more fimple and ufeful Anfwer, tho' the Operation is more complex; for it requires Divifion, the fame way as Addition of mixed Numbers.

III. *Again*; When the Number multiplied is a fimple Number of fuch things as have a fuperiour Species, the Product may be found alfo fimply in its own Species; or may be expreffed in fuperiour Species, as far as it reaches; according to the Method of the following Rule for mixed Numbers. But whereas in Addition we had an eafy way to fupply the Rule of Divifion by pointing, we cannot apply that Method here; and therefore Divifion is indifpenfible, if we would exprefs the Product in its higheft Species, unlefs we make ufe of the Tables defcribed already for Addition; and then the Examples muft be either very fmall Numbers, or the Tables exceeding large. But there is yet another way of expreffing the Product of mixed Numbers fimply without Divifion, which you'll find in the next Cafe. In the mean time, before we proceed to that, there is a fpecial Clafs of mixed Practical Queftions preparatory to it, which are Applications of the firft *Cafe*, comprehended under this Title; *viz.*

REDUCTION from a Higher *to a* Lower *Species*; i. e.

Finding a Number of things of a Lower Species, (Denomination, or Value) Equivalent to a given Number of a Higher Species. For which, this is the Rule.

RULE.

RULE. Multiply the given Number by that Number of the Species to which you would reduce it, which makes an *Unit* of the Species reduced; the Product is the Number sought.

Examp. 1.	*Examp.* 2.
48 *l.*	34 ℔.
20	12
960 *sh.*	408 *oz.*
12	20
11520 *d.*	8168 *dw.*
4	24
46080 *f.*	196032 *gr.*

Exam. 1. To reduce 48 *l.* to *sh.* it is = 960 *sh.* and this again to *d.* is = 11520 *d.* and again to *far.* is = 46080 *f.* According to the Operation in the Margin; each Step being in effect a new Question.

Examp. 2. To reduce 34 ℔ *Troy* Weight; it is = 408 *oz* = 8168 *dw* = 196032 *gr.*

The *Reason* of this Practice is obvious: For if 1 *l* = 20 *sh.* then 48 *l.* is = 48 times 20 *sh.* or 20 times 48 *sh.* which is the same thing. (The like Reason will be found in other Cases.) Therefore you are to *observe*, That tho' the Denomination of *l.* is left with the given Number 48, as it stands in the Operation, yet it is performed in a quite different view; for we are to consider it not as the Multiplication of 48 *l.* by 20, the simple effect of which would certainly be, 960 *l.* But from the Reasons now explained, we consider it as the Multiplication of 48 *sh.* by 20, or of 20 *sh.* by 48; which is = 960 *sh.* And so of the rest. Which takes away the seeming Absurdity of naming the Product differently from the *Multiplicand.*

SCHOLIUMS.

I. If it is required to reduce a Number to a Species which is not immediately the next to it, as *l.* to *d.* We may either do it by Steps thro' all the intermediate Species, as above; or it may be done at one Multiplication, if we know how many *Units* of that Lower makes 1 of the Higher. For Example; To reduce *l.* to *d.* we multiply by 240; because 1 *l* = 240 *d.* And for this purpose, it is ordinary to have Tables of Reduction, shewing how many *Units* of any Species (of common Use in Business) make 1 of any other; which are easily made by the preceding Rule, from the known Relations betwixt each Species and the next, which you have in the Tables of *Addition*. One or two Tables will be sufficient here to explain this Matter; and you may make the like for all other mixed Numbers, at your pleasure.

TABLES *of* REDUCTION.

1. *Money.*

l.	*sh*	*d*	*f.*
1 =	20 =	240 =	960
	1 =	12 =	48
		1 =	4

2. *Troy Weight.*

℔	*oz*	*dw*	*gr.*
1 =	12 =	240 =	5760
	1 =	20 =	480
		1 =	24

The Use of these Tables is plain; for under every Denomination you have 1, and in the same Line the Number to which this 1 is equal of each inferiour Species: So, for Example, to reduce *l.* to *f.* at once, the Multiplier is 960.

But you'll find that it will generally be as convenient to reduce any given Number to any inferiour Species by reducing it gradually thro' all the Denominations, especially for this

this Reaſon, that it were impoſſible to remember all theſe Tables, and too much trouble to turn to them upon every occaſion; whereas the gradual Relations of the ſeveral Species are eaſily kept in mind. And laſtly, the Reductions that moſt frequently happen, are of mixed Numbers, which neceſſarily muſt be done gradually; as in the following Article.

II. If a *mixed Number* is propoſed to be reduced to the loweſt Species expreſſed in it, [or to the loweſt poſſible;] begin at the higheſt, and reduce it to the next Species, adding to the Product the given Number of that Species. Then reduce this Sum to the next Species, and ſo on thro' them all, taking in always the given Number of every inferiour Species: As in the following Example; which needs, I think, no further Explanation.

Examp. 1.

l : *ſh* : *d* : *f.*
724 : 17 : 09 : 2
20
14480 *ſh.*
17 add
14497 *ſh.*
12
173964 *d.*
9 add
173973 *d.*
4
695892 *f.*
2 add
695894 *f.*

Obſerve, The Numbers of the inferiour Species may be taken into the Product of their Species without the pains of writing them down and adding them, by adding them Figure by Figure to the like places of the Product, as they are found in the Multiplying. You'll eaſily underſtand the Method by examining the following Examples. I ſhall only further obſerve, that it is beſt to take them all in upon the Multiplication with the Units place of the Multiplier, in caſe when this has two Places, you do the Work at length. But in the following Examples I have made the Product at once.

Ex. 2. *Troy Weight.*

℔ : *oz* : *dw* : *gr.*
48 : 09 : 16 : 23
12
585 *oz.*
20
11716 *dw.*
24
281207 *gr.*

N.B. In this *Ex.* 2. as you may reduce the *dw.* at two Steps; multiplying firſt by 6, and then by 4, (becauſe 6 × 4 = 24) you muſt mind that the 23 *gr.* are to be taken in with the laſt Multiplication.

Ex. 3. *Averdupoiſe Weight.*

St : ℔ : *oz* : *dr* : *qr.*
256 : 12 : 14 : 06 : 2
16
4108 ℔.
16
65842 *oz.*
16
1053474 *qr.*

Theſe Examples ſhew the Practice ſufficiently; and we need give no other, but leave Examples of other kinds of Things to the Student's own Choice and Exerciſe.

You'll obſerve here alſo the great Difference betwixt multiplying a *mixed Number*, and reducing it; tho' this is performed by Multiplication: For multiplying it, is a Repetition of the Whole ſo many times, or finds a Number which contains every Part of the given mixed Number ſo many times; but reducing is only finding a Number equal to the given mixed Number in the loweſt Denomination; in which every part of it is differently multiplied, and the laſt part not multiplied at all. So in the firſt Example, 724 *l.* is multiplied by 20, and 12, and 4 continually, that is by 960. But 17 *ſh.* is only multiplied by 12 and 4, or 48. and 9 *d.* only by 4. The Anſwer of the Queſtion being 695894 *f.* Whereas the Product of 724 *l* : 17 *s* : 9 *d* : 2 *f.* by any Number would be equal to that Number of times 695894 *f.*

CASE

CASE II. *To multiply a* Mixed Number.

RULE. Begin at the loweſt Species of the *Multiplicand*, and having multiplied that Number, reduce the Product to the next Species; *i. e.* find by Diviſion [in the manner already explained in *Addition of Mixed Numbers*] how many Units of the next ſuperiour Species it is equal to, and what remains over; ſet what is over as a Part of the Anſwer of the Denomination multiplied. Then multiply the given Number of the next ſuperiour Species, and to the Product add that Number to which the Product of the preceding Species was reduced; and reduce this Sum to the next ſuperiour Species; marking the Remainder, or what is over, as a Part of the Anſwer of the Species multiplied; and go on thus thro' all the Species of the *Multiplicand.*

Examp. To multiply 236 *l* : 14 *ſh* : 9 *d.* by 26.

In the annex'd Scheme you ſee the Method of the Work of this Rule; except that the Effect of the Reduction of the ſeveral Products is ſet down without the Operation by which it was done; theſe being ſuppoſed to be done a-part by themſelves, and transferred to this Scheme. But you'll find another way immediately, wherein Diviſion is uſed, and the whole Operation appears in one Scheme, without any confuſion.

l	*ſh*	*d.*	
236 :	14 :	09	
		26	
6136 :	364 :	234	Prod. of each Species.
	19 :	6 . .	= 234 *d.*
	383		= 364 + 19 *ſh.*
19 :	3		= 383 *ſh.*
6155			= 6136 + 19 *l.*
6155 :	03 :	06	Total Product.

Another Method without Diviſion.

Reduce the Multiplicand to the loweſt Species, as has been already taught; then multiply this Number by the given Multiplier, and the Product is the Number of that Species equivalent to the propoſed Number of times the Multiplicand.

Operation.

l	*ſh*	*d.*
48 :	16 :	8
20		
976 *ſh.*		
12		
11720 *d.*		
42		
492240 *d.*		

Examp. To multiply 48 *l* : 16 *ſh* : 8 *d.* by 42, it is equal to 492240 *d.* as in the Margin.

This Anſwer is the only proper and natural Effect of Multiplication. And if it is required to know the Value of it in higher Species, this is properly a Queſtion of Diviſion, to be performed in the manner already explained; which is to divide by the Number of every Species which makes an Unit of the next above. But I ſhall refer you to Diviſion to ſee the beſt and neateſt Method of ordering the Operation. And here only obſerve theſe two things.

1. That with large Multipliers, this laſt Method (*viz.* of reducing to the loweſt Species by Multiplication, and then to the higher by Diviſion) will generally prove a more convenient Method than that of the firſt Rule. But, 2. If the Multiplier is a ſingle Digit, or any Product of two Digits, the Work may, in moſt Caſes, be eaſily performed according to the firſt Rule, without writing the Diviſions: As in the following Examples.

Ex.

	l	*sh*	*d.*	
Ex. 1.	68	14	09	3
				7
	483	03	8	1

Here I say, $3 \times 7 = 21$; which is 1 *f.* over 20, or 5 times 4: therefore I write down 1, and carry 5. Then $7 \times 9 = 63$, and 5 carried, is 68 *d*; which is 8 over 60, or 5 times 12: so I write 8 *d.* and carry 5. Then $7 \times 14 = 98$, and 5 is 103; for which I write 3, and carry 5, (for 5 times 20 is 100.) Then multiply the 68 *l.* and add the 5 from the Shillings.

	C	qr	lb	
Ex. 2.	37	3	18	by 28.
			4	
	148	2	16	
			7	
	1040	2	00	

For the 2d *Examp.* I resolve the Multiplier into 4 and 7. Then beginning with 4; I say, $4 \times 18 = 72$, which is 2 times 28 ($= 56$,) and 16 over; or resolving the 4, I say $2 \times 18 = 36$, which is 28, and 8 over; consequently in 2 times 36 ($= 72$) there are 2 times 28, and 2 times 8, or 16 over. The rest is easy. Then for the 7, I consider that 7 being the 4th Part of 28, and 4 the 4th Part of 16; therefore 7×16, or 16 times 7, is equal to 4 times 28; for which I write 00, and carry 4. Then $7 \times 2 = 14$, and 4 carried is 18; for which I write 2 *qr.* and carry 4 *C.*

By such means as these, one may by Practice easily perform any Questions of this kind.

SCHOL. As to the Solution of other mixed Questions, there is no other general Direction can be given whereby to know when Multiplication is to be applied, but only this, *viz.* To consider, that the true Effect of Multiplication being the repeating of any Number, or taking it a certain number of times; therefore whenever the Sense and Reason of a Question requires that any given Number of things be repeated, or that a Number be found equal in Value to a certain given Number of things, repeated or taken as oft as some other given Number in the Question contains Unity; then Multiplication is the Work required. As in the following Examples.

Mixed Practical Questions in Multiplication.

Quest. 1. There is in each of 28 Bags, 44 *l* : 16 *sh* : 8 *d.* How much is in the Whole? *Answ.* 1255 *l* : 6 *sh* : 8 *d.*

l	*sh*	*d.*
44	16	8
		4
179	06	8
		7
1255	06	8

Here the Nature of the Question plainly requires that 44 *l.* : 16 *sh* : 8 *d.* be multiplied by 28, the Number of Bags; for if 1 Bag contains so much, 28 Bags must contain 28 times so much, which imports a Multiplication by 28; which is taken abstractly in the Operation, tho' it is applied to Bags in the mixed Proposition. As to the Manner of working, I have chosen 4 and 7 as *Factors*; because $4 \times 7 = 28$.

Quest. 2. At 3 *l* : 6 *sh* : 4 *d. per* Yard, what is the Value of 465 Yards? The Reason of this Question shews, that 3 *l* : 6 *sh* : 4 *d.* must be taken 465 times; or multiplied by 465. For the Value of 465 Yards must be 465 times as much as the Value of 1 Yard. And to do the Work, I reduce 3 *l* : 6 *sh* : 4 *d.* to *d.* it is $= 796$ *d.* which multiplied by 465, produces 370140 *d.* which is again equal to 1542 *l* : 5 *sh.* by Division. As you will learn afterwards.

l	*ſh*	*d.*
16	6	8
		6
98	0	0
18		
1764		
7		
12348		

Queſt. 3. There are 7 Cheſts of Drawers; in each of which are 18 Drawers; and in each of theſe are 6 Diviſions; in each of which there is 16 *l* : 6 *ſh* : 8 *d.* How much Money is in the Whole? *Anſw.* 12348 *l.*

It is plain there muſt be a Sum of Money equal to the continual Product of 16 *l* : 6 *ſh* : 8 *d.* by 6, 18, and 7.

Queſt. 4. If 1 *l.* give 4 *ſh.* of Intereſt in any time; How much will 346 *l.* give in the ſame time? *Anſw.* 1384 *ſh* = 346 × 4 *ſh.* Here the 346 is applied to *l.* in the Propoſition, but is an abſtract Number in the Operation; which is not multiplying 4 *ſh.* by 346 *l.* but by the abſtract Number 346.

If this Queſtion be propoſed, *viz.* If 1 *ſh.* yield 4 *ſh.* Intereſt; How much will 4 *l* : 10 *ſh.* yield in the ſame time? It is plain it muſt be 4 *ſh.* taken as oft as 4 *l* : 10 *ſh.* contains 1 *ſh.* *viz.* 90 times, (for 4 *l* : 10 *ſh* = 90 *ſh.*) which makes 360 *ſh* = 90 × 4 *ſh.* But this is not multiplying 4 *ſh.* by 4 *l* : 10 *ſh.* which would be an abſurd Propoſition.

Again, Suppoſe the Queſtion were; If 1 *l.* yield 4 *ſh.* what will 4 *l* : 10 *ſh.* yield? The
four Anſwer is 18 *ſh.* equal to 4 *ſh.* taken as oft as 4 *l* : 10 *ſh.* contains 1 *l.* which is one and a half. But this, and all other Queſtions where Fractions come in, are not ſimple Queſtions of Multiplication. And as either of theſe Queſtions have an equal right to be called the Multiplication of 4 *s.* by 4 *l* : 10 *s.* it ſhews us how unreaſonable ſuch Propoſitions are, ſince it is the mixed Circumſtances of the Queſtion that determine how the Multiplication is to be made, which is different in different Circumſtances.

§. 5. DIVISION *of* WHOLE *and* APPLICATE NUMBERS.

INTRODUCTION.

BEfore we enter upon this Application, we muſt conſider the various Senſes that may be put upon *Diviſion.*

In the Definition, *Chap.* 6. there is but one Senſe expreſſed; but there are other three ways of propoſing a Queſtion in *Diviſion*, ſo dependent upon that in the Definition, that the ſame Number ſolves the Queſtion in all the Senſes in which it is a poſſible Queſtion.

The firſt Senſe is that in the Definition, *viz.* To find how oft one Number is contained in another. The ſecond is to find, What Part of the Dividend the Diviſor is equal to. The third is, To find a Number which is contained in the Dividend as oft as the Diviſor expreſſes. The fourth is, To find a Number which is ſuch a Part of the Dividend as the Diviſor expreſſes or denominates.

Now it will eaſily appear, That the Anſwers to all theſe Queſtions, or the Impoſſibility of ſome of them in ſome Caſes, is diſcovered by dividing according to the preceding Rule taken in the firſt Senſe: Thus,

1. Let us firſt ſuppoſe, That the Diviſion is without a Remainder, and all is plain: For the Number ſhewing how oft one Number is contained in another, (which is the firſt Senſe,) does, from the nature of an *aliquot* Part, denominate what Part the Diviſor is of the Dividend, (which is the ſecond Senſe.) Again, the ſame Quote is a Number contained in the Dividend at oft as the Diviſor expreſſes; (which is the third Senſe,) as has been ſhewn in the *Proof of Diviſion.* And hence, laſtly, it is ſuch a Part of the Dividend as the Diviſor denominates; (which is the fourth Senſe.)

Examp.

Examp. $12 \div 3 = 4$, and nothing remaining; that is, 3 is contained 4 times in 12, (the first Sense.) And it is the 4th Part of 12, (the second sense.) Again, 4 is a Number contained 3 times in 12, (the third Sense.) And it is the 3d Part of 12, (the fourth Sense.)

2. Suppose in the next place, That the Division has a Remainder; or, that the Divisor is not an *Aliquot* Part of the Dividend, [which includes that Case wherein the Divisor is greater than the Dividend.] Then the Question is Possible or Impossible, according to different Views and Limitations; as I shall here explain.

Let the Divisor be 3, and the Dividend 14; the Question is possible in the first Sense; and the Answer is 4 times, if we confine it to the number of times that the whole Divisor is contained in the Dividend: but taking it in a larger Sense, the Answer is $4 + \frac{2}{3}$. And in this View the Divisor may be greater than the Dividend: So if we ask how oft 14 is contained in 3, the Answer is $\frac{3}{14}$ Parts of a Time; the plain Sense of which is, that 3 contains $\frac{3}{14}$ Parts of 14.

In the second Sense, the Question supposes the Divisor is an *Aliquot* Part of the Dividend; and is therefore impossible when it is not so. But if we take a Part more largely for Part *Aliquot* or *Aliquant*, and ask what Fraction the Divisor is of the Dividend, then the Question is possible. But there is no new Question, strictly speaking; for it coincides with the first Question, changing the Dividend and Divisor, and taking *how oft* in the largest Sense. Thus; if we ask what Fraction 3 is of 14, the Answer is $\frac{3}{14}$; which is also the Answer, if we ask *how oft* 14 is contained in 3. And therefore I had rather in this Case call 3 the Dividend, and reduce it to the former Case; especially for this Reason, That the same Quote may be the Answer to all the really different Senses of the Question, while the Names of Divisor and Dividend are applied to the same Numbers.

In the third Sense, If we ask what is the greatest Number contained 3 times in 14; then if we limit it to a Whole Number, the Answer is 4. And if the Divisor is greater than the Dividend, the Question under this Limitation is impossible; as it also is if we should ask, what Whole Number is contained without a Remainder 3 times in 14; for this is contrary to supposition. But if we enlarge the Sense of the Question, and ask what Number, Whole, or Fraction, or Mix'd, is contained in the Dividend precisely as oft as the Divisor expresses, (so that the Product of the Divisor and Quote is equal to the Dividend,) the Question is always possible. Thus, as 3 is contained in 14, $4\frac{2}{3}$ times; so $4\frac{2}{3}$ is contained 3 times in 14; that is, 4 times any thing, and $\frac{2}{3}$ Parts of that thing is contained 3 times in 14 of that thing. For it has been already shewn, (in *Chap.* 6. *Schol.* 4. after the Definition of *Division*,) that B times $\frac{A}{B}$ Parts of any thing, is equal to $\frac{A}{B}$ Parts of B times that thing; therefore as 3 is contained in 14, $\frac{14}{3}$ times; so $\frac{14}{3}$ is contained 3 times in 14. Or, because $\frac{14}{3} = 4\frac{2}{3}$; therefore as 3 is contained $4\frac{2}{3}$ times in 14, so $4\frac{2}{3}$ is contained 3 times in 14.

In the fourth Sense, the Question is impossible in Pure and Integral Numbers, when the Divisor is not an *Aliquot* Part of the Dividend; so because 3 is not an *Aliquot* Part of 14, there is no Number which is a third Part of 14; for if there were, 3 would be such a Part of 14 as that Number expresses. But taking the Question more largely, and admitting Fractions, it is in all Cases possible: So $4\frac{2}{3}$ Parts of any thing is a 3d Part of 14, because it is contained in it 3 times.

As to the Circumstance which makes the third and fourth Sense possible, when the Divisor is not an *Aliquot* Part of the Dividend; it is remarkable, That the Subject of the Question is not pure Numbers, but such Quantitys expressed by Numbers as are divisible, either really or imaginarily, into Parts or lesser Quantitys: for in pure Numbers 14 has not a third Part; but considering the 14 as applied to some divisible Subject, the Quantity expressed by 14 has a third Part, which is expressed by $4\frac{2}{3}$; therefore the Question is possible only in Applicate Numbers.

 From

From what's explain'd we see evidently, that as all the four Questions or Views of Division, are possible when the Divisor is an Aliquot Part of the Dividend; so when it is not an Aliquot Part, there are but three really different Questions; and which are all possible when the Subject of the Question is not pure Numbers, *i. e.* When we admit another Consideration than that of the Number of Things expressed, *viz.* their Divisibility into Parts or lesser Quantities: For then a Fraction comes naturally into the Answer, and makes a compleat Quote.

Now, from these different Views and Senses of Division, we learn what Variety there can possibly be in the particular Application of Numbers for a Question of Division; of which there can only be two *Cases*.

1. To make a Question in the first or second Sense, the Divisor and Dividend must both be applicate, and both to things of the same kind: And mutually, if the Divisor and Dividend are so applied, the Question admits only the first or second Sense; and the Quote is an abstract Number, shewing how oft the Divisor is contained in the Dividend, or denominates what Part the Divisor is of the Dividend, if there is no Remainder. For, as it is a reasonable Question to ask, How oft one Number of any kind of Things is contained in, or what Part it is of another of the same kind of Thing; so a Question being proposed in this manner, and either Divisor or Dividend being applied to a particular kind of Thing, the Nature of the Question imports, that the other is also applied, and to the same kind of Thing; since it's absurd to ask, How oft a Number of one kind of Thing, as 3 Pounds, is contained in a Number of another kind of Thing, as 14 Days?

2. To make a Question in the third or fourth Sense, the Dividend must be an Applicate Number, and the Divisor Abstract, denominating what Part of the Dividend the Quote is, or how oft it is contained in it: so that the Quote is a Number applicate to the same kind of Thing as the Dividend; the Part of any Thing being of the same Nature as the whole. Again, mutually the Dividend being considered as applied, and the Divisor as abstract, the Question can admit only of the third or fourth Sense.

That the Application must be ordered in the manner now explained, may be also deduced from the Connection and Dependence of Multiplication and Division: For, since in Multiplication the Product and one Factor must be applicate to the same kind of Thing, the other Factor being abstract; and in Division the Divisor and Quote produce the Dividend: it follows, that the Dividend, with the Divisor or Quote, are alike applied, the other being abstract.

Again, *Observe*, That in mix'd Questions it will happen that two Numbers which in the Proposition are applied to different Things, must be divided one by the other: But in this Case, you'll always find that the Number made Divisor is considered in the Operation, as abstract, denominating what part of the other the Nature and Reason of the Question requires to be taken. So that in all Cases it's true, that the Divisor is either abstract, or applied to the same kind of thing as the Dividend.

We shall next explain the simple Practice in Division of *Applicate Numbers*.

CASE I. The Divisor and Dividend being both applied to the same kind of Thing.

Rule. Reduce, (if need be) the Divisor and Dividend to simple Numbers of one Name, (the lowest expressed in either Term;) then divide these Numbers by the general Rule. The Quote shews how oft the Divisor is contained in the Dividend, or what Part the Divisor is of the Dividend, when there is no Remainder.

Examp.

Examp. 1. 3*l.*) 24*l.* (8

Examp. 2. 3*l.*) 20*l.* ($6\frac{2}{3}$

Examp. 3. 3*sh.*) 12*l.*
or,
3*sh.*) 240*sh.* (80

Examp. 4. years, mo. ye. mo. da.
3 : 4) 24 : 7 : 18
By Reduction at 13 Months to a Year, and 28 days to 1 Month.
1204 *days*) 8950 *da.* ($7\frac{522}{1204}$
8428
———
522

Examp. 5. 13 *oz.* 8 *dr.*) 128 lb.
By Reduction,
216 *dr.*) 30208 *dr.* ($139\frac{184}{216}$
216
———
860
648
———
2128
1944
———
184

The *Reason* of this *Rule* is plain; for the Divisor and Dividend expressing Things of the same Value and Name, it is evident the Operation is to be managed as with abstract Numbers, by the General *Rule*; *i. e.* The Quantity expressed by the Divisor is contained in the Quantity expressed by the Dividend, as oft as the Divisor is contained in the Dividend, taken purely as Numbers. *Again*, If the Divisor and Dividend express Things of the same general Nature, which can be said to contain one another, then tho' they are not of one particular Species or Name, yet the Question is possible, only it requires that they be reduced to Numbers that express Quantities of one Species or Name; and then it is manifest, that the Division of these Numbers by the General *Rule*, gives the true Quote. So in *Exam.* 1. 3 *l.* is contained in 24 *l.* as oft as 3 in 24. But in *Examp.* 2. 3 *s.* is oftner contained in 12 *l.* than 3 in 12; for it is as oft as 3 in 240, the Number of Shillings equal to 12 *l.* And, because the Divisor and Dividend are then only in a State to be managed as pure Numbers, when they are both simple Numbers of one Name, this shews the Reason of reducing mix'd Numbers.

CASE 2. The Dividend being *Applicate*, and the Divisor *Abstract*.

Rule. If the Dividend is a simple Number, greater than the Divisor, divide it by the General Rule; the Quote is a Number of the same things as the Dividend: and if there is no Remainder, the Operation is finished; but if there is a Remainder, reduce it to the next Denomination, and divide; and so on, as long as there is a Remainder, and any lower Denomination, and make a Fraction of the last Remainder. Thus you have the Answer in one Species or several; which is an Applicate Number contained in the Dividend as oft as the Divisor expresses, or is such a part of the Dividend as the Divisor denominates. (See *Ex.* 1.)

Again, If the Dividend is a simple Number, less than the Divisor, you must first reduce it to a lower Species, till it be equal to, or greater than the Divisor, and then divide and proceed with the Remainder as before. (See *Ex.* 3, 4.) If it's not equal to the Divisor in any Species, then the Answer is a Fraction of the given Species, whose Numerator is the given Dividend. (*Ex.* 6.)

Lastly, If the Dividend is a mix'd Number, you may do the Work two ways: Either (1.) Reduce it to a simple Number of the lowest Species, and then divide; so you'll have the Answer in that Species, (which may be reduced again to superiour Species by Division, as has been formerly explained, and will be more particularly by and by.) But it will be better to proceed in this manner: (2) Begin with the Number of the highest Species in the Dividend; divide it, and reduce the Remainder to the next Species, taking in the given Number of that next Species; then divide; and so go on. (See *Ex.* 2.) But if the Number of the highest Species is less than the Divisor, reduce it, taking in the given Number of the next Species, and so on, till you have a Number equal to, or greater

greater than the Divisor: (*Ex.* 5.) And if that be not in any of the known Species, then the Answer is only a Fraction, whose Numerator is the Dividend, reduced to the lowest Species, and refers to an Unit of that Species. (See *Examp.* 7.)

Examp. 1.

3) 14*l.* (4*l.*
12
2 Rem.
20
40*s.* (13*s.*
39
1 Rem.
12
12*d.* (4*d.*

Examp. 2.

l. : *s.* : *d.*
6) 23 : 10 : 8 (3*l.*
18
5 Rem.
20
110*s.* (18*s.*
108
2 Rem.
12
32*d.* (5*d.*
30
2 Rem.
4
8*f.* (1$\frac{2}{6}$*f.*

Examp. 3.

24) 18*l.*
20
360*s.* (15*s.*
24
120
120

Examp. 4.

42) 16*l.*
20
320*s.* (7*s.*
294
26 Rem.
12
312*d.* (7*d.*
294
18 Rem.
4
72*f.* (1$\frac{30}{42}$*f.*
42
30

Examp. 5.

l. : *s.*
14) 8 : 15
20
175*s.* (12*s.*
168
7 Rem.
12
84*d.* (6*d.*
84

Examp. 6.

3460) 3*l.* ($\frac{3}{3460}$ of 1*l.* because 3 reduced to Farthings is but 2880*f.*

Examp. 7.

3460) 3*l.* 12*s.*
20
72 ($\frac{72}{3460}$ of 1*s.*

The *Reason* of this Practice is plain: For, if we find any proposed Part of all the Members of which any Number or Quantity is composed, we have the like Part of the Whole; since the Whole is nothing else but all the Parts. And if the Dividend is less than the Divisor, yet if reduced to another Species, it is equal to, or greater than the Divisor; it's plain that the equivalent Number in another Species being divided by the same Divisor, gives the true Answer: Thus 18*l.* = 360*s.* therefore the 24th part of 18*l.* is the 24th Part of 360*s.*

SCHOLIUM. What's already done shews the Practice of Division, or the Solution of simple Questions, where the Proposition is directly and simply to divide one Number by another, (in any of the Senses above explained.) And as to the Solution of mix'd Questions, all the further Direction can be given for knowing when Division is to be applied, is to *consider* well the Effect of Division as above explained: Which may be reduced to two principal Views, *viz.* Finding how oft one Number is contained in another, or finding a proposed aliquot Part of a Number. Then, when the Reason and Nature of a Question

Queſtion requires, that you find how oft one Number, ſimple or mix'd, of any kind of Thing, is contained in another Number of the ſame kind of thing; or, that you find ſuch a Part of any Number of things, as another Number denominates, or as an Unit of any Species of Things is of a certain given Number of the ſame Things; then is Diviſion your Work, as in the following Examples.

MIX'D PRACTICAL QUESTIONS *for* DIVISION.

A ſpecial Claſs of ſuch Queſtions is comprehended under Title of,

***REDUCTION** from a lower to a higher Species,* i. e. *To find a Number of things of a higher Species, equal in Value to a given Number of a lower Species; or, at leaſt, to find the greateſt Number of the higher contained in the Number of the lower, with what remains over of that lower Species: Suppoſing always, that an Unit of the lower Species is an aliquot Part of an Unit of the higher. For which this is the*

Rule. Divide the given Number by that Number which expreſſes how many Units of the Species to be reduced, are contained in an Unit of the Species to which it is to be reduced. The Quote is the Number ſought of that higher Species, and the Remainder is a Number of the Species reduced.

Thus you may reduce gradually from the loweſt to the higheſt Species; or all at once to the higheſt, if the Number of Units of the loweſt, which make one of the higheſt, is known. (As you may know by the Reduction Tables, explained in §. 4.)

In all the following Queſtions, I have performed the Diviſions by the contracted Methods, explained in *Chap.* vi. §. 2.

Examp. 1. In Money.

4	74608395 *f.*
12	18652098 *d* : 3 *f.*
20	1554341 *s* : 6 *d.*
	77717 *l* : 1 *s.*

Here 74608395 *f.* being divided by 4, Quote 18652098 *d.* and 3 *f.* over; theſe *d.* divided by 12, Quote 1554341 *s.* and 6 *d.* over; theſe *ſh.* divided by 20, Quote 77717 *l.* and 1 *s.* over. Wherefore 74608395 *f.* = 18652098 *d.* 3 *f.* = 1554341 *s.* 6 *d.* 3 *f.* = 77717 *l.* 1 *s.* 6 *d.* 3 *f.*

By this you know how to explain the following Examples.

Examp. 2. Troy Weight.

24 {	4	3476589426 *grains.*
	6	869147356 — 2
	20	144857892 *dw.* : 18 *gr.*
	12	7242894 *oz.* : 12 *dw.*
		603574 lb. : 6 *oz.*

Examp. 3. Long Meaſure.

	12	46320689372589 *inches.*
	3	3860057447715 *feet* : 9 *inch.*
220 {	11	1286685815905 *yd.* : 00
	20	116971437809 — 6
	8	5848571890 *furl.* : 105 *yd.*
		731071486 *miles* : 2 *furl.*

The *Reaſon* of Dividing in theſe Caſes is obvious: For *Ex.* Since 4 Farthings = 1 *d.* therefore as many times as 4 *f.* are contained in any Number of *f.* ſo many *d.* is that Number of *f.* equal to. In which obſerve, that the immediate Effect of the Diviſion is an abſtract Number, ſhewing how oft 4 *f.* is contained in a greater Number of *f.* and we apply the Name of *d.* to the Quote, from the Reaſon of the Queſtion, as now explain'd.

Queſt. 1. 8 Men have equal Shares of a Stock of 146 *l.* 16 *s.* what is each Man's Share? Anſwer, 18 *l.* 7 *s.* *viz.* the 8th Part of 146 *l.* 16 *s.* For the Nature of the Queſtion plainly directs us to take an 8th Part. Where obſerve, that tho' 8 expreſſes a Number of Men in the mixt Propoſition, yet in the Operation it is conſidered abſtractly as the Denomination of that Part of 146 *l.* 16 *s.* which the Nature of the Queſtion requires to be taken.

8) 146 *l.* : 16 *s.*
18 : 7

Quest. 2. A certain Number of Persons, each of whose Ages is 15 Years 3 Months (reckoning 13 Months to 1 Year) make up in all 182 Years, 10 Months; how many Persons are there? Answer, 12. *viz.* The Number of times that 15 Years, 3 Months, are contained in 182 Years, 10 Months.

```
y.  m.   y.   m.
15 : 3 ) 182 : 10
 By Reduction.
m.     m.
198 ) 2376 ( 12
      198
      ---
       396
       396
```

Quest. 3. What is the Value of 1 Yard of Cloth, whereof 48 Yards cost 15 *l.* 10 *s.* 4 *d.* Answer, 6 *s.* 5 *d.* 2 $\frac{1}{3}$ *f.* = $\frac{1}{48}$ Part of 15 *l.* 10 *s.* 4 *d.*

```
      l.    s.   d.   f.
  {6| 15 : 10 : 4 : 0
48{8|  2 : 11 : 8 : 2 4/6
    |      :  6 : 5 : 2 16/48 or 1/3
```

For it's plain, the Value of 1 Yard of 48 Yards, must be the 48th Part of the Value of the whole 48 Yards, which directs us to Division, or taking a 48th Part of 15 *l.* 10 *s.* 4 *d.* And so this 48, which in the mix'd Proposition is applied to Yards, is considered abstractly in the Operation; which is therefore not a Division of Money by Yards, which cannot be made in any Sense, but taking such a Part of the Money as 1 Yard is of 48 Yards, *viz.* a 48th Part.

Observe, Had it been proposed, in the last Question, to find the Value of 1 Quarter of a Yard, we may do it either by finding first the Value of 1 Yard, and then the 4th Part of this is the Value of 1 Quarter; or, by reducing 48 Yards to Quarters, which make 192; and taking the 192^d Part of the given Money. The Reason for which is the same as for the other Case.

The following Question *requires all the four Operations of Arithmetick.*

Quest. A Father left among 5 Sons an Estate, consisting of 500 *l.* in Cash; with 5 Bills, each of 48 *l.* 10 *s.* 6 *d.* He ordered 20 *l.* to be bestowed upon his Burial, and his Debts to be paid, amounting to 164 *l.* Then his free Estate to be divided in this manner, *viz.* The eldest Son to have the 3^d Part, and the other 4 Sons to have equal Shares. What is the Share of each Son? Answer 186 *l.* 4 *s.* 2 *d.* to the eldest; and 93 *l.* 2 *s.* 1 *d.* to each of the rest.

```
   Operation.                 l.
 48l. : 10s. : 6d.            20
              5 Bills.       164
 -------------------         ---
  242 : 12 : 6               184
  500 : 00 : 0 Cash.
 -------------------
  742 : 12 : 6 Total
  184 : 00 : 0 deduced.
 -------------------
3)558 : 12 : 6 Free Estate.
  186 :  4 : 2 Eldest Son.
 -------------------
4)372 :  8 : 4 Remains.
   93 :  2 : 1 the Share of
     each of the other 4.
```

SCHOLIUM. As Questions may be variously mix'd, so the Solution will depend upon a due Consideration of the several Parts of the Question, and what Operation each may require, according to the Sense and Effect of the different Operations in Arithmetick. But there are mix'd Applications of Multiplication and Division, which require other Rules and Directions, to know when and how they are to be made; these you'll learn in **Book** 4 and especially in *Book* 6. under the Name of *Proportion.* What has been already done in this Book, being sufficient for explaining the Nature of the fundamental Operations, and their more simple Applications in whole Numbers. For the Doctrine of Fractions, you have it in the next Book.

CHAP.

CHAP. VIII.

Containing the more particular Rules of the LITERAL ARITHMETICK, *neceſſary in the following Parts of this Work.*

I. *For* ADDITION *or* SUBTRACTION.

CASE I. If the Numbers to be added or ſubtracted are expreſſed all by the ſame Letter, multiplied by certain Numbers, as, $3a$, or $5b$, add or ſubtract the Coefficients (*i. e.* the Numbers by which the Letters are multiplied) and multiply the ſame Letter into the Sum or Difference, it is the Sum or Difference ſought.

Examp. $3a + 5a = 8a$. *Ex.* $2b + 3b + b = 6b$. *Ex.* $5n - 2n = 3n$.
Ex. $4ab + 3ab = 7ab$. *Ex.* $3ab - ab = 2ab$.

SCHOLIUM. In order to underſtand the other Caſes, we muſt premiſe this *Obſervation*, viz. After the Addition or Subtraction of one Number to, or from another, we may ſuppoſe another added to, or ſubtracted from the preceding Sum or Difference; and another added to, or ſubtracted from the laſt Sum or Difference, and ſo on: Then is this whole Work expreſſed by ſetting the Numbers, or Letters repreſenting them, in the ſame order, with the proper Signs of the ſeveral Operations betwixt them. Thus, if b is added to a, and from the Sum c is ſubtracted, and from this Difference d ſubtracted, and to this laſt Difference e added; the Reſult of all this is expreſſed thus, $a + b - c - d + e$. But again, *Obſerve*, That if the ſame Numbers can be added or ſubtracted in any other order, the final Reſult or Effect will ſtill be the ſame; which, in all the poſſible Orders wherein the Operations can be made, is plainly the Difference betwixt the Sum of all theſe Terms that are added in the ſeveral Steps, and the Sum of all theſe that are ſubtracted: Becauſe in whatever order any Numbers are added and ſubtracted, it's evident there is ſo much in whole added, as the Sum of all theſe that are added in the ſeveral Steps, and as much ſubtracted in whole, as the Sum of all that are ſubtracted in the ſeveral Steps: Wherefore, the final Reſult is the Difference of theſe Sums. Whence, again, this follows, That 'tis no matter in what order we place the ſeveral Terms of a mix'd Expreſſion, if we always prefix the ſame Signs to the ſame Letters, and alſo take the Meaning of the Expreſſion to be univerſally the ſubtracting all theſe Terms that have $-$ prefix'd, out of the Sum of all theſe that have $+$ prefix'd: So that when the Operations can be performed in a propoſed Order, we may explain the Expreſſion either according to that Order, or in the preceding general way, (if that is not the propoſed Order.) And if they cannot be performed in the propoſed Order, then we explain it after the general way, as the univerſal Meaning of all ſuch Expreſſions; for tho' ſome may be explain'd another way, yet the final Reſult is always equal to this.

For *Example*, If b is greater than a, then $a - b + c$ can't be explained in the Order of theſe Letters and Signs; and if it is at all poſſible, it is ſo in this Order, $a + c - b$; yet it may repreſent the ſame thing ſtanding thus, $a - b + c$, while we do not ſo much regard

the Order, as the general Meaning of the Signs, which is as if it were in this Order, $a+c-b$.

CASE II. If any complex Expression [whether it is a Sum, having all its Terms joined by the Sign $+$; or a Difference, having its Terms partly $+$, partly $-$] is to be subtracted from another, (expressed simply or complexly) the Difference sought may be expressed two ways.

Rule 1. By drawing a Line over the whole Terms of the Subtractor, and joining it to the Subtrahend with a Mark of Subtraction between them. Thus, the Difference of a and $b+c$ is expressed $a-\overline{b+c}$, signifying that b and c both, or their Sum, is taken from a, which is a quite different thing from $a-b+c$ without the Line, which would signify the Difference betwixt $a+c$ and b. *Again*, if the Subtractor is the Expression of a Difference, as $b-c$ from a, the Difference sought is expressed $a-\overline{b-c}$, signifying, that the Difference of b and c is taken from a; *i. e.* That c is taken from b, and the Remainder taken from a, which is a different thing from $a-b-c$, which expresses the Difference of a and b, c both, *i. e.* That b and c are both taken from a.

Now, tho' this Method is sometimes convenient, yet it would often prove too general and indefinite, which is supplied by another *Rule*, whereby, from the simple Terms of the given Subtractor and Subtrahend, another Expression is found equal to the Difference sought. Thus,

Rule 2. Change the Signs of all the Terms of the Subtractor, and join them to the Subtrahend without a Line over them; and this expresses a Number equal to the Difference sought. Thus, if the Subtractor is a Sum, as $b+c$, the Difference of this and a is $a-b-c$ $(=a-\overline{b+c}.)$

The *Reason* is plain; for the Sum is subtracted, when all the Parts are subtracted one after another; since the Parts are equal to the Whole.

Again, If the Subtractor is a Difference, as if $b-c$ is to be taken from a, the Remainder is $a-b+c$ $(=a-\overline{b-c})$ which more directly represents the Difference betwixt $a+c$ and b, yet is equal to the Difference of a and $b-c$.

The *Reason* is plain; for by adding the lesser Term c, and then taking away the greater
c b a, we take away as much as was before added, and also all that a exceeds b: Or, if b does not exceed a, we may first take b from a, and to the Difference add c; for thus we restore all that was taken away, except so much as b exceeds c: And so both ways, the Thing really taken away is precisely what b exceeds c, or their Difference. If the Subtractor is a more complex Expression, or consists of more than two Terms added and subtracted, the Reason of the Rule is still the same, from what has been explained, *viz.* That such Expressions signify no more in Effect, than the Difference of the Sum of all that are added, and the Sum of all that are subtracted. Wherefore, by what's now shewn, all that are added in the Subtractor must, in expressing the Difference sought, be subtracted; and all that are subtracted in it, must be added: Thus, $a-\overline{b+c-d-e}=a-b-c+d+e$, or $a+d+e-b-c$.

SCHOLIUM. The last *Case* of this *Rule* may be considered as a *Theorem*, and expressed thus: If the Difference of two Numbers is subtracted from a third Number, the Remainder is the same, as if we added the lesser of these two Numbers to the third, and from the Sum subtracted the greater. So $a-\overline{b-c}=a+c-b$.

Sum of

CASE III. If any complex Expression of a Difference is to be added to any other Expression, join them without changing their Signs, or any Line over them. *Examp.* If to a we add $b+c$, the Sum is $a+b+c$. Again, to a add $b-c$, the Sum is $a+b-c$. To $a-b$ add $c-d+e$, the Sum is $a-b+c-d+e$ $(=a+c+e-b-d.)$

The

The *Reason* of this *Rule* is evident, when the Expression added has all its Terms $+$, as in *Examp* 1. And if they are some $+$ some $-$, as in *Ex.* 2. the Reason is this; we are to add $b-c$; but to add b would be too much by c, therefore out of the Sum $a+b$, we must take away c: which is the same in Effect, as if the c had been first taken from b, and the Difference added to a. And here *observe*, that tho' a Line is drawn over the complex Expression added, it alters not the Effect: So $a+b-c$ is the same in Effect as $a+\overline{b-c}$; since both ways the Difference of b and c is added to a, as has been explained.

Now, to sum up these *Cases* and *Rules* in one *General Rule*, it is this: To *Add*, join the Expressions without changing their Signs; and to *Subtract*, join them, changing all the Signs of the Subtractor; or draw a Line over the Subtractor, without changing the Signs, only join the whole Expression thus united (and, as it were, made one Expression by the Line) by the Sign $-$ betwixt it and the Subtrahend. *Lastly*, If the Numbers added or subtracted are expressed by the same Letters with Coefficients, or particular Numbers prefix'd, add or subtract these Numbers, and prefix the Sum or Difference to the common Literal Part.

II. For Multiplication.

Case I. When two Numbers are expressed by any Letters, with particular Numbers prefix'd, (or, multiplying them) then if two or more such are to be multiplied together, multiply all the Coefficients, and prefix the Product to the Product of the Literal Expressions. Thus, $4a\times 3b=12ab$; also, $2ab\times 5ac=10a^2bc$.

The *Reason* is obvious; for it is only a continual Multiplication, which may be done in any Order.

Case II. If the Multiplier and Multiplicand, one or both, are complex Expressions, the Product may be expressed by the general Rule, thus: Draw a Line over the complex Terms, and join them by the general Sign of Multiplication $\times$. *Example*, To multiply $a+b$ by $c+d$, the Product is $\overline{a+b}\times\overline{c+d}$; or, $a+b$ by $c-d$ makes $\overline{a+b}\times\overline{c-d}$. Which would be very different Expressions, if any of the complex Terms wanted the Line (or *Vinculum*, as the Algebraists call it;) thus $a+b\times\overline{c+d}$ would be the Sum of a, and the Product of b into $\overline{c+d}$; and $\overline{a+b}\times c-d$, is the Difference of d, and the Product $\overline{a+b}\times c$.

So that we are to understand the Sign of Multiplication to refer only to the first of the simple Terms on either hand, unless two or more of them are joined by a *Vinculum*. And here too *observe*, That when several Letters stand together, with, or without the Sign of Multiplication, (whereby they also express the Product of these Letters) this is to be reckoned but one Term, with respect to the following or preceding Sign, whether of Multiplication, Addition, *&c.* as $ab+d$, or $\overline{a+b}\times cd$. And mind also, that all the Terms joined together by Multiplication or Division, upon the Right-hand of the Sign of Addition or Subtraction, make but one Term to which that Sign refers; as $a+bc\times\overline{c+d}$ which is not to be understood as if $a+bd$ were multiplied by $\overline{c+d}$, which then would be made $a+bc\times\overline{c+d}$; but it is the Sum of a and the Product of bc by $\overline{c+d}$.

Observe, Tho' this general way of representing the Products of complex Expressions is often convenient, yet there is another *Rule* more useful, whereby the Product is not expressed so indefinitely, but all reduced to more simple Terms without any *Vinculum*.

Another RULE *for* Complex Expressions.

Multiply each simple Term of the Multiplier by each simple Term of the Multiplicand; (according to the general Rule.) And if the Signs before each of the two simple Terms, multiplied together, are the same or like (*viz.* both $+$ or both $-$) prefix the Sign $+$ to the Product; but if they are unlike (*viz.* the one $+$ and the other $-$) prefix the Sign $-$. The following *Examples* sufficiently shew the Application.

Examp. 1. $a \times \overline{b+c} = ab+ac$. *Ex.* 4. $\overline{a+b} \times \overline{c-d} = ac-ad+bc-bd$.

2. $\overline{a+b} \times \overline{c+d} = ac+ad+bc+bd$ 5. $\overline{a-b} \times \overline{c-d} = ac-ad-bc+bd$.

3. $a \times \overline{b-c} = ab-ac$.

DEMONSTRATION.

1. Where all the Signs are $+$, as in *Examp.* 1. and 2. the Reason is plain, and it has also been shewn in *Lemma* 3. *Chap.* 5. *Book* 1.

2. If one of the Terms is simple, or a Sum, and the other a Difference, as in *Ex.* 3, 4. then, for *Ex.* 3. to multiply $\overline{b-c}$ by a, it is evident that ab is too much; for we must take only a times the Difference of b and c; or what b exceeds c; therefore if we take ac out of ab, the Remainder is a times the Number by which b exceeds c. Or thus, Let $b=c+d$, then $ab = a \times \overline{c+d} = ac+ad$ (by the 1st *Ex.*) from which take ac, there remains ad, *viz.* the Product of a into the Difference of b and c. *Again*, If a Difference $\overline{c-d}$ is to be multiplied by a Sum $\overline{a+b}$, the Reason is the same for the Multiplication of each Term of the Sum into the Difference: As in *Ex.* 4.

3. If both Multiplier and Multiplicand are Differences, as in *Ex.* 5. then having multiplied $\overline{c-d}$ by a, (as in *Ex.* 3.) the Product is $ac-ad$, which is a times $\overline{c-d}$, or $\overline{c-d}$ times a: But, if instead of $\overline{c-d}$ times a, we ought to take only $\overline{c-d}$ times $\overline{a-b}$, therefore, reasoning as in *Ex.* 3. it's plain we have taken too much by $\overline{c-d}$ times b, or $cb-db$; and this being taken from the preceding Product $ac-ad$, the Remainder is the true Product, *viz.* $ac-ad-\overline{cb-db} = ac-ad-cb+db$, (by the Rules of Subtraction) which is according to the Rule.

Or we may reason thus: Instead of taking a times $\overline{c-d}$, which is $ac-ad$, we ought to take only $\overline{a-b}$ times $\overline{c-d}$; therefore we have taken too much by the Product of $\overline{c-d}$ by b, which is $cb-db$. Or also thus: Let $a=b+n$, then $a \times \overline{c-d} = \overline{b+n} \times \overline{c-d}$, $= b \times \overline{c-d} + n \times \overline{c-d}$, (by Artic. 1. taking $c-d$ as one single Term.) And n being the Difference of a and b, therefore, $n \times \overline{c-d}$ is the Product sought: Consequently $b \times \overline{c-d} + n \times \overline{c-d}$, or its Equal $a \times \overline{c-d}$, exceeds it by $b \times \overline{c-d} = bc-bd$; which taken from $ac-cd$, leaves $ac-ad-bc+bd$, as before.

Observe, If any of the Terms are more complex than these *Examples*, yet the Reason is the same in all; because they are nothing else but a Sum or a Difference.

III. For DIVISION.

ALL the Use that is to be made of the Literal Division in the following Work, requires only, that to the General Rule I add these two Observations.

1.

1. If the ſame Letter, or Expreſſion whatever, is multiplied into all the Terms of both Diviſor and Dividend, by putting it out of both, the Quote is thereby expreſſed more ſimply. Thus, $\frac{ab}{a \times \overline{c+d}} = \frac{b}{c+d}$ And mind, that if one Term of the Diviſor or Dividend is multiplied into all the reſt, then, in putting out that Multiplier, ſet 1 in the Place where it ſtood alone; thus, $\frac{ab+a}{ac-ad} = \frac{b+1}{c-d}$ and $\frac{3a+6}{3bc} = \frac{a+2}{bc}$, for $6 = 3 \times 2$. The Reaſon of this will appear in the 4th Book. See *Chap.* 1. *General Corollary* 21. And if all the Letters repreſent Integers, ſo that the Quote thus repreſented has the proper Form of a Fraction; the Reaſon of this *Rule* will be ſeen in *Book* 2. *Chap.* 1. *Lemma* 5. *Cor.* 3.

2. If a Quote is expreſſed by the Sign $\div$ or) ſet betwixt Diviſor and Dividend, it's to be referred only to the ſimple Terms on each hand, or which have a *Vinculum*: So $a+b \div c$ is the Sum of a, and the Quote $b \div c$. But however, many ſimple Terms (or ſuch as become one Term by a *Vinculum*) are continuouſly joined by Signs of Multiplication and Diviſion; we may explain them all in order as they ſtand, one after another. Thus $a \times b \div c \times d \div e$, may be underſtood thus, *viz.* a multiplied by b, and this Product divided by c, and this Quote multiplied by d, and this Product divided by e: Or thus, The Product ab divided by the Product cd, and this Quote divided by e: Or, a multiplied by the Quote $b \div c$, and this Product multiplied by the Quote $d \div e$. All which are equivalent. The Reaſon of which will appear from the Rules of Fractions, when we expreſs the Quotes fractionally, thus, $a \times \frac{b}{c} \times \frac{d}{e}$; and this Method is generally moſt convenient, as it leaves no Sign but that of Multiplication.

The Application of this univerſal Notation, and it's Operations, in our Reaſoning about Numbers, is made by means of theſe few ſimple and eaſy Principles, or

AXIOMS.

1. If to any Number another be added, and from the Sum be ſubtracted the Number added, the Remainder is the firſt Number: Or it is the ſame thing, if we firſt ſubtract and then add. Thus, $a+b-b=a$; or, $a-c+c=a$.

2. The ſame or equal Numbers added to the ſame or equal Numbers, make the Sums equal. So if we ſuppoſe $a=b$, then $a+d=b+d$; if $a+b=c+d$, then $a+b+n=c+d+n$; if $a=b-c$, then $a+c=b$. (For $b-c+c=b$ by the laſt.)

3. The ſame or equal Numbers ſubtracted from the ſame or equal Numbers, make the Remainders equal. So if $a=b$, then $a-d=b-d$; if $a+b=c+b$, then $a=c$; if $a+b=c$, then $a=c-b$; if $a+b=c-b$, then $a=c-2b$.

4. If a Number is multiplied by another, and the Product divided by the ſame Number, (or firſt divide and then multiply) the Quote (or Product) is the firſt Number. So $a \times n \div n = a$.

5. If the ſame or equal Numbers are multiplied equally, the Products are equal. So if $a=b$, then $an=bn$; if $a+b-c=dn+e$, then $r \times \overline{a+b-c} = r \times \overline{dn+e}$, or $ra+rb-rc=rdn+re$; If $\frac{a}{b}=q$, then $a=bq$, (for $\frac{a}{b} \times b = a$ by the laſt.) And this Caſe is the ſame as the Proof of Diviſion.

6. If the ſame or equal Numbers are divided equally, the Quotes are equal. So if $a=b$, then $a \div d = b \div d$, or thus, $\frac{a}{d} = \frac{b}{d}$, and if $a=bq$, then $\frac{a}{b}=q$. (For $bq \div b = q$, by the 4th.)

Obſerve, The Truth contained in theſe *Axioms* are univerſal, whether the ſuppoſed Numbers are Integers or Fractions.

BOOK

BOOK II.

Of FRACTIONS.

CHAP. I.

Containing the GENERAL PRINCIPLES *and* THEORY.

WHAT a *Fraction* is, and the *Notation* of it, has been already explained. I have shewn wherein the essential Difference betwixt *Integral* and *Fractional Numbers* does consist; and have observed, that there cannot be any other Operations in *Arithmetick* but those of *Whole Numbers*; and that the Ground and Reason of different Rules for the management of Fractions lies in their relative Nature and Value. But now more particularly *observe*, That from this relative Nature it follows, that the same Quantity may be fractionally expressed under a variety of different Forms, which in most Cases requires some preparatory Work for reducing the Numbers proposed into a like Form, before the common Operations of Addition, &c. can be performed. The first thing therefore to be done, is to explain the several Distinctions of Fractions, with the general *Theory* of their Nature; and then the Reductions of them from one Expression to another.

Observe, For brevity I contract the word *Numerator* into Num^r. and the word *Denominator* into Den^r.

DEFINITIONS.

From a Comparison of the Num^r to the Den^r, as a Part (taken more generally) to the Whole; Fractions are distinguished into *Proper* and *Improper*.

1. A *Proper Fraction* is that of which the Num^r is less than the Den^r, as $\frac{2}{3}$; and is called *Proper* with respect to the relative Integer, because it expresses a Quantity less than it, (as has been already explained in a *Corollary* to the Definition of a Fraction;) as if the true and proper Signification of the word Fraction were, [a Part or Quantity lesser than another.

2. An *Improper Fraction* is that of which the Num^r is equal to, or greater than the Den^r. as $\frac{3}{3}$, or $\frac{4}{3}$, and is called *improper* with respect to the relative Integer, because it expresses a Quantity greater than it, and is therefore not a Part of it in any sense. But taking the word *Fraction* as I have defined it, there is no such Distinction; for each Unit of the Num^r is an *Aliquot* Part of the Integer, and the Whole is a Number of such Parts: And since the applying a relative Value or Denomination to any Number, makes it a fractional

fractional Number; therefore all ſuch are equally true and proper, according to this latter Senſe: ſo that wherever this diſtinction is applied, the meaning of the word *Fraction* is reſtrained; or, without minding that, we may take the Terms *Proper* and *Improper*, to ſignify no more than a Diſtinction of theſe different Circumſtances, *viz.* the *Numerator's* being leſs or not leſs than the *Denominator*.

Fractions are alſo diſtinguiſhed into *Simple* and *Compound*.

3. A *Simple Fraction* is one ſingle Fraction, referred immediately to ſome Integer; as $\frac{2}{3}$, or $\frac{7}{4}$ of any thing.

4. A *Compound Fraction* is a Fraction of a Fraction, conſiſting of two or more ſimple Fractions referred to one another in order, and the laſt referred to ſome Integer; as $\frac{2}{3}$ of $\frac{4}{5}$ of any thing; or $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{7}{8}$ of any thing; the Particle, *of*, being the Mark of a Compound Fraction.

SCHOLIUM. One Fraction may be either an *Aliquot* or *Aliquant* Part of another, as well as one whole Number is of another; ſo that a Fraction which is $\frac{1}{3}$ of $\frac{4}{5}$, is an *Aliquot* Part of $\frac{4}{5}$; but $\frac{2}{3}$ of $\frac{4}{5}$ is *Aliquant*.

5. A *Whole Number* with a Fraction annexed, is called a *Mix'd Number*, and if the Fraction is referred to an Unit of the ſame thing that the whole Number repreſents, then they are ſet together without any mark of Addition, that being underſtood; for Example $46\frac{2}{3}$ *l.* but if the Fraction is not referred to an Unit of the ſame thing, they muſt be ſeparated, that the name of each thing may be diſtinctly apply'd, as if it were 46 *l.* $+\frac{2}{3}$ *h.*

SCHOLIUM. In Abſtract Numbers when there is no particular thing named, a *Mix'd Number* is always underſtood in the firſt ſenſe, (*i.e.* the Fraction is ſuppoſed to relate to an Unit of the ſame thing, which the whole Number repreſents) and ſo it's written without any mark of Addition, as $24\frac{3}{5}$. Obſerve alſo, that if we ſuppoſe (as we ſhall immediately prove) that two Fractions expreſs'd in different Numbers may be equivalent, then the ſame integral Number, with the ſame or equivalent Fraction, makes the ſame or equivalent improper Fraction.

COROLLARIES.

1. Every *Improper Fraction* is equal to ſome whole or mix'd Number, and particularly, if the *Numerator* and *Denominator* are equal, the Fraction is equal to 1; for then you take as many Parts as the Integer contains, that is, the whole Integer or Unity, ſo $\frac{4}{4}=1$. And where the Numr is greater than the Denomr, the Fraction is greater than Unity, (for the Integer.) But how to reduce it, or find the equivalent whole or mix'd Number, we ſhall learn afterwards.

2. Every compound Fraction is equal to ſome ſimple Fraction, for that which is a Part of a Part, is certainly a Part of the Whole, and we ſhall ſee below how to find that ſimple Fraction.

AXIOMS.

1. The like Fractions of two equal Quantities or Numbers are equal; that is, if $A=B$ then $\frac{n}{m}$ of $A=\frac{n}{m}$ of B.

2. If two Fractions are ſuppoſed to be equal; and if alſo one of their ſimilar Terms be equal, the other is ſo too; thus if $\frac{a}{b}=\frac{a}{d}$ then is $b=d$.

All the other common Axioms of Numbers, hold in Fractions as well as Integers.

LEMMA I.

The greater or lesser the Num^r of a Fraction is with the same Den^r, the greater or lesser is the Value of the Fraction. Or thus,

Two Fractions with the same Denom^r and different Num^{rs}, represent quantities of different Value; and the greatest Num^r makes the greatest Fraction. *Examp.* $\frac{3}{5}$ is greater than $\frac{2}{5}$ (of the same thing:) For the Num^r being the direct Number of things expressed, and the Denom^r being the same, the Value of each Unit of the Num^r is the same, and therefore the greater Num^r makes the greater Value in the whole.

But more particularly, if the one Num^r is Multiple of the other, that Fraction is Equi-multiple of the other; so $\frac{4}{7}$ is double of $\frac{2}{7}$ because 4 is double of 2. For the Value of each Unit of the Num^{rs} being equal, the comparison of them is the same as if they were pure abstract Numbers.

COROLL. A Fraction is multiplied or divided by any Integer, if we multiply or divide its Num^r, and this is a proper Multiplication or Division of the direct Number of things represented: So $\frac{3}{7}\times 2=\frac{6}{7}$ and $\frac{6}{7}\div 2=\frac{3}{7}$; and universally $a\times\frac{b}{c}=\frac{ab}{c}$ and $\frac{ab}{c}\div a=\frac{b}{c}$ or the $\frac{3}{a}$ part of $\frac{ab}{c}=\frac{b}{c}$.

Observe, That the Division is supposed here to be without a Remainder, for otherways a Fraction cannot be divided by dividing its Num^r. Because the compleat Quote being a mix'd Number, it cannot be the Num^r of a Fraction in proper terms. *Observe* also that when the Num^r of a Fraction is 1, it is multiplied by any Number, by placing that Number instead of the 1: thus, n times $\frac{1}{a}$ Part is $\frac{n}{a}$ Parts; the truth of which needs not this *Lemma*, but is comprehended in the very Nature and Idea of a Fraction. So $\frac{3}{4}$, or 3 4th Parts, is an equivalent Expression for 3 times $\frac{1}{4}$ Part, as every Number of any kind of things signifies so many Units of that kind.

LEMMA II.

Any Fraction of any Number is equal to the Sum of the like Fractions of all the lesser Numbers of which that Whole is composed. For *Examp.* $20=12+8$. therefore $\frac{1}{4}$ of 20 $(=5)$ is $=\frac{1}{4}$ of 12 $(=3)$ $+\frac{1}{4}$ of 8 $(=2.)$ Also $\frac{3}{4}$ of 20 $(=15)$ is $=\frac{3}{4}$ of 12 $(=9)$ $+\frac{3}{4}$ of 8 $(=6.)$ Or *Universally*, If $M=A+B+C$, *&c.* then $\frac{1}{m}$ of M is $=\frac{1}{m}$ of $A+\frac{1}{m}$ of $B+\frac{1}{m}$ of C. Or, $\frac{n}{m}$ of $M=\frac{n}{m}$ of $A+\frac{n}{m}$ of $B+\frac{n}{m}$ of C. [How several Fractions are added together in one simple Fraction, we learn afterwards; all that is designed here, is a general Truth, concerning a number of Fractions; for whatever way they are expressed, the general Idea is the same thing.

The *Reason* of this Truth is very plain; for the Whole being nothing else but all the Parts, when you have taken the $\frac{1}{4}$ or $\frac{3}{4}$, (or generally, $\frac{1}{n}$ Part, or $\frac{n}{m}$ Parts) of each Member of the Whole, you have taken the like Part or Parts of the Whole.

SCHOL. We may also express this Truth in this manner, *viz.* If one Quantity is made up of a number of other Quantities, $A+B+C$, and another made up of as many $a+b+c$, *&c.* which are respectively lesser than the former, and which are each equal to the same Fraction of their Correspondents in the other, (*i. e.* a of A, and b of B, *&c.*) then is the Whole $a+b+c$, *&c.* equal to the same Fraction of the Whole $A+B+C$, *&c.* that is, the Sum of the like Fractions of any two or more Numbers, is the like Fraction of the Sum of these Numbers.

COROLLARIES.

1. Any Fraction of any Number is equal to that Number of times the like Fraction of 1; for *Examp.* $\frac{2}{3}$ of 2 is $= 2 \times \frac{2}{3}$ of 1 ($= \frac{4}{3}$ of 1 *Corol. Lem.* 1) Also $\frac{3}{7}$ of 2, is $= 2 \times \frac{3}{7}$ of 1 ($= \frac{6}{7}$ of 1.) Or *Universally*, $\frac{n}{m}$ of a is a times $\frac{n}{m}$ of 1 ($= \frac{an}{m}$ of 1, by *Cor. Lem.* 1.) For $\frac{n}{m}$ of a, is the Sum of the $\frac{n}{m}$ Parts of every lesser Number, which make up a, *i. e.* of every Unit in a, which is $\frac{n}{m}$ of 1, as often taken as there are Units in a, or a times $\frac{n}{m}$ of 1; that is, $\frac{n}{m}$ multiplied by $a = \frac{an}{m}$ of 1. (*Cor. Lem.* 1.) [And let it be always minded, that when the Fraction of a Number is proposed which has no such Fraction in pure Numbers, we must have recourse to applicate Numbers, so that the Number proposed is conceived to represent a Quantity subdivisible so as to have the Part proposed.]

SCHOL. In this *Corollary* we have a compleat Demonstration of what we have in part supposed in Division of whole applicate Numbers, and referr'd to this Place, *viz.* That any Part of any Number of Things is equal to that Number of Times the like Part of one of these Things. *Examp.* That $\frac{2}{3}$ of 2 $l = \frac{4}{3}$ of 1 $l.$ or $\frac{1}{a}$ of $n = \frac{n}{a}$ of 1.

2. Any Fraction referred to any Number is equal to a Fraction whose Num^r is that given Number, and its Den^r the given one, and which is referred to the given Num^r as the Whole. Thus, $\frac{3}{5}$ of $2 = \frac{2}{5}$ of 3: or generally, $\frac{a}{n}$ of $m = \frac{m}{n}$ of a. For $\frac{a}{n}$ of m is $= \frac{ma}{n}$ of 1; and $\frac{m}{n}$ of $a = \frac{am}{n}$ of 1; (*per* the last.) Consequently, $\frac{a}{n}$ of $m = \frac{m}{n}$ of a.

3. From the two last this follows again, That any Fraction of any Number is an *Aliquot* Fraction having the same Den^r, and referred to the Product of the Num^r and the given Number, as the Whole: Thus, $\frac{2}{3}$ of $5 = \frac{1}{3}$ of 10. *Universally*, $\frac{a}{n}$ of $b = \frac{1}{n}$ of ab For $\frac{a}{n}$ of $b = b$ times $\frac{a}{n}$ of 1; or, $\frac{ba}{n}$ of 1, (by the first,) and $\frac{ba}{n}$ of 1 is $= \frac{1}{n}$ of ba, (by the last;) therefore $\frac{a}{n}$ of $b = \frac{1}{n}$ of $a\,b$.

4. The Sum of two or more Fractions having the same Den^r, is equal to a Fraction of the same Den^r, whose Num^r is the Sum of the given Num^rs. *Examp.* $\frac{a}{n} + \frac{b}{n} + \frac{c}{n} = \frac{a+b+c}{n}$. For $\frac{1}{n}$ of $a + \frac{1}{n}$ of $b + \frac{1}{n}$ of $c = \frac{1}{n}$ of $\overline{a+b+c}$; but $\frac{1}{n}$ of $a = \frac{a}{n}$ of 1. And so of the rest: Therefore, &c.

5. Here now we have a further Demonstration of that general Truth mentioned in *Schol.* after *Lem.* 2. for demonstrating Division of Whole Numbers; *viz.* That the Sum of the complete Quotes of any Numbers is equal to the complete Quote of the Sum of these Numbers, being all divided by the same Divisor. For $\frac{A+B+C}{n} = \frac{A}{n} + \frac{B}{n} + \frac{C}{n}$; and whatever these Quotes are, whether Whole Numbers or Mix'd, these Fractions express the Quotes of these Numbers by the Den^r n. But again, we see here this more general Truth, *viz.* That whatever Numbers, (Whole or Fractional) any Number (Whole or Fractional) is resolved into, the Sum of the Quotes is equal to the Quote of the Sum, being divided by the same

Integral Divisor: For whatever kind of Numbers A,B,C are, still it is true, that $\frac{1}{n}$ of $A+\frac{1}{n}$ of $B+\frac{1}{n}$ of $C=\frac{1}{n}$ of $\overline{A+B+C}$; and the $\frac{1}{n}$ of any Quantity is the Quote of it divided by n. Afterwards when we learn what the meaning of dividing by a Fraction is, we shall see the same truth hold in that Case also: So that it is Universal for all Cases, whatever the Dividend and Divisor is; as you may easily make Examples of, when you have learned the Operations of Fractions.

6. Any *Aliquot* Part of one Quantity or Number whatever, Whole or Fraction, is the same Fraction of the like *Aliquot* Part of another, as the one Whole is of the other: Or thus, any two Numbers or Quantitys are the same Fractions one of another as their Equimultiples. *Examp.* If $\frac{1}{3}$ of any Quantity is equal to $\frac{2}{5}$ of the $\frac{1}{3}$ of another; then the first Whole is $\frac{2}{5}$ of the other: Or if $\frac{1}{n}$ of one Quantity is $=\frac{a}{b}$ of $\frac{1}{n}$ of another, the first Quantity is $=\frac{a}{b}$ of the other: For each Whole being composed of equal Parts, they are represented thus, $A+A+A$, *&c.* and $a+a+a$, *&c.* And the Number of Parts being equal, a and A are the like *Aliquot* Parts of their Wholes; and a is the like Fraction of A, as the Sum $a+a$, *&c.* is of $\overline{A+A}$, *&c.*

7. If we compare any Fraction, as $\frac{n}{m}$ of any Quantity, or Number whatever A, and the like Fraction of another Number B; then is $\frac{n}{m}$ of A equal to the same Fraction of $\frac{n}{m}$ of B, as A is of B; which is plain from the last: for $n\times\frac{1}{m}$ of A, and $n\times\frac{1}{m}$ of B, are Equimultiples of $\frac{1}{m}$ of A, and $\frac{1}{m}$ of B: And by the last, Equimultiples, or like *Aliquot* Parts of any two Quantitys are the like Fractions one of another as these Quantities are; *i. e.* $n\times\frac{1}{m}$ of A ($=\frac{n}{m}$ of A) is the same Fraction of $n\times\frac{1}{m}$ of B ($=\frac{n}{m}$ of B,) as $\frac{1}{m}$ of A is of $\frac{1}{m}$ of B: Also $\frac{1}{m}$ of A, is the same Fraction of $\frac{1}{m}$ of B, as A is of B. Hence lastly, $\frac{n}{m}$ of A is the same Fraction of $\frac{n}{m}$ of B as A of B.

LEMMA III.

The Difference betwixt the like Fractions of two Quantitys or Numbers whatever, is equal to the like Fractions of the Difference of these Numbers. *Examp.* $\frac{2}{3}$ of 15 ($=10$) $-\frac{2}{3}$ of 6 ($=4$) is $=\frac{2}{3}$ of $\overline{15-6}$ ($=6$;) or generally, $\frac{n}{m}$ of $A-\frac{n}{m}$ of $B=\frac{n}{m}$ of $\overline{A-B}$. Or thus, if a, b are like Fractions of A,B, *viz.* $\frac{n}{m}$ Parts, then $a-b=\frac{n}{m}$ of $\overline{A-B}$.

DEMONST. Let $A-B$ be $=d$; then is $A=d+B$: Wherefore $\frac{n}{m}$ of $\overline{A-B}=\frac{n}{m}$ of d, and $\frac{n}{m}$ of $A=\frac{n}{m}$ of $\overline{d+B}$; which is $=\frac{n}{m}$ of $d+\frac{n}{m}$ of B, (by last *Lemma.*) Consequently, $\frac{n}{m}$ of $A=\frac{n}{m}$ of $d+\frac{n}{m}$ of B; out of each of these take $\frac{n}{m}$ of B; then is $\frac{n}{m}$ of $A-\frac{n}{m}$ of $B=\frac{n}{m}$ of d. But $\frac{n}{m}$ of $\overline{A-B}=\frac{n}{m}$ of d, (as above;) therefore $\frac{n}{m}$ of $A-\frac{n}{m}$ of $B=\frac{n}{m}$ of $\overline{A-B}$.

I COROL.

COROL. The Difference of two Fractions of the same Den^r is equal to a Fraction of that Den^r, whose Num^r is the Difference of the given Numbers. *Example*, $\frac{a}{n} - \frac{b}{n} = \frac{a-b}{n}$, for $\frac{1}{n}$ of $a = \frac{a}{n}$ of 1, and $\frac{1}{n}$ of $b = \frac{b}{n}$ of 1, and $\frac{1}{n}$ of $\overline{a-b} = \frac{a-b}{n}$ of 1: Therefore $\frac{a}{n} - \frac{b}{n} = \frac{a-b}{n}$.

LEMMA IV.

The more (equal) Parts any Number is divided into, the smaller these Parts are; and the fewer the Number of Parts, the greater is the Part. For *Examp.* $\frac{1}{3}$ Part is greater than $\frac{1}{4}$ Part; and so of others: For it is plain, you cannot divide the Whole into more Parts without breaking the former Parts into Pieces or smaller Parts. But more particularly, if the Den^r of one Part is Multiple of the Den^r of another, then this Part is Equimultiple of the first. For *Examp.* $\frac{1}{3}$ of any thing is double of $\frac{1}{6}$, because 6 is double of 3. And it is evident that the same reason must hold in all Cases.

Or we may express it thus; If any Quantity is divided into any Number of equal Parts, and the same (or an equal) Quantity is divided into 2, 3, 4, &c. times as many Parts, then the Part of this last Division is but $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. of the Part of the former. For you cannot make 2, 3, &c. times as many Parts, otherways than by breaking or dividing each of the former Parts into 2, 3, &c. whereby they will become $\frac{1}{2}$, $\frac{1}{3}$, &c. of the Part divided; and reciprocally if the last Division is into $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. of the Number of Parts of the first, then is the Part of the last Division 2, 3, 4, &c. times as great as that of the first. *Universally*, $\frac{1}{a}$ Part of any thing is equal to r times $\frac{1}{ra}$ Part of that thing; and Reciprocally, $\frac{1}{ra}$ Part is but the $\frac{1}{r}$ Part of $\frac{1}{a}$ Part: So that if the Den^r of an *Aliquot* Fraction is the Product of two Numbers, that Fraction is equal to the Compound of two *Aliquot* Fractions whose Den^{rs} are these Numbers. Thus, if $a = rn$, then is $\frac{1}{a} = \frac{1}{r}$ of $\frac{1}{n}$.

LEMMA V.

If two Fractions have the same Num^r with different Den^{rs}, they represent Quantities of different Values, and that which has the greatest Den^r is the least Fraction. For *Examp.* $\frac{2}{5}$ is less than $\frac{2}{3}$, because it represents the same Number of lesser Parts, (*Lem.* 3.) But more particularly, if the greater Den^r is equal to 2, or 3, &c. times the lesser, that Fraction is but $\frac{1}{2}$ or $\frac{1}{3}$, &c. of the other. And Reciprocally, this Fraction is equal to 2 or 3 times the former; so $\frac{2}{6}$ is $\frac{1}{2}$ of $\frac{2}{3}$ (and $\frac{2}{3}$ is 2 times $\frac{2}{6}$) because 3 is $\frac{1}{2}$ of 6; and $\frac{5}{28}$ is $\frac{1}{4}$ of $\frac{5}{7}$, because 7 is $\frac{1}{4}$ of 28. *Universally*, $\frac{a}{rn}$ is $\frac{1}{r}$ of $\frac{a}{n}$, and $\frac{a}{n}$ is r times $\frac{a}{rn}$, because n is $\frac{1}{r}$ of rn.

The Reason is plain: Thus, $\frac{1}{n}$ of any thing is r times $\frac{1}{rn}$ of that thing, (*Lem.* 4.) *i. e.* $\frac{1}{n}$ of $a = r \times \frac{1}{rn}$ of a; but $\frac{1}{n}$ of $a = \frac{a}{n}$ of 1, and $\frac{1}{rn}$ of $a = \frac{a}{rn}$ of 1, (by *Cor.* 2. *Lem.* 2.) therefore $\frac{a}{n}$ of $1 = r$ times $\frac{a}{rn}$ of 1.

Or we may see the Truth of this somewhat otherwise; thus, $\frac{1}{n}$ and $\frac{1}{rn}$ are the like *Aliquot* Parts of $\frac{a}{n}$ and $\frac{a}{rn}$, *viz.* the $\frac{1}{a}$ Part; (for $\frac{1}{n} = \frac{a}{n} \div a$, or $\frac{1}{a}$ of $\frac{a}{n}$, and $\frac{1}{rn} = \frac{a}{rn} \div a = \frac{1}{a}$ of $\frac{a}{rn}$; (*Corol. Lem.* 1.) But $\frac{1}{rn} = \frac{1}{r}$ of $\frac{1}{n}$, (by *Lem.* 4.) There-

fore

fore that Whole of which $\frac{1}{rn}$ is the $\frac{1}{a}$ Part, ($viz.$ $\frac{a}{rn}$) is alſo the $\frac{1}{r}$ Part of that Whole, of which $\frac{1}{n}$ is the $\frac{1}{a}$ Part, ($viz.$ $\frac{a}{n}$,) *i. e.* $\frac{a}{rn}$ is the $\frac{1}{r}$ of $\frac{a}{n}$, (*Cor.* 5. *Lem.* 2) And Reciprocally, $\frac{a}{n}$ is r times $\frac{a}{rn}$.

COROLLARIES.

1. A Fraction is multiplied or divided by any Integer, if we divide or multiply its Den^r; So $\frac{a}{rn} \times r = \frac{a}{n}$, and $\frac{a}{n} \div r = \frac{a}{rn}$; For $\frac{a}{rn} = \frac{1}{r}$ of $\frac{a}{n}$, (by this *Lemma*.) But the $\frac{1}{r}$ Part of any Quantity multiplied by r, produces that Quantity; therefore $\frac{a}{rn} \times r = \frac{a}{n}$; whence again $\frac{a}{n} \div r = \frac{a}{rn}$.

And take notice, That the Diviſion is ſuppoſed here to be without a Remainder; for otherwiſe the Fraction cannot be multiplied by Diviſion of its Den^r, becauſe the complete Quote is a mix'd Number, and ſo cannot be the Den^r of a Fraction.

SCHOL. As to this *Corollary*, obſerve, That by multiplying or dividing the Den^r of a Fraction, what we call properly the fractional Number is not multiplied or divided, for that is the Num^r; but the mix'd Value or Quantity expreſſed by the Fraction is multiplied or divided; ſo that it is ſtill proper to ſay, the Fraction (*i. e.* the Quantity expreſſed by it) is multiplied or divided. Again, it is manifeſtly the ſame thing in effect, to increaſe or diminiſh a Number of things, keeping the ſame Value of each; or to increaſe or diminiſh the Value of each, keeping the ſame Number; for either way the Quantity or mix'd Value of the Whole is equally increaſed or diminiſhed. Hence,

2. If the Num^r and Den^r of a Fraction are equally multiplied or divided by any Number, the Products or Quotes (where there is no Remainder) make an equal Fraction: Or thus; two Fractions are equal if the Num^r and Den^r of the one are Equimultiples, or like *Aliquot* Parts of the Num^r and Den^r of the other: So $\frac{ra}{rn} = \frac{a}{n}$. For by multiplying or dividing the Number, the Fraction is multiplied or divided in the Number of things directly expreſſed, (*Cor. Lem.* 1.) and by multiplying or dividing the Den^r by the ſame Number, the Fraction is contrarily as much divided or multiplied in the Value of the things expreſſed, (laſt *Cor.*) ſo that what the Fraction gains or loſes in the one Member, it contrarily loſes or gains as much in the other; and conſequently it remains ſtill the ſame Fraction, only in different Terms.

3. If we find a Number which will exactly divide the Num^r and Den^r of a Fraction, we can thereby reduce it to lower Terms, (*i. e.* find another Expreſſion in leſſer Numbers, which is an equal Fraction,) *viz.* by dividing the Num^r and Den^r of the given Fraction by that Number, and taking the Quotes in place of the former Num^r and Den^r. Thus, $\frac{6}{8} = \frac{3}{4}$, by dividing 6 and 8 both by 2.

SCHOL. Such Diviſoss will be eaſily diſcovered in many Caſes; and from the Nature of Numbers we have theſe particular Rules for finding a Number which will divide two other Numbers, (*i. e.* the Num^r and Den^r of a Fraction) *viz.* 1. If both are even Numbers, or have in place of Units 2, 4, 6, 8, or 0; then they are both diviſible by 2; ſo $\frac{14}{28} = \frac{7}{11}$. And after one Diviſion by 2, if they are ſtill even, divide again by 2, and ſo on as long as they are even; thus, $\frac{32}{56} = \frac{16}{28} = \frac{8}{14} = \frac{4}{7}$.

2. If any one or both of them have 5 or 0 in the firſt Place, then will 5 divide them both; or if they have both 0's in the firſt Places on the Right-hand, cut away an equal Num-

Number of o's from both, (which is dividing them both equally by 10 or 100, *&c.* according to the Number of o's cut off;) and after these are cut off, apply some of the other Divisions, if the Case admit it.

Examp. 1. $\frac{15}{240}=\frac{3}{48}$ (*viz.* $15\div5=3$, and $240\div5=48$.) *Ex.* 2. $\frac{70}{210}=\frac{7}{21}$

Ex. 3. $\frac{400}{1800}=\frac{4}{18}=\frac{2}{9}$. *Ex.* 4. $\frac{500}{1750}=\frac{50}{175}=\frac{10}{75}=\frac{2}{15}$.

When none of these Cases occur, yet in small Numbers you will easily discover a Number which will divide both, if there is any such; and tho' 2 will divide them, yet you'll frequently find, at first, a greater Number which will divide them.

So $\frac{7}{35}=\frac{1}{5}$, $\frac{15}{48}=\frac{5}{16}$, $\frac{24}{48}=\frac{1}{2}$, $\frac{1}{54}=\frac{1}{3}$, $\frac{24}{56}=\frac{3}{7}$.

DEFINITION 1. Two Fractions are said to be *reverse* or *reciprocal* to one another, when the Numr and Denr of the one is the Denr and Numr of the other, as $\frac{2}{3}$ and $\frac{3}{2}$, or generally $\frac{a}{b}$ and $\frac{b}{a}$ are Reciprocals; and because any integral Number is made an improper Fraction by making 1 the Denr: Therefore a whole Number has also its Reciprocal, *viz.* a Fraction, whose Denr is that whole Number, and its Numr 1. So 2 and $\frac{1}{2}$, or a and $\frac{1}{a}$ are Reciprocals.

2. Two Fractions, whereof the Terms of the one are the 2 Numrs, and the Terms of the other the two Denrs of other two Fractions, are called, The Alternate Fractions of these other two: Thus $\frac{A}{C}$ and $\frac{B}{D}$ are the Alternate Fractions of these two $\frac{A}{B}$, $\frac{C}{D}$, and these Alternate of those.

LEMMA VI.

If two Fractions are equal, then these Truths follow.

1. The Products made of the Numerator of each multiplied into the Denominator of the other are equal. Thus, if $\frac{a}{b}=\frac{n}{m}$ then is $am=bn$.

Examp. $\frac{2}{3}=\frac{4}{6}$, therefore, $2\times6=3\times4$ $(=12.)$

DEMONST. Since $\frac{a}{b}=\frac{n}{m}$ multiply each by m, the Products must be equal: But $\frac{a}{b}\times m=\frac{am}{b}$ (by *Cor. Lem.* 1.) and $\frac{n}{m}\times m=\frac{n}{1}$ or n, (Corol. 1. Lemma 4.) *that is*, $\frac{am}{b}=n$. Again, multiply each of these by b, the Products are also equal, *viz.* $am=bn$, which also follows from the Proof of Division.

Hence we have learned a certain Rule for trying the Equality or Inequality of two Fractions.

The *Reverse* of this Article is also true, *viz.* That if the Products made of the Numerator of each Fraction, multiplied into the Denominator of the other, are equal, these Fractions are equal. Thus, if $am=bn$, then $\frac{a}{b}=\frac{n}{m}$. For, divide am and bn both by b, the Quotes are equal, *viz.* $\frac{am}{b}=n$; and again dividing both by m, the Quotes are equal *viz.* $\frac{a}{b}=\frac{n}{m}$ (for $\frac{am}{b}\div m=\frac{a}{b}$.)

COROL. If $\frac{a}{b}=\frac{n}{m}$, then are n, m, both less or both greater, or both equal to their Correspondents a, b; for since $am=bn$, if n is less than a, b must be greater than m, else it's plainly impossible that bn should be $=am$; and if n is greater than a, b must be less than m. *Lastly*, if $n=a$, then $b=m$.

2.

2. The *Reciprocal Fractions* are also equal; that is, if $\frac{a}{b}=\frac{n}{m}$, then is $\frac{b}{a}=\frac{m}{n}$. *Ex.* if $\frac{2}{3}=\frac{4}{6}$ then is $\frac{3}{2}=\frac{6}{4}$

DEMONST. Because $\frac{a}{b}=\frac{n}{m}$, therefore $am=nb$, (*Art.* 1.) Divide each by a, and the Quotes are equal, *viz.* $m=\frac{bn}{a}$: then divide each of these by n, the Quotes are equal, $\frac{b}{a}=\frac{m}{n}$.

SCHOLIUM. Because the Reciprocals of two equal Fractions are equal, we may also say, that the Reciprocal of one of two equal Fractions, is a Reciprocal to the other; So if $\frac{a}{b}=\frac{n}{m}$ then are $\frac{b}{a}$ and $\frac{n}{m}$ Reciprocals, also $\frac{a}{b}$ and $\frac{m}{n}$. But then it will be convenient to distinguish betwixt immediate and remote Reciprocals. Thus $\frac{a}{b}$ and $\frac{b}{a}$ are immediate Reciprocals, and $\frac{a}{b}$, $\frac{m}{n}$, are remote Reciprocals. But when we speak of Reciprocals in general, without distinguishing, then either of the kinds may be supposed.

COROL. If the Numerator of two reciprocal Fractions are multiplied together, and also their Denominators, the Products make a Fraction equal to 1. So if $\frac{a}{b}$, $\frac{m}{n}$, are Reciprocals, then $\frac{am}{bn}=1$, for $\frac{a}{b}=\frac{n}{m}$ and $am=bn$, (Art. 1.) hence $\frac{am}{bn}=1$.

3. The *Alternate* Fractions are also equal: That is, if $\frac{a}{b}=\frac{n}{m}$, then also is $\frac{a}{n}=\frac{b}{m}$ and $\frac{n}{a}=\frac{m}{b}$. *Examp.* if $\frac{2}{3}=\frac{4}{6}$, then $\frac{2}{4}=\frac{3}{6}$.

DEMONST. Since $\frac{a}{b}=\frac{n}{m}$ then $am=bn$ (Art. 1.) divide both by n, and the Quotes are equal, *viz.* $\frac{am}{n}=b$; again, divide both these by m, and the Quotes are equal, $\frac{a}{n}=\frac{b}{m}$ (for $\frac{am}{n}\div m=\frac{a}{n}$ and $b\div m=\frac{b}{m}$:) And because $\frac{a}{n}=\frac{b}{m}$ therefore $\frac{n}{a}=\frac{m}{b}$. By *Art.* 2.

COROL. 1. If two Fractions are equivalent, as $\frac{a}{b}=\frac{n}{m}$, the Terms of the one are like Fractions of the corresponding Terms of the other: Thus n, m, are like Fractions of a, b, or a, b, like Fractions of n, m. For $n=\frac{n}{a}$ of a, and $m=\frac{m}{b}$ of b; but $\frac{n}{a}=\frac{m}{b}$: Also $a=\frac{a}{n}$ of n, and $b=\frac{b}{m}$ of m; but $\frac{a}{n}=\frac{b}{m}$.

2. If two Fractions are equivalent, as $\frac{a}{b}=\frac{n}{m}$, the Terms of the one are the Quotes of an equal Division of the correspondent Terms of the other, by some Number, either integral or fractional: For by *Cor.* 2. *Lem.* 2. $\frac{a}{n}$ or $\frac{a}{n}$ of 1 is $=\frac{1}{n}$ of a, and $\frac{b}{m}$ is $=\frac{1}{m}$ of b. Suppose then, that $\frac{a}{n}=d$, then, whether d is a whole Number or Fraction, it's plain, that n times d is $=a$, and consequently d is contained in a, n times, or $a\div d=n$. Again, $\frac{a}{n}=\frac{b}{m}$, therefore, $\frac{b}{m}=d$, and for the same Reason as before, d is contained m times in b, or $b\div d=m$. In the same manner, if we suppose $\frac{n}{a}=d$, then $n\div d=a$; and because $\frac{n}{a}=\frac{m}{b}$, therefore, $\frac{m}{b}=d$, and $m\div d=b$.

Observe,

Observe, When we have learned afterwards, that the Quote of any Number (Whole, or Fraction,) divided by any other Number, is such a Fraction of the Dividend, as the Reciprocal of the Divisor expresses, (*i.e.* the $\frac{1}{a}$ Part, if the Divisor is a whole Number a, or the $\frac{b}{a}$ Parts, if the Divisor is the Fraction $\frac{a}{b}$, then it will be plain that the one of these two *Corollaries* is contained in the other, so as either of them may be deduced from the other.

LEMMA VII.

If any Number A (whole or fractional) is equal to any Fraction (proper or improper) of another Number B, then is B equal to the reciprocal Fraction of A. *Examp.* If A $=\frac{2}{3}$ of B, then is B$=\frac{3}{2}$ of A; universally, if A$=\frac{n}{m}$ of B, then is B$=\frac{m}{n}$ of A.

DEMONST. If A$=\frac{n}{m}$ of B, this supposes, that B being divided into m Parts, A contains n of these Parts; which infers reciprocally, that A being divided into n Parts, B contains m of such Parts.

COROL. Hence we have another Proof of the 2d Article of the preceding *Lemma*, *viz.* That if two Fractions are equal, their Reciprocals are also equal: For if A$=\frac{n}{m}$ of B, then is B$=\frac{m}{n}$ of A; and if $\frac{n}{m}=\frac{c}{d}$, therefore A $=\frac{c}{d}$ of B, and B $=\frac{d}{c}$ of A. But since also B $=\frac{m}{n}$ of A, it follows that $\frac{m}{n}=\frac{d}{c}$, else B would be equal to two different Fractions of A, which is impossible.

CHAP. II.

REDUCTION *of* FRACTIONS.

PROBLEM I.

TO *reduce an improper Fraction to its equivalent Whole or Mix'd Number.*

Rule. Divide the Num^r by the Den^r, the Quote is the Answer.

Examp. 1. $\frac{4}{4}=1$. *Ex.* 2. $\frac{24}{6}=4$, (the Quote of 24 by 6.)

Ex. 3. $\frac{14}{4}=3\frac{2}{4}$, or $3\frac{1}{2}$, (because $\frac{1}{2}=\frac{2}{4}$, *Cor.* 2. *Lem.* 4.)

DEMONST. The Den^r represents the relative Integer or Unit, expressing it by a Number of Parts; therefore as oft as the Num^r contains the Den^r, it's equal to so many times that Integer, (or so many Integral Units;) and what's over in the Division, makes a Fraction of the given Denominator.

PROBLEM II.

To reduce a mix'd Number, to an equivalent improper Fraction.

Rule. Multiply the integral Number by the Den^r of the Fraction, and to the Product add the Num^r; make the Sum a Num^r to the given Den^r, and that is the Fraction sought.

Examp. $6\frac{2}{3} = \frac{20}{3}$; thus, $6 \times 3 = 18$, then $18 + 2 = 20$.

DEMONST. This is plain from the last, for it's only the Reverse of it.

PROBLEM III.

To reduce a Whole Number to an improper Fraction, having any given Denominator.

Rule. Multiply the given Number by the given Den^r, and the Product is the Num^r of the Fraction sought.

Examp. To reduce 8 to a Fraction, having 6 for its Den^r, it is $= \frac{48}{6}$.

DEMONST. This is also plain from *Probl.* 2. being the Reverse of it.

COROL. Every Whole Number is reduced to the Form of a Fraction, by making 1 the Den^r; thus, $4 = \frac{4}{1}$.

SCHOLIUM. The same or equivalent mix'd Number, *i. e.* the same integral Number, with the same or equivalent Fraction, will always make the same or equivalent improper Fraction, only in different Terms, according as the fractional Part is. *And Reversely*, The same or equivalent improper Fraction will always reduce to the same or equivalent mix'd Number.

Examp. $4\frac{6}{9} = \frac{42}{9}$, and $4\frac{2}{3} = \frac{14}{3}$; and because $\frac{6}{9} = \frac{2}{3}$, therefore $4\frac{6}{9} = 4\frac{2}{3}$, consequently, $\frac{42}{9} = \frac{14}{3}$; and from this it follows reversely, that $4\frac{6}{9}$ $(= \frac{42}{9}) = 4\frac{2}{3}$ $(= \frac{14}{3}.)$

Hence we see the Demonstration of a Truth proposed in *Schol.* 2. to Division of whole Numbers, *viz.* That the same Quote will always be expressed by the same Fraction; *i. e.* That if two Numbers are proposed to be divided by other two, if the integral Quotes are the same, when there is no Remainder; and when there is a Remainder, if the fractional Parts are also equivalent, then the Quotes taken fractionally (*i. e.* by setting the Dividend as Num^r over the Divisor) will always be equal; and if the integral or mix'd Quotes are unequal, so will these fractional ones be: And, in fine, whatever Part or Parts the lesser mix'd Quote is of the greater, the same will the equivalent fractional Quotes be. So that in the Comparison of one Quote to another, it's the same to all Intents and Purposes to express them fractionally by the Dividends and Divisors, or to reduce (*i. e.* divide) and express them directly and properly. But the Use and Conveniency of this way of expressing Quotes, we shall learn more particularly afterwards.

PROBLEM IV.

To reduce a compound Fraction to an equivalent simple Fraction.

Rule. Multiply all the Num^{rs} continually, the last Product is the Num^r sought; and multiply all the Den^{rs}, the last Product is the Den^r sought.

Examp. 1. $\frac{2}{3}$ of $\frac{4}{7} = \frac{8}{21}$, (for $2 \times 4 = 8$, and $3 \times 7 = 21$.)

Examp. 2. $\frac{2}{5}$ of $\frac{5}{7}$ of $\frac{8}{9} = \frac{80}{315}$, (for $2 \times 5 \times 8 = 80$, and $5 \times 7 \times 9 = 315$) $= \frac{16}{63}$.

DEMONST. 1. If the compound Fraction consists of two Parts, as $\frac{a}{n}$ of $\frac{c}{m}$; the Reason of the Rule is this: Since $\frac{a}{n}$ Parts of any thing is $= a$ times $\frac{1}{n}$ Part, (*Cor. Lem.* 1.) or

or alſo $= \frac{1}{n}$ Part of a times that thing, (*Cor.* 2. *Lem.* 2.) Thence it's plain, that if we take firſt a times $\frac{c}{m}$, which is $= \frac{ac}{m}$, and of this take $\frac{1}{n}$ Part, which is $= \frac{ac}{nm}$; or firſt take $\frac{1}{n}$ Part of $\frac{c}{m}$, which is $= \frac{c}{nm}$; and then a times this, which is $\frac{ac}{nm}$: We have either way taken $\frac{a}{n}$ Parts of $\frac{c}{m}$, which gives the ſimple Fraction according to the Rule.

2. The ſame Reaſoning holds if the compound Fraction has three or more Members; for the two firſt being reduced to one, that one and the third make the ſame Caſe as that of two Numbers; which being reduced, gives the ſimple Fraction equivalent to the Compound of three given ones, (which will be plainly according to the Rule, *viz.* the continual Product of Num^rs and Den^rs) and ſo on for four or more Members.

Scholiums.

1. It's no matter whether the Members of a compound Fraction be Proper or Improper, the Reduction is done the ſame way, and for the ſame general Reaſon, wherein there is no Regard had to the Diſtinction of *Proper* and *Improper*.

But this is to be obſerved, that if all the Members are *Proper Fractions*, their equivalent ſimple Fractions will neceſſarily be *Proper*; and if they are all *Improper*, it's *Improper*: But if ſome of them are *Proper* and others *Improper*, the ſimple ones will in ſome Caſes be *Proper* and in ſome *Improper*, according as the Value of the *Proper* and *Improper* Members happen to be. But it is not to be known what it will be, otherwiſe than by applying the Rule, and actually finding the ſimple Fraction ſought. So here,

$$\tfrac{2}{3} \text{ of } \tfrac{6}{5} \text{ of } \tfrac{7}{8} = \tfrac{84}{120}; \text{ but } \tfrac{2}{3} \text{ of } \tfrac{9}{4} \text{ of } \tfrac{7}{8} = \tfrac{126}{96}.$$

2. Fractions which are referred to a Number greater than Unity, as $\frac{2}{7}$ of 3, may be alſo conſidered as compound fractional Expreſſions (by putting the whole Number in form of a Fraction, as $\frac{2}{7}$ of $\frac{3}{1}$, reducible to a Fraction of an Unit (of the ſame things) by the ſame Rule, (and for the ſame Reaſons as before;) where it's plain we have nothing to do but multiply the Num^r of the Fraction by the given whole Number, and apply that Product to the given Den^r; ſo $\frac{2}{7}$ of $3 = \frac{6}{7}$, and $\frac{3}{4}$ of $2 = \frac{6}{4}$ $(= \frac{3}{2}.)$ But the more original Reaſon for this Caſe, we have already learn'd in *Cor.* 2. *Lem.* 2. *Obſerve* alſo that here, as in the other kind, the ſimple Fraction will in ſome Caſes be *Proper*, and in ſome improper, even tho' the given Fraction is *Proper*; but muſt always be *Improper*, if the given Fraction is ſo. *Again*, We may have a Fraction referred to a mix'd Number, as $\frac{3}{4}$ of $5\frac{2}{7}$, and the Reduction to a ſimple Form is plainly this; Reduce the mix'd Number by *Problem* 2, and then apply the preſent *Problem*, thus, $5\frac{2}{7} = \frac{37}{5}$, and then $\frac{3}{4}$ of $\frac{37}{5}$ $= \frac{111}{28}$ $(= 3\frac{27}{28}$, *Prob.* 1.)

3. Some Authors propoſe as a kind of compound Fractions, ſuch Expreſſions wherein the Num^r and Den^r are themſelves Fractions pure or mixed; as theſe, $\dfrac{3/4}{2/5}$ or $\dfrac{9\frac{5}{7}}{15\frac{1}{2}}$ or $\dfrac{3}{5\frac{2}{7}}$. But, in my Opinion, we cannot call any of theſe a Fraction with any Propriety; for they expreſs not a certain Number of determinate Parts, which is the true and proper Notion of a Fraction. They are, indeed, reducible to an equivalent Expreſſion in the natural Form of a Fraction; but that does not make them Fractions in the proper Notion, more than a Number of Shillings can be ſaid to be an Expreſſion of *Pence*, becauſe it's reducible to ſuch an Expreſſion, (*i. e.* becauſe a Number of Pence can be aſ-

ſigned equal to the given Number of Shillings.) For this Reaſon I would never conſider theſe Expreſſions as Fractions, but only as a manner of ſignifying that the one is to be divided by the other; or at moſt, as an indefinite way of expreſſing the Quote of the upper Number divided by the under: The finding of which Quote (or the Reduction, if you pleaſe to call it ſo) muſt therefore be learn'd from the Diviſion of Fractions. And upon the Diviſion of Fractions does alſo depend another *Problem*, which ſome Authors bring in among *Reductions*, *viz.* To find of what Number any one given Number (Whole, or Fraction, or Mix'd,) is another given Fraction. *Examp.* To find of what Number $\frac{2}{3}$ is the $\frac{8}{9}$; or to find of what Number 6 is the $\frac{3}{4}$: But theſe we muſt leave to the Rule of Diviſion.

4. The preceding *Rule* is general, and finds the true ſimple Fraction required in all Caſes, as has been demonſtrated; and that ſimple Fraction may be again reduced to lower Terms, in the manner ſhewn in *Cor.* 3. *Lem.* 5. But you may more eaſily, in many Caſes, find the ſimple Fraction required in lower Terms, at the firſt, than the General Rule gives; by this Method: Firſt, ſee if the given ſimple Fractions can be expreſſed lower by the Method of *Cor.* 2. *Lem.* 5. and uſe theſe new Expreſſions in place of the former, which muſt certainly give the true Fraction ſought; becauſe equal Fractions are the ſame Fractions, only differently expreſſed. *Examp.* 1. $\frac{4}{6}$ of $\frac{5}{7}$ $(=\frac{20}{42})$ is the ſame as $\frac{2}{3}$ of $\frac{5}{7}$ $(=\frac{10}{21})$ becauſe $\frac{4}{6}=\frac{2}{3}$. But, 2*ly*, When you cannot reduce the given Fractions, or after you have done it, proceed thus; *viz.* Compare the ſeveral Num^{rs} and Den^{rs} together, and if the Num^{r} of one Fraction and the Den^{r} of another are diviſible by the ſame Number, (which may ſometimes be the leſſer of theſe two Numbers themſelves) take the Quotes and put in the Places of the Numbers divided; and do this with as many as you can; and then apply the General Rule, which will give the Fraction ſought in lower Terms. The *Reaſon* of which is, that by this Method you have done the ſame in effect, as if you had found the ſimple Fraction by the General Rule without ſuch previous Work, and then divided both Num^{r} and Den^{r} by theſe Numbers which were made Diviſors in the previous Work.

The following *Examples* will illuſtrate this ſufficiently. I have made *Examples* only with two Members; but you can eaſily do the ſame when there are more Members: And as for ſuch *Examples* as theſe, where there are but two Members, there will be no need to ſet down the Effect of the preparatory Work, but the Anſwer of the Queſtion all at once, the intermediate Steps being eaſily done without writing. The finding the ſimple Fraction in the ſmalleſt Numbers poſſible, depends upon the next *Problem*, which you are to apply to the Fraction found by the preceding Rule.

Examp. 2. 4 of $\frac{3}{7}=\frac{2}{3}$ of $\frac{3}{7}=\frac{2}{1}$ of $\frac{1}{7}=\frac{2}{7}$. *Ex.* 3. $\frac{2}{3}$ of $\frac{5}{8}=\frac{1}{3}$ of $\frac{4}{5}=\frac{4}{15}$.

Ex. 4. $\frac{5}{7}$ of $\frac{14}{19}=\frac{5}{1}$ of $\frac{2}{19}=\frac{10}{19}$. *Ex.* 5. $\frac{5}{12}$ of $\frac{4}{7}=\frac{5}{3}$ of $\frac{1}{7}=\frac{5}{21}$.

Ex. 6. $\frac{4}{9}$ of $\frac{6}{25}=\frac{4}{3}$ of $\frac{2}{25}=\frac{8}{75}$. *Ex.* 7. $\frac{3}{5}$ of $\frac{20}{27}=\frac{1}{1}$ of $\frac{4}{9}=\frac{4}{9}$.

Ex. 8. $\frac{8}{15}$ of $\frac{5}{12}=\frac{2}{3}$ of $\frac{1}{3}=\frac{2}{9}$. *Ex.* 9. $\frac{6}{7}$ of $\frac{28}{3}=\frac{2}{1}$ of $\frac{4}{1}=8$.

COROLLARIES.

1. In whatever Order the Members of a Compound Fraction are taken, it is ſtill equal: So $\frac{2}{3}$ of $\frac{4}{5}=\frac{4}{5}$ of $\frac{2}{3}$; and $\frac{2}{3}$ of $\frac{4}{5}$ of $\frac{7}{9}=\frac{7}{9}$ of $\frac{2}{3}$ of $\frac{4}{5}$

Or alſo exchanging the Num^{rs} and Den^{rs} of any two of the Members, it is ſtill equal: So $\frac{2}{7}$ of $\frac{4}{5}=\frac{2}{5}$ of $\frac{4}{7}$. In ſhort, the ſame Number of Simple Fractions make an equal Compound one, if the Num^{rs} of the Simples in each, and alſo the Den^{rs} are the ſame Numbers, tho' in ſuch Order as not to make the ſame ſimple Fractions. The Reaſon is, becauſe

the Simple Fraction to which each of these Compounds is reduced, will be the same, being produced by the same Numbers.

2. Hence we learn how to dissolve a Fraction (if possible) into two or more component Parts; *i. e.* to reduce a Simple Fraction to a Compound one: Thus, if we can discover two, or three, or more Numbers, which multiplied together will produce a Number equal to the Num^r of the given Fraction, and as many which will produce a Number equal to the Den^r; then, of these Numbers we may make as many Simple Fractions, which, connected as the Members of one Compound Fraction, will be equal to that Simple Fraction. *Examp.* 1. $\frac{6}{20}=\frac{3}{5}$ of $\frac{2}{4}$. *Examp.* 2. $\frac{1}{16}=\frac{1}{2}$ of $\frac{1}{8}$ *Examp.* 3. $\frac{21}{24}=\frac{1}{2}$ of $\frac{21}{12}$, or $\frac{1}{2}$ of $\frac{3}{4}$ of $\frac{7}{3}$. *Examp.* 4. $\frac{3}{42}=\frac{1}{2}$ of $\frac{1}{3}$ of $\frac{3}{7}$. And this Resolution does not depend upon the Simple Fraction's being reducible to lower Terms; for this Fraction $\frac{8}{15}$, which is not reducible to lower Terms, is yet equal to $\frac{2}{3}$ of $\frac{4}{5}$. In short, as many Numbers as there are which will produce the Den^r, the Fraction is reducible to a Compound having as many Members, whereof these Numbers are the Den^rs; and tho' the Num.^r is the Product of no Numbers but 1 and itself, yet that will afford as many Num^rs for the Members of the Compound Fraction, as in *Examp.* 2, and 4.

DEFINITION. As one of two equivalent Fractions must be in lesser Numbers than the other, (by *Lem.* 6. *Cor.* to *Part* 1.) So that one which is expressed by the lesser Numbers, is said to be in *lower Terms* than the other: And a Fraction is said to be in its *least* or *lowest Terms*, when there cannot be another equal to it expressed in smaller Numbers: So $\frac{2}{3}=\frac{4}{6}$, and $\frac{2}{3}$ is in the lowest Terms.

PROBLEM V.

To reduce a Fraction to its lowest Terms; i. e. *to find an equivalent Fraction expressed in the least Numbers possible.*

We have already in *Lem.* 5. *Cor.* 3. learnt how a Fraction may be reduced to lower Terms, by finding a Number (if there is any such) which will exactly divide both its Num.^r and Denom.^r: But to reduce a Fraction to its lowest Terms, (or find if it is so already) you must take the following

RULE. Divide the greater Term by the lesser, and the Divisor by the Remainder, and the last Remainder by the preceding one, continually till nothing remains. Then by the last Remainder divide the Num.^r and also the Den.^r; (in which Division there will be no Remainders,) the Quotes are the Terms of the Fraction sought. So that if the last Remainder is 1, the Fraction is already in its least Terms.

Examp. 1.

$\frac{144}{560}=\frac{9}{35}$.

Operation.

```
144) 560 ( 3
     432
     ---
     128) 144 ( 1
          128
          ---
           16) 128 ( 8
               128
               ---
               000
```

Then $144 \div 16 = 9$, and $560 \div 16 = 35$.

Examp. 2.

$\frac{7}{27}$ irreducible;

for

```
7) 27 ( 3
   21
   --
    6) 7 ( 1
       6
       -
       1
```

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DEMONST. There are three things to be here demonstrated, *viz.* 1. That the last Remainder will divide the Num^r and Den^r exactly, (or without a Remainder.) 2. That it is the greatest Number that will do so. 3. That the Quotes make the least equivalent Fraction.

For the first, I must premise these Truths, *viz.* 1. If any Number does measure (or divide without a Remainder) each of two or more Numbers, it will also measure their Sum; for it is contained in the Sum precisely as oft as the Sum of the times it is contained in each of the Parts, (*Lem.* 2. *in Division of Whole Numbers.*) Therefore the Number which measures another, will also measure all the Multiples of that other. 2. Every Dividend is the Sum of the Remainder, and that Multiple of the Divisor produced by the Integral Quote, (by the Proof of *Division.*) Hence, 3. If the Remainder of any Division measure the Divisor, it will also measure the Dividend; for it measures the two Parts of the Dividend, *viz.* the Remainder itself, and that Multiple of the Divisor produced by the Integral Quote.

From these Truths we have a clear Demonstration of the first thing proposed: For in the Operation, every Divisor and Dividend (upward from the last) is the Remainder and Divisor of the last Division: Wherefore since the last Remainder exactly divides the last Divisor, it will also measure the last Dividend; but these being the Remainder and Divisor of the preceding Division, it must also measure the preceding Dividend; and for the same Reason, the Dividend preceding that; and so on it must measure every Divisor and Dividend to the first, which are the Terms of the given Fraction; the thing to be proved.

For the second *Article.* The last Remainder is the greatest Number that will measure the Num^r and Den^r. In order to prove this, consider, That if a Number measures the Sum of two Numbers, and also any one of them, it must measure the other; for the Sum and one Part being Multiples of that Number, so is the other Part, (*Corol.* 3. *Lem.* 2. *in Division of Whole Numbers*) and every Number measures itself and its Multiples. But that Number which measures the Divisor, measures any Multiple of it, *viz.* that Multiple produced by the Integral Quote, which is one Part of the Dividend; and if the same Number also measure the Dividend, it must measure the Remainder, which is the other Part of the Dividend. Now then if the last Remainder is not the greatest Number that measures the Num^r and Den^r of the given Fraction; suppose another greater will do it: Then, by what is now shewn, that other will also measure the first Remainder, (which is the second Divisor;) and because the first Divisor (which this supposed Number measures) is the second Dividend, it will also measure the second Remainder; and so on every succeeding Remainder: Consequently it will measure the last Remainder, which is absurd; for this Number is supposed to be greater than the last Remainder: Wherefore the last Remainder is the greatest Number which measures both the Numerator and Denominator.

For the last *Article,* viz. *That the Quotes make the Equivalent Fraction in lowest Terms:* Let the given Fraction be expressed $\frac{A}{B}$, and any Fraction in lower Terms be $\frac{a}{b}$; These Terms a and b are Quotes of an equal Division of A, B, (by *Cor.* 2. *Lem.* 6.) But the greater the Divisor is, the lesser is the Quote. Therefore the greatest Number which measures A and B, makes the least Quotes, and consequently the least Terms of an Equivalent Fraction.

COROL. If a Fraction is not in its least Terms, the Terms of it are Equimultiples of its least Terms; and these like *Aliquot* Parts of those. Hence again, All Equivalent Fractions in different Terms are different Multiples of the least Terms.

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PROBLEM VI.

To reduce two or more Fractions to one Denominator; i. e. *to find as many Equivalent Fractions having all the same Denominator.*

RULE. Multiply all the Denrs continually into one another, the Product is the common Denr sought. Then multiply each of the given Numrs into the Denrs of all the other given Fractions continually; the Product is the Numr of the Fraction sought, equivalent to the Fraction whose Numr was multiplied.

Examp. 1. $\frac{2}{3}, \frac{5}{7}, = \frac{14}{21}, \frac{15}{21}$. Thus, $3 \times 7 = 21$, the common Denr: Then $2 \times 7 = 14$, which makes $\frac{14}{21} = \frac{2}{3}$. And $3 \times 5 = 15$ makes $\frac{15}{21} = \frac{5}{7}$.

Examp. 2. $\frac{5}{9}, \frac{13}{15}, \frac{8}{13} = \frac{975}{1755}, \frac{1421}{1755}, \frac{1080}{1755}$. Thus, $9 \times 15 \times 13 = 1755$, the common Denr. $5 \times 15 \times 13 = 975$, the first Numr. $13 \times 9 \times 13 = 1421$, the second Numr. $8 \times 15 \times 9 = 1080$, the third Numr.

DEMONST. The Numr and Denr of each Fraction is equally multiplied, *viz.* by the Denrs of all the other Fractions; consequently the Fractions produced are equivalent, by *Lem.* 4. *Cor.* 2.

SCHOLIUM. If it is proposed to reduce any Number of Fractions to as many equivalent Fractions in the lowest Terms that can be with a common Denr; it is plain, that having reduced them first according to the preceding Rule, if we can find the greatest Number that will measure the common Denr and all the new Numrs, these being divided by it, the Quotes will make the Fractions sought. But the Demonstration of the Rule for finding that greatest Number must be referred to another Place. To which I shall therefore refer this Part of the *Problem*; and here only *observe*, that tho' the given Fractions are in their lowest Terms, yet being reduced to a common Denr by the present *Problem*, the new Fractions will not always be in their lowest Terms that admit of a common Denr. *Examp.* $\frac{5}{9}, \frac{13}{15}$, are both in their lowest Terms; and being reduced, they are $\frac{75}{135}, \frac{117}{135}$, which are again reducible to these, $\frac{25}{45}, \frac{39}{45}$; for 3 measures 75 by 25, (*i. e.* $3 \times 25 = 75$) and 117 by 39, and 135 by 45.

COROL. Hence we have another Demonstration of *Article* 1. *Lemma* 5. *viz.* That two Fractions are equal when the Products are equal which are made of the Numr of each multiplied into the other's Denr: So $\frac{2}{3} = \frac{6}{9}$, because $2 \times 9 = 3 \times 6$. For when the two Fractions are reduced to one common Denr, by the preceding Rule, these Products are the new Numrs; and it is certain, that when two Fractions are reduced to a common Denr, if the new Numrs are also equal, these new Fractions are equal, and consequently so are the Fractions to which they are equal; so in the preceding Example, $\frac{2}{3}$ and $\frac{6}{9}$ being reduced, are each $= \frac{18}{27}$; therefore $\frac{2}{3}$ and $\frac{6}{9}$, which are each equal to the same, must also be equal to one another.

PROBLEM VII.

To reduce a Fraction to an Equivalent one of any other given Denr (if possible;) i. e. *to find a Numr which with that given Denomr will make an Equivalent Fraction.*

RULE.

RULE. Multiply the given Denomr by the Numr of the Fraction, and divide the Product by its Denr; the Quote (if there is no Remainder) is the Numr sought.

Examp. To reduce $\frac{3}{4}$ to an Equivalent Fraction, having for its Denr 12. It is $\frac{9}{12}$. Thus, $3 \times 12 = 36$, and $36 \div 4 = 9$, the Numr sought.

DEMONST. This follows from *Corol.* to the last. For let the given Fraction be $\frac{a}{b}$, if the Denr to which it is to be reduced be d, suppose the Numr sought is c. And because $\frac{a}{b} = \frac{c}{d}$ by supposition, then $ad = bc$; therefore dividing both by b, it is $\frac{ad}{b} = c$, according to the Rule.

SCHOLIUM. If the Division has a Remainder, the *Problem* is plainly impossible; yet the given Fraction is equal to the Sum of two Fractions, one of which has the given Denr, and its Numr is the Integral Quote of the Dividend directed, by the preceding Rule; and the other has for its Numr the Remainder of the Division, and the Denr is the Product of the given Denr and the Denr of the Fraction reduced. For *Examp.* if $\frac{4}{7}$ is proposed to be reduced to the Denr 5, I take $4 \times 5 = 20$; then $20 \div 7 = 2$, and 6 remains. Whence I conclude, that $\frac{4}{7} = \frac{2}{5} + \frac{6}{35}$. *Universally,* Let it be proposed to reduce $\frac{a}{n}$ to the Denr m. And let $n\,)\,am = q$, and r remaining; then the *Problem* is impossible. But I say, that $\frac{a}{n} = \frac{q}{m} + \frac{r}{nm}$.

DEMONST. Since $\frac{am}{n} = q + \frac{r}{n}$, then dividing both by m, it is $\frac{a}{n} = \frac{q}{m} + \frac{r}{nm}$, by *Lemma* 2. For $\frac{am}{n}$ expressing the Sum of $q + \frac{r}{n}$ the m Part of $\frac{am}{n}$, which is the Sum, is = the Sum of the m Parts of q and $\frac{r}{n}$, *i. e.* $\frac{q}{m} + \frac{r}{mn}$.

PROBLEM VIII.

To reduce a Fraction to an Equivalent one, having a given Numr, (if possible.)

RULE. Multiply the given Numr by the Denr of the given Fraction, and divide the Product by its Numr, the Quote (if there is no Remainder) is the Correspondent Denr sought.

Examp. To reduce $\frac{4}{6}$ to a Fraction having 18 for its Numr; it is done thus, $18 \times 6 = 108$, and $108 \div 4 = 27$: So the Fraction sought is $\frac{18}{27}$. *Universally;* To reduce $\frac{a}{n}$ to the Numr c, take $cn \div a = m$, then is $\frac{a}{n} = \frac{c}{m}$.

DEMONST. By reversing the given Fraction, and taking the given Numr as a Denr, it becomes the same Case with the preceding *Problem*; and it has been shewn, that if two Fractions are equal, they are so when reversed. But we may argue for this the same way as in that *Problem*: Thus, if $\frac{a}{n} = \frac{c}{m}$, then $am = cn$, (*Lem.* 6.) and $m = \frac{cn}{a}$.

SCHOLIUM. If there is a Remainder, the *Problem* is impossible; yet we can find two Fractions, the one of which has the given Numr, and whose Difference is equal to the given Fraction. For which, this is the Rule; *viz.* Having multiplied the given Numr into

into the Den^r of the Fraction, and divided the Product by its Num^r, take the Integral Quote as a Den^r to the given Num^r. And if from this Fraction you subtract another, whose Num^r is the Remainder, and the Den^r is the Product of the Den^rs, (*viz.* of the given Fraction and that last found) this Difference is equal to the given Fraction.

Examp. If it's proposed to reduce $\frac{5}{7}$ to a Fraction whose Num^r is 9; work thus, $7 \times 9 = 63$. Then $63 \div 5 = 12$, and 3 remains. And then I say, $\frac{5}{7} = \frac{9}{12} - \frac{3}{84}$, (84 being $= 7 \times 12$.) *Universally*, If it's proposed to reduce $\frac{a}{b}$ to the Num^r n; and if $\frac{bn}{a} = q$, with r remaining, then $\frac{a}{b}$ is not reducible to such a Num^r. But I say, $\frac{a}{b} = \frac{n}{q} - \frac{r}{bq}$.

DEMONST. Since $\frac{bn}{a} = q$, and r remaining, then is $bn = aq + r$, (by the Proof of Division.) Hence dividing equally by b, it is $n = \frac{aq + r}{b}$. And again dividing by q, it is, $\frac{n}{q} = \frac{aq + r}{bq} = \frac{aq}{bq} + \frac{r}{bq}$ (*Lem.* 2.) But $\frac{aq}{bq} = \frac{a}{b}$, (*Corol.* 2. *Lem.* 5.) Wherefore $\frac{n}{q} = \frac{a}{b} + \frac{r}{bq}$. Hence lastly, by equal Subtraction, $\frac{n}{q} - \frac{r}{bq} = \frac{a}{b}$. According to the Rule.

Observe, The preceding *Problems* relate all to *Abstract Fractions*, *i. e.* the Fraction reduced, and that to which it is reduced, are supposed to have the same absolute Denomination, or all to be applied to the same Integer; therefore there is none mentioned. The following *Problems* concern *Fractions* as they are specially *Applicate*.

PROBLEM IX.

To reduce a Fraction of an Unit of a higher Value, to an Equivalent Fraction of an Unit of a lower Value, these Units having a known Relation to one another, i. e *the lesser being equal to a certain known aliquot Part of the other.*

RULE. Take the Reciprocal of the Fraction which expresses what Part or Parts the lower Unit is of the higher, and making that with the given Fraction (of the higher) the two Members of a Compound Fraction, reduce it to a Simple, [by *Prob.* 4.] *i. e.* multiply the two Num^rs together and the two Den^rs, the Products make the Fraction sought. And *observe*, if the lower is an *Aliquot* Part of the higher, we have no more to do but multiply the Num^r of the given Fraction of the higher by the Den^r of that Part.

***Examp.* 1.** To reduce $\frac{2}{3}$ of 1 *l.* to a Fraction of 1 *sh.* it is $\frac{40}{3}$ of 1 *sh.* for 1 *sh.* is $\frac{1}{20}$ Part of 1 *l.* and 1 *l.* is 20 *sh.* Therefore $\frac{2}{3}$ of 1 *l.* is $\frac{2}{3}$ of 20 *sh.* by the Rule.

Examp. 2. To reduce $\frac{4}{5}$ of 1 *l.* to a Fraction of 1 Merk, it is $\frac{12}{10} = \frac{6}{5}$: Thus, 1 Merk is $\frac{2}{3}$ of 1 *l.* Therefore 1 *l.* is $\frac{3}{2}$ of 1 Merk, (*Lem.* 7.) So that $\frac{4}{5}$ of 1 *l.* is $= \frac{4}{5}$ of $\frac{3}{2}$ of 1 Merk, which, according to the Rule, makes $\frac{12}{10}$, the Fraction sought.

DEMONST. In the preceding Examples, I have made the Reason obvious. But to demonstrate it more *Universally*; let it be proposed to reduce $\frac{a}{b}$ of a higher Unit to a Fraction of a lower, which is $\frac{n}{m}$ of the higher. I say it is $\frac{a}{b}$ of $\frac{m}{n}$ of the lower: for since the lower is $\frac{n}{m}$ of the higher, this must be $\frac{m}{n}$ of the other, (*Lem.* 7.) Therefore $\frac{a}{b}$ of the higher is $= \frac{a}{b}$ of $\frac{m}{n}$ ($= \frac{am}{bn}$) of the lower, according to the Rule. And if

the

the lower Unit is an *Aliquot* Part of the higher, all we have to do, is to multiply the Numr of the given Fraction (of the higher) by the Denr of the *Aliquot* Part. So, it is $\frac{am}{n}$.

SCHOL. If a Compound Fraction, or a Fraction of a Number greater than Unity, is proposed, first reduce it to a Simple Fraction, and then proceed as above.

PROBLEM X.

To reduce a Fraction of a lower Unit to a higher, (the lower having a known Relation to the higher.)

RULE. Make a Compound Fraction of the given Fraction (of the lower,) and that Fraction which expresses what Part or Parts the lower is of the higher; and reduce this Compound to a Simple, you have the Fraction sought.

Exam. 1. To reduce $\frac{2}{3}$ of 1 *sh.* to the Fraction of 1 *l.* it is $\frac{2}{60}$ *l.* $=(\frac{1}{30}.)$ For 1 *sh.* being $\frac{1}{20}$ *l.* therefore $\frac{2}{3}$ *sh* $=\frac{2}{3}$ of $\frac{1}{20}$ of a *l.* which is $=\frac{2}{60}$ *l.* according to the Rule.

Examp. 2. $\frac{4}{5}$ of 1 Merk $=\frac{8}{15}$ of 1 *l.* For 1 Merk $=\frac{2}{3}$ *l.* therefore $\frac{4}{5}$ of 1 Merk is $=\frac{4}{5}$ of $\frac{2}{3}$ of 1 *l.* $=\frac{8}{15}$ *l.* according to the Rule.

The *Reason* of this Rule is obviously the same in all Cases, and needs not be farther insisted on.

SCHOL. In either of the two last *Problems*, if there are any intermediate Species betwixt the two given Units; and if instead of the Relation betwixt the higher and lower, there be given the several Relations betwixt the Extremes and the Intermediate Species, then reduce the given Fraction to the first intermediate Species, and from that to the next, till you come to the Species required. *Examp.* $\frac{2}{3}$ of 1 *l.* reduced to the Fraction of 1 Farthing, is $\frac{1920}{3}$; which is found either all at once by knowing that 1 Farthing is $\frac{1}{960}$ of 1 *l.* or by degrees thus, $\frac{2}{3}\,l=\frac{40}{3}\,sh=\frac{480}{3}\,d=\frac{1920}{3}$ *farthings.* By multiplying the Numerators gradually by 20, 12, and 4.

PROBLEM XI.

To express any Applicate Whole Number, simple or mixed, by a Fraction of some superiour Integer.

CASE 1. For a Simple Number, make it the Numr, and for Denr take the Number of the inferiour Species which is equal to 1 of the superiour; and that is the Fraction sought. So 8 *d.* is $\frac{8}{12}$ of 1 *sh.* or $\frac{8}{240}$ of 1 *l.*

CASE 2. For a mixed Number, reduce it to the lowest Species expressed in it, and make that the Numr; and the Number of that lower Species which is equal to 1 of the given superiour Species make the Denr, and that is the Fraction sought.

Examp. To express 12 *sh.* 8 *d.* 3 *f.* by the Fraction of a *l.* it is $\frac{611}{960}$ *l.* for the mixed Number is 611 *f.* and 1 *f.* is $\frac{1}{960}$ *l.* therefore 611 *f.* is 611 times $\frac{1}{960}$ *l.* $=\frac{611}{960}$ *l.*

PROBLEM XII.

To find the Value of a Fraction of any Unit (or other Number) of a given Name, in Integers of lower Species, (where there are any such.)

RULE. The given Fraction being (or made) a Simple Fraction, reduce it to a Fraction of the next lower Species, (by *Prob.* 9.) which being improper, reduce it (by

(by *Prob.* 1.) and the Integral Quote is the Anſwer in that Species, if there is no Remainder; but if there is a Remainder, it makes a Fraction of that Species; with which you are to proceed to the next Species, and reduce as before; and ſo on to the loweſt: then the Integral Number found in each Species, with the Fraction of the lower, if there is a Remainder, make up the complete Anſwer. [And *obſerve*, if the Fraction of the firſt, or any ſucceeding lower Species is *Proper*, it is plain you can have no Integer of that Species; and ſo you muſt proceed, and reduce it to the next continually till you have an *Improper* Fraction: And if you never find ſuch a Fraction, then the given Fraction is not expreſſible in Integers.]

Examp. 1. $\frac{2}{3}$ of 1 *l.* is = 13 *ſh.* 4 *d.* which I find thus, $\frac{2}{3}$ *l* = $\frac{40}{3}$ *ſh* (*Prob.* 10.) = 13 *ſh.* + $\frac{1}{3}$ *ſh.* (*Prob.* 1.) and $\frac{1}{3}$ *ſh* = $\frac{12}{3}$ *d.* (*Prob.* 10.) = 4 *d.* (*Prob.* 1.)

The *Reaſon* of this Rule is evident of itſelf.

Scholiums.

1. This *Problem* ſuppoſes the given Fraction a Proper one; but for an Improper, firſt reduce it, and the Integral Quote is the firſt Part of the Value ſought. Then proceed with the Remainder according to the Rule.

2. This Rule is accommodated to all Caſes, whether the lower Units be *Aliquot* or *Aliquant* Parts of the higher. But becauſe in the Caſes which moſt commonly occur, they are *Aliquot* Parts, therefore the Operation is the more Simple; and the Rule may be expreſſed thus, *viz.* Reduce the Num^r of the given Fraction (as an Integer) to the next lower Species, till the Product be equal to, or greater than the Den^r, then divide by the Den^r, the Integral Quote is the Part of the Anſwer in that Species: Reduce the Remainder to the next Species, and divide as before (by the Den^r) and ſo on to the loweſt Species; and you have the Anſwer either in a Simple Whole Number of one Species, or Mixed of different. And if there is a Remainder upon the laſt Species, it makes that Part of the Anſwer belonging to that Species a Mixed Number with a Fraction. And this in effect is the ſame as the preceding Rule.

Examp. 2. To find the Value of $\frac{24}{35}$ *l.* it is 13 *ſh.* 8 *d.* 2 $\frac{10}{35}$ *f.*

Operation:

```
    24 l.
    20
35 ) 480 ſh ( 13 ſh.
     35
     130
     105
      25 Rem.
      12
35 ) 300 d ( 8 d.
     280
      20 Rem.
       4
35 ) 80 f ( 2 10/35 f.
     70
     10 Rem.
```

This way of ordering the *Operation* is diſtinct and eaſy; and it is exactly according to the *General Rule* of the *Problem*, which you will readily perceive by comparing. For this Reduction of the Remainder, and then the Diviſion of the Product, is exactly the *Operation* whereby the Fraction made of that Remainder is reduced to the Fraction of the next Species, and that again reduced to a Whole Number.

3. If

3. If the Fraction to be valued refers to a Whole Number, greater than 1, as $\frac{2}{3}$ of 5 *l.* or to a Mix'd Fraction, as $\frac{2}{3}$ of $6\frac{3}{4}$ *l.* let the Expreſſion be reduced to a ſimple Fraction, and then find the Value. If that ſimple Fraction is improper, reduce it to its equivalent Whole Number, and then find the Value of the Remainders in inferiour Species. For the valuing the Fraction of a Mix'd Whole Number, as $\frac{2}{3}$ of 48 *l.* 14 *ſh.* 8 *d.* it is to be done by multiplying the Mix'd Number by the Num^r of the Fraction, and dividing the Product by the Den^r; for $\frac{2}{3}$ of any kind of Quantity is the ſame as $\frac{1}{3}$ of 2 times that Quantity. Or *Generally*, $\frac{a}{b}$ of any Quantity is $\frac{1}{b}$ of a times that Quantity, which ſhews the Reaſon of this Rule.

CHAP. III.

ADDITION *of* FRACTIONS.

DEFINITION.

ADDITION of FRACTIONS, is finding a Fraction equal to all the given Fractions taken together.

PROBLEM. *To add two or more Fractions into one Sum.*

Rule. Reduce all the given Fractions to ſimple Fractions, of one Unit, and one Den^r, (if they are not ſo already;) then the Sum of the Num^rs being made a Num^r to the common Den^r, makes the fractional Sum ſought, (which may be further reduced as the Caſe admits.)

SCHOLIUM. In the following *Examples*, I thought it ſuperfluous to write down the Operations; but I have ſet down the Effect of every Step in the Work, ſeparating them from each other by the Mark of *Equality*, ſhewing that what follows is equivalent to what precedes; being only the ſame Fractions reduced (where it was neceſſary) to a different State, according to the Direction of the Rule: Which therefore being compared with the Rule, all will be clear and manifeſt.

Examp. 1. $\frac{2}{7} + \frac{3}{7} = \frac{5}{7}$. *Ex.* 2. $\frac{2}{5} + \frac{3}{7} = \frac{14}{35} + \frac{15}{35} = \frac{29}{35}$.

Ex. 3. $\frac{2}{3}$ of $\frac{4}{5} + \frac{13}{15} = \frac{8}{15} + \frac{13}{15} = \frac{21}{15} = 1\frac{6}{15}$.

Ex. 4. $\frac{3}{4}$ of $\frac{2}{7} + \frac{5}{9}$ of $\frac{1}{2} = \frac{6}{28} + \frac{5}{18} = \frac{108}{504} + \frac{140}{504} = \frac{248}{504}$.

In the preceding *Examples* the Integer is ſuppoſed to be the ſame in all the given Fractions, therefore I have named none; but in the following we ſhall make them different.

ſh. d. f.

Examp. 5. $\frac{3}{7}$ *l.* + $\frac{2}{7}$ *ſh.* = $\frac{60}{7}$ *ſh.* + $\frac{2}{7}$ *ſh.* = $\frac{62}{7}$ *ſh.* = $8\frac{6}{7}$ *ſh.* = 8 : 10 : $1\frac{1}{7}$.

Ex. 6. $\frac{3}{5}$ *l.* + $\frac{4}{7}$ *ſh.* = $\frac{60}{5}$ *ſh.* + $\frac{4}{7}$ *ſh.* = $\frac{420}{35}$ *ſh.* + $\frac{20}{35}$ *ſh.* = $\frac{440}{35}$ *ſh.* = $12\frac{20}{35}$ *ſh.*

Ex. 7. $\frac{2}{3}$ *l.* + $\frac{3}{4}$ of $\frac{5}{6}$ *ſh.* = $\frac{2}{3}$ *l.* + $\frac{15}{24}$ *ſh.* = $\frac{60}{3}$ *ſh.* + $\frac{15}{24}$ *ſh.* = $\frac{1440}{72}$ *ſh.* + $\frac{45}{72}$ *ſh.* &c.

Again,

Again, when there are mix'd Fractions, as $4\frac{2}{3}+7\frac{1}{2}$, we may either reduce these to improper Fractions, and proceed by the General Rule; or, add the fractional Parts by themselves, and the integral, and then join both their Sums. Thus in the preceding *Example*, the Fractions added make $\frac{7}{6}=1\frac{1}{6}$, and the Integers make 11; so the Total is $12\frac{1}{6}$. Again, take this *Example* to add $24\frac{3}{4}$ *l.* and 16 *l.* $10\frac{2}{5}$ *sh.* The Sum of the two Fractions, *viz.* $\frac{3}{4}$ *l.* and $\frac{2}{5}$ *sh.* will be found 15 *sh.* 4 *d.* $3\frac{1}{5}$ *f.* The Sum of the whole Numbers is 40 *l.* 10 *sh.* and the Total is 41 *l.* 5 *sh.* 4 *d.* $2\frac{1}{5}$ *f.* *Observe* also, that if the relative Integers of two Fractions are not of one general Nature, so as to have some relation, there can be no Addition.

DEMONST. It is already shewn, in *Cor.* 2. *Lem.* 2. that if several Fractions have one Den^r, the Sum of their Num^rs applied to that Den^r, is a Fraction equal to their Sum; but without that *Lemma*, this Truth will appear very simply and easily thus: The given Fractions being such, or reduced to such a State, that all the Num^rs represent things of the same Denomination, both absolute and relative, [*i. e.* of the same Species and Value in all respects,] their Sum is therefore a Number of things of the same kind, or a Number of such Parts as the common Den^r expresses of the same common Integer.

CHAP. IV.

SUBTRACTION *of* FRACTIONS.

DEFINITION.

SUBTRACTION is the finding a Fraction equal to the Difference of two given Fractions.

PROBLEM. *To subtract one Fraction from another.*

Rule. Reduce them both to simple Fractions of one Unit, and one Den^r, (if they are not so) then subtract the one Num^r out of the other; and make the Remainder a Num^r to the common Den^r, and you have the fractional Difference sought.

The *Reason* of this *Rule* is founded upon the same Principle as that of *Addition*, which need not be repeated. Or may also be deduced from *Lemma* 3. where it's shewn that $\frac{1}{n}$ of A $-$ $\frac{1}{n}$ of B $=$ $\frac{1}{n}$ of $\overline{A-B}$, that is $\frac{A}{n}-\frac{B}{n}=\frac{A-B}{n}$; because $\frac{1}{n}$ of A $=\frac{A}{n}$, and $\frac{1}{n}$ of B $=\frac{B}{n}$, and $\frac{1}{n}$ of $\overline{A-B}=\frac{A-B}{n}$ (*Cor.* 2. *Lem.* 2.)

Examp. 1. $\frac{3}{5}-\frac{2}{5}=\frac{1}{5}$. *Ex.* 2. $\frac{3}{5}-\frac{4}{7}=\frac{21}{35}-\frac{20}{35}=\frac{1}{35}$.

Ex. 3. $\frac{3}{4}-\frac{2}{5}$ of $\frac{1}{7}=\frac{3}{4}-\frac{2}{35}=\frac{105}{140}-\frac{8}{140}=\frac{97}{140}$.

Ex. 4. $\frac{3}{4}$ *l.* $-\frac{5}{6}$ *s.* $=\frac{60}{4}$ *s.* $-\frac{5}{6}$ *s.* $=\frac{360}{24}$ *s.* $-\frac{20}{24}=\frac{340}{24}$ *s.* = &c.

Ex. 5. $\frac{2}{3}$ of $\frac{4}{7}$ *l.* $-\frac{6}{7}$ *s.* $=\frac{8}{21}$ *l.* $-\frac{6}{7}$ *s.* $=\frac{160}{21}$ *s.* $-\frac{6}{7}$ *s.* $=\frac{1120}{147}$ *s.* $-\frac{126}{147}$ *s.* $=\frac{994}{147}$ *s.* = &c.

When there is a Whole Number concerned, either in the *Subtractor* or *Subtrahend*, or both, the Difference may be found also by the General Rule; after reducing Whole and

and Mix'd Numbers to Improper Fractions: But ſuch Caſes may be ſolved eaſier, by reducing only the fractional Parts according to the Rule, and then ſubtracting Fraction from Fraction, and Whole Number from Whole Number. See *Ex.* 6, 7. below. Obſerving this, that where the abſolute Denomination is the ſame, and the fractional Part of the Subtractor is greater than the Subtrahend, borrow Unity; *i. e.* add the Den^{r} (which repreſents the integral Unit) to the Num^{r}, and then ſubtract, and for that carry and add 1 to the Whole Number of the Species to which the Fraction refers: And if there is no Fraction in the Subtrahend, ſuppoſe one, whoſe Num^{r} and Den^{r} are equal each to the Den^{r} in the Subtractor, and the Fraction therefore equal to 1; and ſubtract from it (that is, ſubtract the Num^{r} of the Subtractor from its Den^{r}) and for that add 1 to the Whole Number of that Species, (*Ex.* 8, 9.) And if there is no Whole Number of that Species in the Subtrahend, or leſs than that to be ſubtracted from it, (*Ex.* 12.) you muſt ſupply it as in Subtraction of Whole Numbers. And *laſtly*, mind that the Unity borrowed for the integral Part, muſt be repaid to the Integrals of the next Species, (*Ex.* 11.) and alſo the Unity which may happen to be borrowed for the Fraction of that next Species, (*Ex.* 13.) But the following *Examples* will make all clear.

Examp. 6.

Subtrahend.	$7\frac{3}{4} = 7\frac{9}{12}$
Subtractor.	$4\frac{2}{3} = 4\frac{8}{12}$
Differ.	$3\frac{1}{12}$

Examp. 7.

	$24\frac{3}{5} = 24\frac{18}{30}$
	$19\frac{5}{6} = 19\frac{25}{30}$
Diff.	$4\frac{23}{30}$

Examp. 8.

	14
	$8\frac{3}{7}$
Diff.	$5\frac{4}{7}$

Examp. 9.

	46
	$\frac{7}{13}$
Diff.	$45\frac{6}{13}$

Examp. 10.

	$22\frac{3}{4}$
	16
Diff.	$6\frac{3}{4}$

Examp. 11.

	l.	*ſh.*	*d.*
	32	12	2
		6	$0\frac{5}{13}$
Diff.	32	05	$\frac{8}{13}$

Examp. 12.

	l.	*ſh.*	*d.*
	28	$14\frac{6}{17}$	9
	6	$10\frac{3}{17}$	$2\frac{2}{9}$
Diff.	22	$4\frac{3}{17}$	$6\frac{7}{9}$

Examp. 13.

	l.	*ſh.*	*d.*		*l.*	*ſh.*	*d.*
	82	$09\frac{3}{7}$	00	=	82	$09\frac{9}{21}$	00
	60	$14\frac{2}{3}$	$08\frac{5}{6}$	=	60	$14\frac{14}{21}$	$08\frac{5}{6}$
Diff.					21	$1[illegible]\frac{16}{21}$	$\frac{1}{6}$

Such *Examples* as the 12th and 13th may perhaps never (or very ſeldom) occur in Buſineſs, yet they are an uſeful Exercife to compleat one's Notion of the Nature of Fractions.

Subtraction of Fractions is proved by Addition; the ſame Way, and for the ſame Reaſon, as in Whole Numbers.

CHAP

CHAP. V.

MULTIPLICATION *of* FRACTIONS.

DEFINITION.

TO multiply any Number or Quantity by a Fraction, is no other thing than taking ſuch a Part or Parts of it as that Fraction expreſſes.

PROBLEM. *To multiply one Fraction by another.*

Rule. Reduce both to ſimple Fractions (if they are not ſo) and then multiply the two Num.rs together, and the two Den.rs; their Products make a Fraction, which is the Product ſought.

Examp. 1. $\frac{2}{3} \times \frac{5}{7} = \frac{10}{21}$. *Ex.* 2. $\frac{3}{5}$ of $\frac{9}{13} \times \frac{5}{12} = \frac{135}{780}$ $(= \frac{27}{156}.)$

When either of the two Factors is a Whole Number, and the other a Fraction or Mix'd, or both being Mix'd; theſe Caſes come alſo under the preceding Rule, if the Whole or Mix'd Numbers are firſt reduced to the Form of ſimple Fractions, (as in the following *Examples.*) So that when there is a Whole Number to be multiplied into a Fraction, it's plain we have no more to do but multiply the Num.r by that whole Number; ſo 4 by $\frac{3}{5} = \frac{12}{5}$: And the Reaſon of this we have alſo learn'd before, *Cor. Lem.* 1. for $\frac{3}{5}$ of 4 is 4 times $\frac{3}{5} = \frac{12}{5}$; ſo that it's no matter which of theſe you call the Multiplier.

Examp. 3. $24 \times \frac{3}{5} = \frac{72}{5}$ $(= 14\frac{2}{5})$

Ex. 4. $37\frac{2}{3} \times 6\frac{7}{8} = \frac{113}{3} \times \frac{55}{8} = \frac{6215}{24}$ $(= 258\frac{23}{24})$

Ex. 5. $64 \times 8\overline{\tfrac{2}{3} \text{ of } \tfrac{5}{7}} = 64 \times 8\frac{10}{21} = \frac{64}{1} \times \frac{178}{21} = \frac{12104}{21}$ $(567\frac{8}{21}.)$

SCHOLIUM. After the Product is found by the General Rule, it may be reduced to lower Terms: But this Multiplication being nothing elſe than the Reduction of a compound Fraction to a ſimple, we may apply the Directions in *Schol.* 4. *Probl.* 4. *Reduction,* for finding the Product in lower Terms than the General Rule gives it; of which ſee the *Examples* there explained, which will be needleſs to repeat here.

DEMONST. To multiply by a Fraction ſignifies no more by the Definition, than to take ſuch a Part or Parts of the Multiplicand as that Fraction expreſſes; *i.e.* plainly taking the given Fractions as Members of a compound Fraction, and reducing it to a Simple; which is done (ſee *Probl.* 4. *Reduction*) preciſely according to the Rule here given: So to multiply $\frac{2}{3}$ by $\frac{4}{5}$ is no other thing than taking $\frac{4}{5}$ of $\frac{2}{3} = \frac{8}{15}$. (by *Probl.* 4. and this Rule.) And becauſe it's equivalent in what Order the Members of a compound Fraction are taken, therefore it's the ſame which of them is called the Multiplier or Multiplicand.

For the *Proof* of this Work, it cannot have any which is a more ſimple or eaſier than itſelf; but it has a counter Operation in Diviſion, as we ſhall explain when we come to it.

For APPLICATE NUMBERS.

Any absolute Denomination may be applied to one of the Terms; but the other must be abstract by the Definition; for it can signify only what Part or Parts of the other are to be taken, and the Product is Applicate to the same Things, which is to be further reduced as the Case requires. For *Example* $\frac{3}{4}$ *l.* $\times \frac{5}{6} = \frac{15}{24}$ *l.* $=$ 12 *sh.* 6 *d.*

If the Multiplicand is a Mix'd, Applicate, Whole Number, and the Multiplier a Fraction, then reduce the former to the lower Species; and if there is in that Term a Fraction, make the whole an improper Fraction, and then apply the Rule.

Examp. 6. To multiply 24 *l.* 12 *s.* 8 $\frac{3}{7}$ *d.* by 4 $\frac{5}{9}$ By Reduction they are equal to

$$5912\tfrac{3}{7}\ d. \times 4\tfrac{5}{9} = \frac{41387}{7} \times \frac{41}{9} = \frac{1696867}{63}\ d. = 26934\tfrac{31}{63}\ d. = \&c.$$

But here if the Multiplier is a Whole Number, and a small one, so that the Multiplication can easily be performed without previous Reduction, let it be done that way; beginning with the Fraction in the lowest Species. *Examp.* 7. 14 *l.* 12 *s.* 10 $\frac{2}{5}$ *d.* by 4, $=$ 58 *l.* 11 *s.* 5 $\frac{3}{5}$ *d*: Thus $\frac{2}{5} \times 4 = \frac{8}{5} = 1\frac{3}{5}$ *d.* which 1 is carried to the Product of Pence. *Examp.* 8. 78 *l.* 14 *s.* 6 *d.* by $\frac{5}{9}$: Multiply the Mix'd Number by 5, then divide the Product by 9; and this is a Question of that same kind which we have seen already in *Schol.* 3. *Probl.* 12. *Reduction.*

GENERAL SCHOLIUM.

The word Multiplication, more properly and strictly taken, signifies the encreasing of a Number by Repetition; whereas to multiply by a proper Fraction (according to the preceding Rule) does plainly find a Number less than the Multiplicand; which does therefore rather divide than multiply it. But more particularly, when the Multiplier is an Aliquot Fraction, as $\frac{1}{3}$, the Effect of this Operation is plainly nothing else but Division, *viz.* of the Multiplicand by 3, which finds $\frac{1}{3}$ of it. *Again,* If the Multiplier is a Fraction of any other kind, proper or improper, as $\frac{2}{3}$ or $\frac{7}{5}$, the Operation is mix'd, whereby the Multiplicand is first multiplied and the Product divided; for we take a certain Part of a certain Multiple of it. But this Difference is remarkable, *viz.* That if the Multiplier is a proper Fraction, the Division prevails, and the Number said to be multiplied is really lessened: But if it's an Improper Fraction, the Multiplication prevails, and the Multiplicand is encreased. The first is therefore in some sense more properly a Division, and the last a Multiplication; tho', according to the Definition, they are both called Multiplication; (nor does the first agree to the Definition in Division, as we shall see in the next Chapter.) *Again,* Take notice, that if a Whole Number and a Proper Fraction are multiplied together, the Fraction is, in a strict and proper Sense, multiplied; but the Whole Number is lessened, and is only multiplied in that Sense in which Multiplication by a Fraction is here defined. And now at last if you enquire, How the Name of Multiplication comes to be applied to a Work which really diminishes? it seems to be from this Consideration, *viz.* That whether the Multiplier is a Fraction or Whole Number, the Number found has the same relation to the one Factor as the other has to 1; *i. e.* it contains it as oft, or as many Parts of it, as the other expresses, or as it contains Unity or Parts of Unity; which is plain from the Definition. So in Whole Numbers, $a \times b$ (or the Product of a by b) contains a, b times, and so does b contain 1, b times. In Fractions the Product of a by $\frac{n}{m}$ is $\frac{n}{m}$ Parts of a, as $\frac{n}{m}$ expresses $\frac{n}{m}$ Parts of 1: And because of this general Likeness in the Effect, both are called Multiplication; which is also defined in this general manner, *viz.* Finding a Number which shall have the same relation (above explained) to one of the two given Numbers, as the other has to Unity: tho' the Effect of this is in some Cases really Division, and in all others, is mix'd of Multiplication and Division, taking these in their more strict and proper Sense.

CHAP.

CHAP. VI.

DIVISION *of* FRACTIONS.

DEFINITION.

DIVISION is taken here in the ſame Senſe as already explained in Whole Numbers, *viz.* finding how oft one Fraction is contained in another.

GENERAL SCHOLIUM.

We ſee now by comparing the General Nature of Multiplication and Diviſion by Fractions, as it appears in their Definitions, that they are the ſame way oppoſite in their Effects, as in Whole Numbers. For, let any Number A (Whole or Fraction) be multiplied by any Number B, (Whole or Fraction) and let the Product be repreſented by D; then D contains A, B times, or ſuch a Fraction of A as B expreſſes, (according to the Nature of Multiplication by Fractions;) conſequently, if we divide D by A, *i. e.* enquire how oft A (or what Fraction of it, if it's greater than D) is contained in D, the Quote muſt be B: And reverſely, if any Number (Whole or Fraction) is contained ſo oft (or ſuch a Fraction of it) in D, as B expreſſes, then muſt ſo many times (or ſuch a Part of) A, as B expreſſes, be equal to D. Therefore, whatever be the Rule for finding how oft any Fraction is contained in another Number, this is certain, that the Quote multiplied into the Diviſor (according to the Rules of Fractions) muſt produce the Dividend, or its equivalent; for it may ariſe in different Terms, as we ſhall preſently ſee.

PROBLEM. *To divide one Fraction by another.*

Rule. Reduce both to ſimple Fractions, then take the Dividend and the Reciprocal of the Diviſor, as Members of a Compound Fraction, and multiply them together; the Simple Fraction produced is the Quote ſought; *i. e.* the Quote is equal to ſuch a Fraction of the Dividend as the Reciprocal of the Diviſor expreſſes: Or thus, (which is the ſame thing) Multiply the Num.^{r} of the Dividend into the Den.^{r} of the Diviſor, then the Den.^{r} of the Dividend into the Num.^{r} of the Diviſor; make the firſt Product the Num.^{r}, and the other Den^r of a Fraction, and it's the Quote ſought; which is to be further reduced according to the Circumſtances and Senſe of the Queſtion.

Examp. 1. $\frac{3}{39}$) $\frac{6}{13}$ ($\frac{234}{39} = 6$. *Ex.* 2. $\frac{2}{3}$) $\frac{8}{9}$ ($\frac{24}{18} = 1\frac{6}{18} = 1\frac{1}{3}$.

Ex. 3. $\frac{3}{4}$) $\frac{2}{5}$ ($\frac{8}{15}$. *Ex.* 4. $\frac{1}{3}$ of $\frac{5}{6}$) $\frac{8}{15}$, or $\frac{5}{9}$) $\frac{8}{15}$ ($\frac{72}{75}$.

If there is a Fraction any way concerned in either of the Terms, *i. e.* if either of them is a whole Number, and the other a Fraction or mix'd, or both mix'd Numbers, they come both under the preceding Rule, if the Whole or Mix'd Number is firſt reduced to the Form of a Simple Fraction, (as in the following *Examples.*) So that when either of them is a Whole Number, we have no more to do but multiply the Num.^{r} or Den.^{r} of the other by it, according as that whole Number is the Dividend or Diviſor.

Examp. 5. $24)\ 368\frac{5}{6}$ or $\frac{24}{1})\ \frac{2213}{6}\ (\frac{2213}{144} = 15\frac{53}{144}$.

Ex. 6. $8)\ \frac{24}{39}\ (\frac{24}{312} = \frac{1}{13}$.   *Ex.* 7. $2\frac{3}{4})\ 6\frac{5}{7}$ or $\frac{11}{4})\ \frac{47}{7}\ (\frac{188}{77} = 2\frac{34}{77}$.

Observe. The preceding ***Examples*** are all in abstract Numbers; and for the Management of applicate Numbers, where Fractions are concerned, I shall consider them by themselves, because of some things that require to be particularly explained as to the Sense and Meaning of Division applied in some Cases; but I shall first demonstrate the preceding General Rule.

DEMONST. The Reason of this Rule may be variously deduced thus: (1.) That is the true Quote, which, multiplied by the Divisor, produces the Dividend, (by what's shewn in the preceding *Gen. Schol.*) Now if $\frac{c}{d}$ is divided by $\frac{a}{b}$, the Quote according to the Rule is $\frac{bc}{ad}$, which multiplied by the Divisor $\frac{a}{b}$ (*i. e.* take $\frac{a}{b}$ of $\frac{bc}{ad}$) the Product is equal to $\frac{abc}{bad}$. But this being reduced to lower Terms, *viz.* by dividing both Num^r and Den^r by ab, (or ba,) becomes equal to $\frac{c}{d}$ the given Dividend; therefore $\frac{bc}{ad}$ is the true Quote.

Or, 2. We may prove it thus: Suppose $\frac{c}{d} \div \frac{a}{b} = \frac{n}{m}$, that is, $\frac{a}{b}$ is contained in $\frac{c}{d}$ $\frac{n}{m}$ times; wherefore $\frac{n}{m}$ times $\frac{a}{b}$, or $\frac{n}{m}$ of $\frac{a}{b} = \frac{c}{d}$: But $\frac{n}{m}$ of $\frac{a}{b} = \frac{a}{b}$ of $\frac{n}{m}$, which is therefore $= \frac{c}{d}$; and hence (by *Lem.* 7.) $\frac{n}{m} = \frac{b}{a}$ of $\frac{c}{d}$, which, according to the Rule, is the Quote of $\frac{c}{d} \div \frac{a}{b}$.

Now, tho' either of these two is a strict Demonstration of the Rule, yet being deduced only from the Opposition betwixt Multiplication and Division, it will be useful to see the Reason and Invention of it more directly and immediately from the Nature of Division itself. Thus,

3. Let it be required to divide $\frac{c}{d}$ by $\frac{a}{b}$. If we first enquire how oft is a contained in $\frac{c}{d}$, the Quote is $\frac{c}{da}$ (by *Cor.* 1. *Lem.* 5. *Chap.* 1.) But because we ought to enquire, how oft $\frac{a}{b}$ or $\frac{1}{b}$ of a is contained; and this must be b times as oft, therefore multiply the last Quote $\frac{c}{da}$ by b, the true Quote is $\frac{cb}{da}$ ($= \frac{b}{a}$ of $\frac{c}{d}$,) according to the Rule.

4. We have yet a more simple View of it, thus: Suppose the Divisor and Dividend have (or are reduced to) one common Den^r, then it's evident that the Dividend contains the Divisor as oft, or as many Parts of it, as its Num^r does the other; for having one Den^r, they are in the same State with respect to one another, as Whole Numbers: So that the Num^r of the Dividend, set fractionally over the Num^r of the Divisor, expresses the true Quote. For it's evident, that $\frac{a}{n}$ contains $\frac{b}{n}$ as oft as a contains b, since these Numbers express Units of the same Value. Now, tho' there is no word in the Rule of reducing to one Den^r, yet the fractional Quote, found by the Rule, is plainly the same as that now mentioned; for the Operation is the same as that by which the Num^rs are found, when they are reduced to a common Den^r. Thus, if $\frac{a}{b}$ and $\frac{c}{d}$ are reduced to a common Den^r, they

they are $\frac{ad}{bd}$ and $\frac{bc}{bd}$, and the Fraction made of their Num^rs; *i. e.* the Number of times $\frac{ad}{bd}$ is contained in $\frac{bc}{bd}$, is truly expreſſed by $\frac{bc}{ad}$, which is the Quote of $\frac{c}{d}$ divided by $\frac{a}{b}$, according to the Rule.

COROLLARIES.

If the Diviſor and Dividend, being ſimple Fractions, (or reduced to that State) have a common Den^r, their Quote is the Quote of the Num^rs taken by the Rule of Diviſion of whole Numbers. So $\frac{3}{7}$) $\frac{5}{7}$ ($\frac{5}{3}$ and $\frac{3}{7}$) $\frac{6}{7}$ ($\frac{6}{3} = 2$.

II. From the General Rule this plainly follows: That the Reciprocal of any Quote will be the Quote when the Diviſor and Dividend are changed. For *Example*, $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$, and $\frac{c}{d} \div \frac{a}{b} = \frac{bc}{ad}$. Or thus, if $\frac{a}{b} \div \frac{c}{d} = \frac{n}{m}$, then is $\frac{n}{m}$ of $\frac{c}{d} = \frac{a}{b}$, and $\frac{c}{d} = \frac{m}{n}$ of $\frac{a}{b}$; therefore $\frac{c}{d} \div \frac{a}{b} = \frac{m}{n}$. And the ſame thing being true in whole Numbers alſo, we learn this general Truth, *viz.* That if any two Numbers are divided, either by the other, the Quote of the one by the other, is the Reciprocal of the Quote of that other by the former.

SCHOLIUM. When the Quote is found, it may be reduced to lower Terms; but if you conſider the Dividend and the Reciprocal of the Diviſor, as Members of a compound Fraction, (according to the Rule) then you may apply the Directions given in *Schol.* 4. *Probl.* 4. *Reduction*, for finding the Quote in lower Terms. But the Rules for the more uſeful of theſe Caſes, applied to Diviſion, may be expreſſed in this manner, *viz.*

(1.) If the Num^r and Den^r of the Dividend are Multiples of the Num^r and Den^r of the Diviſor, divide them, and make the firſt Quote Num^r and the other Den^r, and that Fraction is the Quote ſought.

Examp. $\frac{2}{5}$) $\frac{4}{15} = \frac{2}{3}$.

(2.) If the Num^r of the Dividend is Multiple of the Num^r of the Diviſor, but not the Den^rs, then take the Quote of the Num^rs and multiply it into the Fraction made by ſetting the Den^r of the Diviſor over the Den^r of the Dividend.

Examp. $\frac{2}{3}$) $\frac{4}{7}$ ($2 \times \frac{3}{7} = \frac{6}{7}$.

(3.) If the Den^r of the Dividend is Multiple of the Den^r of the Diviſor, but not the Num^rs, then take the Fraction made by ſetting the Num^r of the Dividend over the Num^r of the Diviſor, and divide it by the Quote of the Den^rs; *i. e.* multiply its Den^r by that Quote.

Examp. $\frac{2}{3}$) $\frac{5}{6}$ ($\frac{5}{2} \div 2 = \frac{5}{4} = 1\frac{1}{4}$.

(4.) If the Num^r of the Dividend is Multiple of the Num^r of the Diviſor, and the Den^r of the Diviſor Multiple of the Den^r of the Dividend, divide the Num^rs one by the other, alſo the Den^rs, the Product of thoſe Quotes is the Quote ſought.

Examp. $\frac{3}{39}$) $\frac{6}{13} = 6$, for $6 \div 3 = 2$, and $39 - 13 = 3$.

The *Reaſon* of all theſe Rules is contained in what is explained in the place above referred to; and were ſuperfluous to repeat.

 For

For APPLICATE NUMBERS.

There is the same Variety in Division of Applicate Numbers where Fractions are considered, as when all are Whole Numbers. But before we make any Examples of this kind, it will be proper that we first reflect upon the four different Senses explained in Division of Whole Numbers, and consider how they are to be applied with Fractions.

First, If the Divisor is a Whole Number, the Dividend being a Fraction either pure or mixed, the different Senses are Applicable the same way as if the Dividend were also a Whole Number. For we may enquire, 1. How oft the Divisor is contained in the Dividend. 2. What Part it is of the Dividend. Or, 3. What Number (Whole or Fraction) is contained in the Dividend as oft as the Divisor expresses. Or, 4. What is that Number which is such a Part of the Dividend as the Divisor denominates. In all which there is no matter what the Dividend is; for the Answer of the Question in the first Sense will answer it in them all; except in the second, when the Divisor is not an *Aliquot* Part of the Dividend; as has been already explained.

In the next Place, let the Divisor be a Fraction pure or mixed; whatever the Dividend is, the first two Senses are applicable without any Variation: for we may reasonably ask, 1. How oft a Fraction is contained in any Number. Or, 2. What Part it is of any Number. *Observing* this, That whether the Divisor is an Integer or Fraction, if it is not an *Aliquot* Part, yet we may ask what Fraction it is of the other. 3. We may enquire what is that Number which is contained in the Dividend such a Fraction of a time (*i. e.* of which the Dividend contains, or is equal to such a Fraction) as the Divisor expresses. It is plain, the Quote taken in the first Sense will answer this also; because the Quote and Divisor produce the Dividend. For let $\frac{c}{d} \div \frac{a}{b}$ be $= \frac{n}{m}$, which in the first Sense signifies that $\frac{a}{b}$ is contained in $\frac{c}{d}$, $\frac{n}{m}$ Parts of a time; *i. e.* that $\frac{c}{d} = \frac{n}{m}$ of $\frac{a}{b}$. But $\frac{n}{m}$ of $\frac{a}{b} = \frac{a}{b}$ of $\frac{n}{m}$, (by *Cor.* 1. *Probl.* 4. of Reduction.) Therefore $\frac{c}{d} = \frac{a}{b}$ of $\frac{n}{m}$, *i. e.* $\frac{c}{d}$ contains $\frac{n}{m}$, $\frac{a}{b}$ Parts of a time. Wherefore $\frac{n}{m}$ is the Quote in this third Sense also. And this Question applied to a fractional Divisor, is parallel to the third Sense applied to an Integral Divisor. And to comprehend both without distinguishing, we may ask what is the Number which is contained in the Dividend so many times, or such a Fraction of a time as the Divisor expresses.

4. The fourth Sense in Whole Numbers is finding a Number which is such a Part of the Dividend as the Divisor denominates; which is in effect multiplying by its reciprocal Fraction; for it's plainly taking such a Fraction of the Dividend as the Reciprocal of the Divisor expresses. So to divide by 4, in this Sense, is to take $\frac{1}{4}$. And to make a Question like this with a fractional Divisor, we must seek a Number which is such a Fraction of the Dividend as the Reciprocal of the Divisor expresses; which is the immediate effect of the Rule given for finding the Quote in the first Sense: So that it is plain, the Quote which answers the Question in the first Sense, does so in this Sense also; *i. e.* is a Number which is such a Fraction of the Dividend as the Reciprocal of the Divisor expresses. This therefore is a general Truth, that whatever the Divisor and Dividend be, the Quote found by the general Rule, is such a Fraction of the Dividend as the Reciprocal of the Divisor expresses.

But lastly *observe*, That if a Question is thus proposed, *viz.* to find $\frac{b}{a}$ Parts of $\frac{c}{d}$, it is directly in the Form of Multiplication, and so does not appear as a Question of Division; nor is it so in any other Sense than as the Answer of it is equal to the Number of times

times that its Reciprocal $\frac{a}{b}$ is contained in $\frac{c}{d}$: Or, as it is the Anſwer of a Queſtion of Diviſion made with the Diviſor $\frac{a}{b}$ in the third Senſe.

Now then, as in Diviſion of Applicate Whole Numbers, ſo here in Fractions there are but two Caſes: For either,

1. The Diviſor and Dividend are both Applicate to the ſame kind of thing; and the Queſtion is always in the firſt and ſecond Senſe: Therefore the Quote is an Abſtract Number. And to find it, you muſt reduce the given Numbers to one abſolute Denomination or Integer, and then apply the preceding general Rule. See *Examp.* 8, 9, 10. *below.*

2. The Dividend is Applicate, and the Diviſor Abſtract; then the Queſtion can be propoſed only in the third and fourth Senſe. Minding that the fourth Senſe is to be called Diviſion only as it is the Anſwer of the Queſtion propoſed in the third Senſe, with the Reciprocal of the Diviſor, See ***Examp.*** 14, 15, 16. *below.*

Examples of CASE I.

Ex. 8. $\frac{2}{3}$ *l.*) $\frac{8}{9}$ *l.* ($\frac{4}{3} = 1\frac{1}{3}$.   ***Ex.*** 10. $3\frac{5}{9}$ *l.*) 18 $\frac{3}{7}$ *ſh.* Or, $\frac{32}{9}$ *l.*) $\frac{129}{7}$ *ſh.*

Ex. 9. $\frac{3}{5}$ *ſh.*) $\frac{2}{3}$ *l.*   Or, $\frac{640}{9}$ *ſh.*) $\frac{129}{7}$ *ſh.* ($\frac{1161}{4480}$.

By Reduction.   ***Ex.*** 11. $3\frac{2}{5}$ *ſh.*) 26 *l.* Or, $\frac{17}{5}$ *ſh.*) $\frac{520}{1}$ *ſh.* ($\frac{2600}{17} = 152\frac{16}{17}$.

$\frac{3}{5}$ *ſh*) $\frac{40}{3}$ *ſh* ($\frac{200}{9} = 22\frac{2}{9}$.

If either Diviſor or Dividend is a mix'd Whole Number of different Species, reduce both to a Simple Number of the loweſt Species; and if there is a Fraction in one or both, they muſt alſo be reduced to one Integer, and that the ſame to which the Integral Numbers are reduced. But this will be beſt done by reducing to an Improper Fraction, and then reducing this to the Integer required.

Ex. 12. $4\frac{3}{5}$ *ſh.*) 48 *l.* 18 *s.* 9 *d.* By Reduction, $\frac{276}{5}$ *d*) $\frac{11745}{1}$ *d* ($\frac{58725}{276} = 212\frac{214}{276}$.

Ex. 13. 3 *l.* $4\frac{2}{7}$ *ſh.*) 38 *l.* 12 *ſh.* $8\frac{1}{5}$ *d.*

By Reduction.

$64\frac{2}{7}$ *ſh.*) $9272\frac{1}{5}$ *d.* Or, $\frac{450}{7}$ *ſh.*) $\frac{46361}{5}$ *d.* Or, $\frac{450}{7}$ *ſh.*) $\frac{46361}{60}$ *ſh.* ($\frac{324527}{27000} = 12\frac{527}{27000}$.

Examples of CASE II.

Where the Dividend is Applicate, and the Diviſor Abſtract; i. e. *wherein is ſought a Number, of which ſuch a Fraction or Multiple as the Diviſor expreſſes, is equal to the Dividend.*

Examp. 14. $\frac{2}{3}$) $\frac{8}{9}$ *l.* ($\frac{4}{3}$ *l.* $= 1\frac{1}{3}$ *l.* $=$ 1 *l.* 6 *ſh.* 8 *d.*

Examp. 15. $4\frac{3}{5}$) 24 *l.* 16 *s.* 9 *d.* Or, $\frac{23}{5}$) $\frac{5961}{1}$ *d.* ($\frac{29805}{23}$ *d.* $= 1295\frac{20}{23}$ *d.* $=$ 5 *l.* 7 *s.* $11\frac{20}{23}$ *d.*

If

If the Divisor is a Whole Number, you need not always reduce the Dividend when it is a mixed Number, but carry on the Division by degrees to the lowest Species, where the Fraction is; and what is Remainder upon dividing the Integral Part of that Species, annex to the Fraction, and reduce them to an Improper Fraction, and then divide as is done in the following Example.

Examp. 16. 4) 18 *l.* 13 *sh.* $9\frac{2}{3}$ *d.* (4 *l.* 13 *sh.* $5\frac{5}{12}$ *d.*

GENERAL SCHOLIUM.

As we observed before upon the multiplying by a proper Fraction, that it diminishes the Number multiplied by it, contrary to the effect of an Integral Multiplier, and to the more limited Sense of the word Multiplication; so here, to divide by a proper Fraction does always quote a Number greater than the Dividend, contrary to what is done with an Integral Divisor; and contrary also to the more strict Sense of the word Division, which imports the lessening of a thing: and so the Quote will in some Cases be a Whole or Mix'd Number, tho' the Dividend is a Fraction; because one Fraction may be equal to another taken once, twice, or any Number of Times, or Parts over. Now, that the Quote must be always greater than the Dividend, is plain from the Rule. For the Divisor being a Proper Fraction, its Reciprocal is an Improper Fraction greater than Unity, by which the Dividend is multiplied, which is therefore taken more than once to make the Quote. Again more particularly, if the Divisor is an *Aliquot Fraction*, the Operation is in effect a proper Multiplication; so to divide by $\frac{1}{4}$, is no other thing than multiplying by 4. And the Divisor being a Fraction of any other kind, the Operation is mix'd of a proper *Multiplication* and *Division*, in which the Multiplication prevails, if the Divisor is a proper Fraction; but the Division prevails, if the Divisor is an improper Fraction. Now, because the Quote may be in some Cases greater than the Dividend, we may enquire, How the name *Division* came to be applied to an Operation which really increases a Number: And the Reason of this is probably, (as before we alledged in Multiplication,) the general likeness in the effect of dividing by a Whole Number and by a Fraction, *viz.* That the Quote has the same Relation to the Dividend that Unity has to the Divisor, *i. e.* that the Quote contains the same Part or Parts of the Dividend as Unity does of the Divisor. This is manifest in Whole Numbers; for if B is divided by A, the Quote is $\frac{1}{A}$ of B, and 1 is $\frac{1}{A}$ of A. In Fractions the same truth will exactly appear; Thus, let the Dividend be D, (either a Whole Number, or Fraction,) and the Divisor $\frac{a}{b}$; by the Rule, the Quote is $=\frac{b}{a}$ of D; but 1 is also $=\frac{b}{a}$ of $\frac{a}{b}$; (these being Reciprocals, *Cor. Lem.* 5.) And from hence some define *Division* in this manner, viz. *Finding a Number which shall have the same Relation* (as before explained) *to the Dividend, as Unity has to the Divisor.*—— But we have also this Difference to observe, in the Definitions which I have applied to *Multiplication* and *Division*: That in *Division*, the Definition is the same for Fractions and Whole Numbers without changing one Word, *viz.* finding how oft the *Divisor* is contained in the *Dividend*. But in *Multiplication* there is Variety; for in Whole Numbers it is repeating the Multiplicand, and taking the Whole of it a Number of Times; but with a Fraction it is only taking a Part, or certain Parts of it. Yet this is still true, That the Operations of *Multiplication* and *Division* are reverse to one another; so that the one undoes what the other did, whereby they prove one another. And hence we have this

GENERAL COROLLARY, *To the Rules of Multiplication and Division, both by Whole Numbers and Fractions*; viz.

To multiply by any Number, Whole, or Fraction, and to divide by its Reciprocal, have always the same effect, or brings out the same Number; so $8 \times 3 = 8 \div \frac{1}{3}$; or generally, $A \times m = A \div \frac{1}{m}$, each being (by their proper Rules) $= Am$; or, $\frac{A}{B} \times m = \frac{A}{B} \div \frac{1}{m}$, each being $= \frac{Am}{B}$. And $A \times \frac{n}{m} = A \div \frac{m}{n}$; for each is $= \frac{An}{m}$. Also $\frac{A}{B} \times \frac{n}{m} = \frac{A}{B} \div \frac{m}{n}$, each being $\frac{An}{Bm}$.

The last thing I have to *observe*, is, That some may ask, Why the finding, for Example, $\frac{2}{3}$ of a thing is not called *Division*, since $\frac{1}{3}$ is found by dividing, (*viz.* with 3.) The Answer is plainly this, Dividing by 3 is the same as multiplying by $\frac{1}{3}$. So that to find $\frac{1}{3}$ we do multiply according to the Rules of Fractions, which makes the Product a Fraction; and when this happens to be an Improper Fraction, (as it is always when the Dividend is a Whole Number greater than 3) the Division by 3 is only the Reduction of that Improper Fraction. And for the same Reason to find $\frac{2}{3}$ will also require a Division, if the Fraction produced be Improper: But to make the Comparison aright, let them consider the Difference betwixt dividing by 3 and by $\frac{1}{3}$; these cannot have the same Effect. And if dividing by 3, finds $\frac{1}{3}$ of the Dividend; dividing by $\frac{1}{3}$ does not also find $\frac{1}{3}$ of it. Indeed, the finding $\frac{1}{3}$ is in some sense *Division*; but it is not Division by $\frac{1}{3}$, but by 3. Or it is not finding how oft $\frac{1}{3}$, but how oft 3 is contained in the Dividend. In the same manner, finding $\frac{2}{3}$ is in a sense *Division*, but it is not Division by $\frac{2}{3}$, but by $\frac{3}{2}$ its Reciprocal; in the same manner as $\frac{1}{3}$ is found by dividing by 3, the Reciprocal of $\frac{1}{3}$. In short, we must take *Multiplication* and *Division* according to their established Definitions; about which I have said all that is necessary to make them be clearly understood.

CHAP. VII.

Of the more special Application of Fractions.

I Have in the preceding Chapter not only explained the general Principles and Operations in Fractions, but also made Application to particular things by Examples in all the Operations. These Examples are indeed Simple, and but pure Suppositions: And if we consider that Fractions require more Operation than absolute Numbers, it is unreasonable to bring them into Business without necessity. It remains then that I make a few further Reflections, and shew you how they necessarily occur in Business. In the first place observe, that in what we call mix'd Whole Numbers, the inferiour Species are indeed Fractions, but such as we may call *tacite ones*; because their Denominations are never express'd, it being thought more convenient to distinguish them by common Names than by numeral Denominations; yet these are always understood, and really applied in all Operations: And the Rules given about them in the *first Book*, are either the very same, or deduced from the general Rules of Fractions explained in this *Book*; and so have the same effect: For since each of these Rules is demonstrated to be true and right,

right, the Effect of each muſt be the ſame, when applied to the ſame things. But I ſhall alſo ſhew it by a ſhort Compariſon, thus:

In *Addition* of mix'd Whole Numbers, we add the Numbers of each Species by themſelves without any further preparation, becauſe they are Fractions of the ſame Integer, *viz.* the next higher Species, having all one common Denominator, (or numeral Relation to the next Species,) which is exactly the Rule of Addition of Fractions; and then if the Sum is greater than that Denʳ, it's an Improper Fraction; and accordingly we do in effect reduce it to an Integral Number of that Species, in the very Addition, (which is equivalent to dividing it by the common Denʳ;) leaving the Remainder in the loweſt Species to which it belongs. So Pence are 12ᵗʰ Parts of a Shilling, and therefore we carry the Number of 12's contained in the Sum of Pence to the Shillings; and ſo on.

In *Subtraction* we do the ſame in effect as the Rule for Fractions requires; for the like Species are Fractions having the ſame Denʳ, (with reſpect to the next above,) and when the Number in the Subtrahend is leaſt, we borrow Unity from the next Species, *i. e.* the Denʳ, (as in *Examp.* 7, 8, 9. *Subtraction of Fractions.*)

In *Multiplication*, which is but a reiterated Addition, the Compariſon is the ſame as in Addition, if the Work is performed by beginning at the lower Species, and multiplying upwards; and if we reduce firſt the Mix'd Number to a Simple Number of the loweſt Species, then it is a Fraction of the higheſt, whoſe Denʳ is the Relation betwixt the loweſt Species and higheſt. And this Fraction is multiplied by multiplying its Numʳ, (*i. e.* the Number produced by the Reduction,) and the Product makes an Improper Fraction; the Reduction of it being nothing elſe but finding its equivalent Mix'd Number, and again reducing the Integral Part to the higher Species.

In *Diviſion*, if the Dividend is a Mix'd Whole Number, and the Diviſor a ſimple Abſtract Number, the Compariſon is the ſame as in *Multiplication*, if we take the Method of reducing the mix'd Number to the loweſt Species; and if we divide from the higheſt Species gradually, then it is the ſame as if we expreſs'd each Species as a Fraction of the next above it, and divided each Member by itſelf. Again, if the Diviſor and Dividend are both Applicate, whether both are mix'd, or only one, the Reduction of both to the ſame loweſt Species, is making them both Fractions of the higheſt Species and of one Denʳ, (*viz.* the Relation betwixt the higheſt and loweſt,) and then dividing their Numʳˢ, which is according to the Rule of Fractions.

But again, as in Whole Mix'd Numbers each Species has a Relation to all the reſt, ſo the ſeveral Members of a Mix'd Whole Number may be expreſſed as Fractions of the higheſt Species; and then if theſe are all added together, they will make with the Number of the higheſt Species, a Mix'd Fractional Number. For Example, 8 *l.* 6 *ſh.* 4 *d.* is equal to *l.* $8 + \frac{6}{20}$ *l*: $+ \frac{4}{240}$ *l.* and this reduced, is equal to *l.* $8\frac{76}{240}$. And when ſeveral Mix'd Numbers of one kind are thus expreſs'd, the Addition or Subtraction by the Rules of Fractions, will bring out for the Sum or Remainder, a Number equal to what will ariſe from expreſſing the Sum or Remainder got by the common Rules, in the ſame fractional manner with reſpect to the higheſt. And the ſame thing will hold for the Products and Quotes in *Multiplication* and *Diviſion*; for if this were not true, either the Rules given for fractional Operations, or thoſe for Whole Mix'd Numbers muſt be falſe. But each of theſe are demonſtrated to be true.

But again, *Obſerve*, that the common Rules for Mix'd Whole-Numbers do make eaſier and diſtincter Work than what would happen by that way of expreſſing the inferiour Species, except upon certain Suppoſitions of their mutual Relations, as we ſhall immediately explain. But keeping to the common Subdiviſions at preſent inſtituted, it is better to expreſs them as Mix'd Whole Numbers, and uſe the Rules given about theſe in the firſt *Book*, and never bring in Fractions when they can be avoided: But this cannot always be done; for ſince Fractions neceſſarily ariſe from Imperfect Diviſion, therefore they will

will unavoidably happen upon the Divifion of Numbers of any loweft Species, or of things for which no inferiour Species are inftituted.

Now, tho' the Operations with Mix'd Whole Numbers, according to the prefent Subdivifions, are eafier than what would be if the Numbers of their inferiour Species were exprefs'd fractionally, and the general Method of Fractions applied; yet there is a Suppofition in the Subdivifion of Quantitys, *i. e.* a certain Species of Fractions, according to which the *Divifion* being made, it would be more eafy and convenient to exprefs the leffer always as Fractions of the greater: and that Species is the *Decimal Fraction* already mentioned; whofe Principles and Operations I fhall firft explain, and then more particularly their Ufe and Application.

CHAP. VIII.

Of DECIMAL FRACTIONS.

§. 1. *DEFINITION.*

IF we fuppofe any Integer divided into 10 Parts, and each of thefe again into 10 Parts, making of the Whole 100 Parts; and each of the laft Parts again into 10 Parts, making of the Whole 1000, and fo on: thefe Parts are called *Decimal Parts*; and any Number of them is called a *Decimal Fraction*: Whofe Definition is therefore this, *viz.* a Fraction whofe Den^r^ is 10, or 10 × 10, or 10 × 10 × 10, *&c. i. e.* 10, or 100, or 1000, *&c.*

Examp. $\frac{3}{10}$, $\frac{14}{100}$, $\frac{46}{1000}$, *&c.*

It is plain therefore that the Den^r^ of any *Decimal Fraction* is 1, with one or more o's on the Right-hand of it. And in this lies the effential Difference betwixt *Decimal Fractions* and all others. But there is alfo another Difference, which is in the *Notation* of them: For 'tho' they may be written in the Vulgar Form, yet from this Property of the Den^r^, we have a Method of Notation different from and eafier than the General or Common Way ufed in the preceding Chapters. And hence alfo we have Operations as Simple and Eafy as thofe of Whole Numbers.

§. 2. NOTATION *of* DECIMALS.

THE *Numerator* and *Denominator* of a *Decimal Fraction*, whether Proper or Improper, being known, write firft down the Num^r^, then confider how many Cyphers, or o's belong to the Den^r^; and beginning at the Right-hand or Place of Units of the Num^r^, reckon towards the Left-hand one Figure or Place for every o in the Den^r^. And if there are not as many, fupply the Defect with o's, fet on the Left-hand; and fet before them a Point, (reprefenting the 1 belonging to the Den^r^) called the *Decimal Point*; which is therefore the Mark of a *Decimal Fraction*: As in the following Examples, written both in the *Vulgar* and *Decimal* Form.

Examp. 1. $\frac{3}{10}$ is .3 *Examp.* 2. $\frac{24}{100}$ is .24 *Examp.* 3. $\frac{426}{10000}$ is .0426

Examp. 4. $\frac{28}{10000}$ is .0028 *Examp.* 5. $\frac{46}{10}$ is 4.6 *Examp.* 6. $\frac{3467}{100}$ is 34.67

This *Notation* is arbitrary, and requires no Demonstration, but only to shew that the Num^r and Den^r are thus distinctly marked, tho' not altogether separated from one another: And this is very obvious; for the Num^r is completely expressed, (what Cyphers are in some Cases, set on the Left-hand of it, changing nothing of its Value,) and because the Den^r consists of a Number of o's on the Right-hand of 1, we want only to know how many are of these o's: and this we know by numbering the Places that stand on the Right-hand of the Point. Therefore

To read a Decimal which is written in its proper Form;

Take the whole Rank of Figures, which together make one Number, (*i.e.* the whole Rank excluding the o's that stand on the left of all) for the Num^r; and for the Den^r, reckon as many o's as there are Figures before the Point on the Right-hand. So .057 is $\frac{57}{1000}$, and .004607 is $\frac{4607}{1000000}$.

SCHOL. The same *Problems* and *Rules* of *Reduction* belong to Decimals as to any other Fractions, which need not be repeated. There are only these few particular things to be remarked, which are consequences of the Nature and Notation of Decimals, and the general Rules of *Reduction of Fractions* already explained; and which will serve as Principles for the Demonstration of the following Rules of Operation.

COROLLARIES.

1. A proper Fraction in Decimals can have no Figures standing on the Left-hand of the Point, but all upon the Right; for the Den^r must necessarily have more Places than the Num^r, and so the Point must fall without them on the Left-hand: And an Improper one must have Figures on both hands of the Point; for because the Den^r cannot have more Places than the Num^r, therefore the Point cannot fall without. *Examp.* .046, is Proper; and 4.62, is Improper.

2. A mix'd Decimal Number and its equivalent Improper Fraction have the same *Notation* in Decimal Form, and therefore require no Operation to reduce them; and so the Distinction can only be made in the reading them. For Example, $\frac{3467}{100} = 34 + \frac{67}{100}$, have the same Decimal Notation, *viz.* 34.67. For this is the Reduction of the improper Fraction, by the General Rule; and it is plain the mix'd Number can have no other Notation, except that a Mark of Addition may be put betwixt the Integers and Fraction thus, 34 + .67. But this Mark is superfluous. Hence any Decimal Expression, where there are Figures standing on the Left-hand of the Point, may be read either as an Improper Fraction, or a Mix'd Number; thus, as an Improper Fraction, taking the whole Rank as it stands (without minding the Point) for the Num^r, and for the Den^r reckon as many Cyphers as stand on the right of the Point; or as a Mix'd Number, by taking all the Figures on the left of the Point as a Whole Number, and those on the right as a Decimal Fraction. So this other *Example*, 234.08, is either $\frac{23408}{100}$, or $234 + \frac{8}{100}$

3. It is manifest, that all Expressions wherein there are no Figures, but o's after the Point, are pure Whole Numbers composed of the Figures standing before the Point. So 34.00 is = 34. Yet such Expressions are equal to, and may be read as an improper Decimal Fraction, whereof the Num^r is all the Rank of Figures on each side the Point, and the Den^r has as many o's as stand after the Point; so the preceding is $\frac{3400}{100}$, which, ac-

according to the Decimal Notation is 34.00. Wherefore a Whole Number is expreſſed in Decimal Form by ſetting any Number of o's after it, and a Point between; which is to be read as a Fraction in the manner explained.

4. Cyphers ſtanding in Places next the Right-hand of a Decimal, make the Fraction the ſame as if they were not there; ſo .3400 = .34. Becauſe theſe Cyphers being reckoned both to the Num^{r} and Den^{r}, if they are taken away, do equally divide both; ſo .3400 being $\frac{3400}{10000}$, divide both Terms by 100, (or take two Cyphers from each) it is $\frac{34}{100}$, or .34

5. Two Decimal Fractions having the ſame Number of Figures on the right of the Point, have the ſame common Den^{r}; ſo .34 and .06 have the ſame Den^{r}, *viz.* 100. Hence two or more Decimals are reduced to one Den^{r} by adding as many Cyphers on the right of thoſe that have fewer Places, till they have all an equal Number of Places. So theſe, .4, .25, .067, are equal to, .400, .250, .067; having a common Den^{r} 1000.

6. Every proper Decimal Fraction is equal to the Sum of ſo many leſſer ones, whoſe Num^{rs} are the ſeveral ſignificant Figures of the given Num^{r}, their Den^{rs} having as many o's as there are Places from the Point to that Figure. For *Examp.* 1. .34 = .3 + .04, for .3 = .30, which has the ſame Den^{r} with .04; and therefore that Sum is .34. *Ex.* 2 .04608 = .04 + .006 + .00008

§. 3. ADDITION *of* DECIMALS.

RULE.

WHether the Numbers given are pure or mix'd Decimals, or ſome of them Whole Numbers, write them down under one another in ſuch order that the Decimal Points ſtand all in a Column, and the Figures all in diſtinct Columns, in order as they are removed from the Point either on the Right or Left-hand; then beginning at the Column on the Right-hand, add the Figures in every Column together, and carry forward 1 for every 10 in the Sum, as in Whole Numbers; placing a Point in the Sum under the Points of the given Numbers.

Examp. 1.	*Examp.* 2.	*Examp.* 3.
.24	.004	36.24
.378	.9	450.058
.057	.4067	378.62
.9356	.08	8923.9
.6827	.235	42.007
2.2933	1.6257	9830.825

If ſome of the Numbers given are Whole Numbers without a Fraction, the Work is the ſame, ſetting theſe Whole Numbers on the left of the Column of Points; and if the Sum of the firſt Column in the decimal Part is a Number of 10's, the o need not be written down, but proceed; and do the ſame, if the Figure to be ſet down in the next Column happens alſo to be o. But mind that after a ſignificant Figure comes in the Sum, ſuch o's belonging to the following Columns muſt not be neglected.

Examp. 4.	*Examp.* 5.	*Examp.* 6.
468.	.456	67.004
24.06	.08	8.206
48.724	37.604	.018
2370.	478.26	.4
8.4	.8	.28
2919.184	517.2	75.908

DEMONST. If you conceive as many o's to be set on the Right-hand of the Decimal Point in each of the Numbers added, till they have all an equal Number of Figures after the Point, they are thereby reduced to Fractions (Proper or Improper) having a common Den^r, (*Corol.* 5.) and the Sum of the Num^{rs} is the Num^r of the Sum sought to be applied to that common Den^r; but these o's adding nothing to the Sum, need not be filled up, and therefore the Sum of the Num^{rs} is truly found by the Rule; and by setting a Point in the Sum under the Column of Points in the Numbers added, the common Den^r is right applied, therefore the Rule is good.

Or we may deduce the Reason of this Rule thus: All the Figures of the fractional Parts standing in one Column, are Num^{rs} of Fractions having the same common Den^r, (by *Corol.* 6.) and from the nature of a Decimal Den^r, each 10 in the Sum of any Column is equal to a Fraction whose Num^r is 1, the Den^r having one o less than the Den^r of the Column added; *i.e.* .010 = .01. Therefore it is plain, that adding and carrying of every 10 from the several Columns according to the Rule, gives the true Sum.

SCHOL. In *Applicate Numbers*, the Decimals thus added must all refer to one Integer, or be reduced to that state; and in doing this, reduce always the higher Species to the lower, by multiplying the Numerator, or the lower to a higher by dividing the Num^r, if it can be done; and either way the Fraction found will be a Decimal. But if you reduce any other way, the Fraction will not always be a Decimal. *Examp.* .6 Yards + .08 Quarters, are, by Reduction, .24 qr + .08 qr. Or, .6 y^d + .02 y^d. But if instead of .08 qr, you put .07 qr. then if you reduce this to a Fraction of a Yard, it will not be a Decimal; for it can only be made $\frac{7}{400}$ Yards. And therefore reducing the Higher Denomination to the Lower is the General Rule that will keep the Expression in Decimals. But such Examples seldom occur; as we shall see afterwards in explaining the Use of Decimals.

§. 4. SUBTRACTION *of* DECIMALS.

RULE.

ORder the Subtrahend and Subtractor the same way as directed in *Addition*, then subtract as in Whole Numbers, every Figure of the Subtractor from what stands over it in the Subtrahend, (supposing o to stand where there is no Figure in that Place of the Subtrahend;) and set a Point in the Remainder, in the same Column with the Points of the given Numbers; and when o's fall next the Right-hand of the Remainder, they need not be set down.

Ex. 1.	*Ex.* 2.	*Ex.* 3.	*Ex.* 4.
.84	68.28	48.	478.
.32	24.057	9.86	.037
.52	44.223	38.14	477.963

Ex.

Examp. 5.	*Ex.* 6.	*Ex.* 7.
82.0642	234.075	28.24
29.7562	83.94	16.
52.308	150.135	12.24

The *Demonſtration* of this Rule ſtands upon the ſame Principles as that for Addition. The ſame *Scholium* is alſo applicable here, which was made after Addition.

§. 5. MULTIPLICATION *of* DECIMALS.

RULE.

TAKE the Numbers propoſed (*i. e.* the Rank of Figures in each) as Whole Numbers, and multiply them one by the other as ſuch, neglecting the 0's that ſtand next the Left-hand, as uſeleſs in the Multiplication; then take the Number of Places or Figures, whatever they are, that ſtand after the Point, both in the Multiplier and Multiplicand, and numbering as many Places from the Right-hand of the Product, ſet a Point before them; and if there are not as many, ſupply the Defect with Cyphers, and you have the Product duly expreſſed; which, according to the Circumſtances of the Factors, may be either a Pure Fraction, (*Examp.* 1, 2, 3.) or a Mix'd one, (*Examp.* 4, 5, 6.) or a Whole Number, (*Examp.* 7.)

Examp. 1.	*Ex.* 2.	*Ex.* 3.	*Ex.* 4.
.46	42.78	.000486	6.08
.28	.00094	.021	.47
368	17112	1944	4256
92	38502	972	2432
.1288	.0402132	.0000011064	2.8576

Ex. 5.	*Ex.* 6.	*Ex.* 7.
723.246	578.32	456
48.003	64	2.75
2169738	231328	2280
5785968	346992	3192
2892984	37012.48	912
34717.977738		1254.00

DEMONST. The *Reaſon* of this Rule is obvious; for if you conceive the two given Numbers as Fractions, Proper or Improper, the Work of the Rule is plainly multiplying their Num^rs together, [which are the Rank of Figures propoſed, taken as whole Numbers;] and applying to the Product a Decimal Den^r, equal to the Product of the given Den^rs: [Which is done by taking as many Places from the Product of the Num^r for Decimal Places, as there are in the Den^rs of both the Factors; for the Sum of theſe is the Number of Places in the Product.] And this Fraction is the Product ſought, by the general Rule of multiplying Fractions, (*Chap.* 5.)

SCHOL. To multiply any Decimal (Pure or Mix'd) by 10, or 100, &c. (*i. e.* any Number expreſſed by 1, with 0's after it,) it is plain we have no more to do, but remove the Point as many Places towards the Right-hand, as there are 0's in the Multiplier; and then, if all happen to be 0's on the Left, they are uſeleſs and to be neglected.

Examp.

Examp. 1. $.46 \times 10 = 4.6$ *Ex.* 2. $.004 \times 100 = .4$ *Ex.* 3. $82.0375 \times 100 = 8203.75$

It is plain that this has the ſame Effect as the preceding Rule; according to which we ſhould ſet as many o's after the Multiplicand as there are in the Multiplier; and then taking from the Product as many Decimal Places as are in the Multiplicand, (for we now ſuppoſe none in the Multiplier) the Point will be removed as many Places to the Right, as the Number of o's annex'd; and theſe being ſuperfluous on the Right of a Decimal, need not be ſet down. But if there are not as many Places after the Point, the Defect muſt be ſupply'd with o's, and the Product will be a Whole Number. So $4.6 \times 100 = 460$. Or the Reaſon of this Practice is alſo clear by conſidering, that by ſetting the Point a Place nearer the Right, every Figure is thereby 10 times the Value it was; and conſequently, ſo is the whole.

How to CONTRACT *Multiplication.*

When, betwixt the Multiplicand and Multiplier, there are more decimal Places than we incline to have in the Product, then we may find the Product true to as many decimal Places as we pleaſe, (or very nearly true) without producing all the reſt of the Figures, which will in many Caſes make a great Abridgement of the Work. For which take this

Rule. Conſider how many decimal Places you would have in the Product; ſet the Figure in Unit's Place (*viz.* of Integers) of the Multiplier, under that decimal Figure of the Multiplicand, whoſe Den^r is what you would have in the Product, (*i. e.* under the 1^{ſt}, 2^d, or 3^d, *&c.* Place after the Point, if you would have only one, two, or three decimal Places in the Product.) Then ſet the other Figures of the Multiplier in the reverſe Order from that; and multiply by every Figure in the Multiplier: In doing which, begin only at the Figure of the Multiplicand, under which the multiplying Figure ſtands, neglecting all towards the Right. But at the ſame time conſider what would have been carried from the Product of the preceding Figures on the Right, (which will be found in moſt Caſes by multiplying the two next preceding Figures) that it may be added to the Product which is firſt written down. *Again,* Let all the partial Products be ſet under one another, ſo as the firſt Figures in each ſtand in one Column, and the reſt in the ſame Order. *Laſtly,* In adding theſe partial Products together, you muſt judge as near as you can what would have been carried from the preceding Columns, if we had neglected none of the Figures of the Multiplicand; ſo as to add that to the firſt Column written down. See the following *Examples.*

Examp. To multiply 47.32685 by 8.463, and have three Places of Decimals in the Product.

Operation at large.	Abridged.
47.32685	47.32685
8.463	364.8
14198055	378614
28396110	18930
18930740	2839
37861480	141
400.52713155	400.527

As the Allowance for what may be carried from the Columns neglected is altogether a Gueſs, we may very often make the Product leſs than it ought to be, by 1 or 2 in the laſt Place; which can ſcarcely be help'd otherwiſe than by making one or two more Columns than the Number of Decimal Places you would have in the Product, and then you may cut off the two laſt Places from the Product.

Examp.

Examp. 2. To multiply 463.25639 by 67.864, ſo as two Places of Decimals ſhall be true in the Product.

Operation at large.	Abridged.
463.25639	463.25639
67.864	468.76
185302556	27795383
277953834	3242794
370605112	370605
324279473	27795
277953834	1853
31438.43165096	31438.43

In this *Example* I have ſet the 7 which is in the Unit's Place of the Multiplier under the 3[d] decimal Place of the Multiplicand, tho' I wanted only two Places in the Product, that by that other Place, the two firſt Places may be true.

Obſerve, If there be not as many decimal Places in the Multiplicand as are wanted in the Product, you muſt ſupply them with o's. As if in the preceding *Example* the Multiplicand were 463256.39 and I wanted four decimal Places in the Product, then I write the Multiplicand thus: 463256.3900 (or with one o more) and ſet the 7 of the Multiplier under the laſt o.

Again, If the Multiplier is all a decimal Fraction, imagine a o in the Unit's Place, and ſet the other Figures in order from that on the Left.

For the *Reaſon* of this Practice it is ſeen in the Compariſon of the Work at large and abridged, which you ſee is but the former reverſed.

§. 6. DIVISION *of* DECIMALS.

RULE.

TAKE the Numbers propoſed as Whole Numbers, *i. e.* ſuch a Whole Number as the Rank of Figures would make, without regard to the Point; (in which View o's next the Left-hand will be altogether uſeleſs) and as ſuch, divide the one by the other. If there is a Remainder, which is neceſſarily leſs than the Diviſor, ſet a Cypher after it, and then divide again: But when the Remainder, with one o added, makes a Number leſs than the Diviſor, ſet o in the Quote, and add another o; and ſo on, till the Remainder, with the o's added, make a Number greater than, (or equal to) the Diviſor. And thus continue, adding Cyphers to the Remainder, and dividing, till there be no Remainder. But as this will not happen in every Caſe, the Diviſion is to be thus carried on to a greater or leſſer Number of Figures, according as the Circumſtances of the Queſtion require, as ſhall be further explained in the uſe of *Decimals*. Obſerve alſo, that if at the beginning of the Work, the Dividend makes a leſſer Number than the Diviſor, when both are conſidered as Whole Numbers, then ſet as many o's after it, till it be greater than (or equal to) the Diviſor; and then begin the Diviſion, proceeding with the Remainders, as before directed.

When the Diviſion is finiſhed, or carried on as far as you think fit, the Quote muſt be qualify'd in this manner; *viz.* Conſider how many Decimal Places (or Figures after the Point on the Right-hand) there are in the Diviſor, and alſo in the Dividend; (among which laſt are to be reckoned all the o's added to the Dividend, and to the Remainders: and if the Dividend is a Whole Number, the o's added are reckoned the Decimal Places of it.) Then, 1. If the Number is equal in both, the Quote is a Whole Number, (*Ex.* 1.) 2. If the Number in the Dividend is greateſt, take the Difference, and ſeparate as many for Decimal Places from the Right of the Quote, (ſupplying the Defect with o's) by a Point ſet before them; and then

then the Quote becomes either a *Proper Fraction* or *Improper*, *i. e.* a Mix'd Number. (*Examp.* 2, 3, *&c.*) 3. If the Number in the Divisor is greatest, take the Difference, and set as many o's after the Quote, and take all for a Whole Number. (*Examp.* 7, 8.)

Examp. 1.

```
.004) .128 (32
       12
       ---
       008
         8
       ---
         0
```

Ex. 2.

```
.32) .152 (.475
      128
      ---
      240
      224
      ---
       160
       160
       ---
       000
```

Ex. 3.

```
.64) .8476 (1.3243
      64
      ---
      207
      192
      ---
       156
       128
       ---
        280
        256
        ---
         240
         192
         ---
Rem.      48
```

Ex. 4.

```
2.7) 109.35 (40.5
     108
     ---
       135
       135
       ---
       000
```

Ex. 5.

```
.8) 2.04 (2.55
    16
    --
     44
     40
     --
      40
      40
      --
      00
```

Ex. 6.

```
4.6) .028 (
     or
4.6) .0280 (6086
      276      True Quote,
      ---        .006086
       400
       368
       ---
        320
        276
        ---
Rem.     44
```

Ex. 7.

```
.024) .48 (20
```

Ex. 8.

```
.567) 2721.8 (4800
      2268
      ----
       4538
       4536
       ----
Rem.      2
```

Of Valuing the Remainder, and compleating the Quote.

There remain yet, as a Part of this Rule, some further Considerations about the Remainder and Compleating of the Quote. For tho' in the Application and Use of Decimals, as we shall afterwards learn, the Remainder is neglected, yet what I am now to add, is not only fit to be known as a Part of the Theory, but necessary for our judging aright how far the Division ought to be carried on in different Cases, that the Defect of the Quote, arising from the Neglect of the Remainder, may not be too great. For this is certain, that where there is a Remainder the Division is not perfect; so that the Quote found, and qualify'd by the preceding Rule, will be deficient of the compleat Quote; and this Deficiency depending on the true Value of the Remainder, we shall first see how that is to be found, and then what is to be added to the Quote already found, to make the compleat Quote.

1. The Remainder is to be valued thus: Make it the Num^{r} of a Fraction, whose Den^{r} is that of the Dividend, (taking in all the Cyphers added in the Operation.) So in *Ex.* 3. the Remainder in its true Value is .000048; in *Ex.* 6. it is .0000044, and in *Ex.* 8. it is .2.

2. If you demand the compleat Quote, (when there is a Remainder) *i. e.* which multiplied by the Divisor will produce the Dividend (respecting their true Values) you may find it by the general Rule in *Chap.* 6. But if you would keep the Quote already found as one distinct Part, and would know what is to be added to it that the Sum may be the compleat Quote; then take the Remainder in its true Value (as above,) and divide it by the

the Divifor taken alfo in its true Value, by the Rule of *Chap. 6.* and that Quote is the thing fought. Or thus: Firft make a Fraction of the Remainder and Divifor (as Whole Numbers, without refpecting their true Value) and then fet as many o's after the Den^r as the Decimal Places of the Dividend exceed in number thofe of the Divifor; or if thofe in the Divifor be moft, fet as many Cyphers as the Difference after the Num^r: But if they are equal in Number, you have nothing more to do. And thus you have the thing fought, which will be the fame as that found by the Rule of *Chap. 6.* only in lower Terms, as will eafily appear by comparing them.

Thus in *Ex.* 3. the compleat Quote is $1.3243 + \frac{48}{3400005}$; in *Ex.* 6. it is $.006086 + \frac{44}{480000000}$; in *Ex.* 8. it is $4800 + \frac{202}{587}$. See alfo the following *Examples*, where I have only written down the Quotes without the Operation.

Examp. 9. $.23)\,46.8\,(187 + \frac{5}{23}$.

Ex. 10. $.0432)\,342.8\,(7900 + \frac{15200}{432}$.

Ex. 11. $.008)\,2.68\,(330 + \frac{40}{8}$.

Ex. 12. $.08)\,2742\,(34200 + \frac{600}{8}$.

Obferve again, that this additional Member to the Quote will always be a proper Fraction, when the Number of Decimal Places in the Dividend is equal to, or greater than that in the Divifor; for the Remainder which is the Num^r, is lefs than the Divifor which is the Den^r, and the o's are added to the Den^r: But if it's lefs, then it will be an improper Fraction in fome Cafes, (*Ex.* 10, 11, 12.) which fhews, that the Divifion being carried further on, the integral Number of the Quote would become greater; and particularly if you reduce that improper Fraction, then, as many Figures as its equivalent Whole Number contains, after fo many more Steps in the Divifion, the Quote would have in it all the whole Number that can poffibly belong to it, fo that the additional Member will be after that a proper Fraction: So in *Ex.* 10. the compleat Quote being $7900 + \frac{15200}{432}$, and this Fraction being $= 35\frac{80}{432}$, makes the compleat Quote $7935\frac{80}{432}$. Again, if the additional Member is an improper Fraction, equal to fome whole Number, it fhews, that after fo many more Steps as that Whole Number has Figures, the Divifion would have been perfect without a Remainder. So in *Ex.* 11. the additional Number of the Quote is $\frac{40}{8} = 5$, and the compleat Quote is 335; and in *Ex.* 12. the additional Member is $\frac{600}{8} = 85$, making the compleat Quote 34275.

Wherefore, that the additional Member of the Quote may be always a proper Fraction, (and fo the firft Part never want an Unit of the compleat Quote) carry on the Divifion till the Number of Decimal Places in the Dividend are equal to, or greater than that in the Divifor; unlefs the Divifion is finifh'd without a Remainder before you come to that; for then the Quote found, and qualify'd according to the Rule, is compleat.

DEMONSTRATION *of the preceding* Rule.

The Divifor and Dividend being confidered as Whole Numbers in the Operation, and the o's added to the Dividend and Remainder as belonging to the Dividend; then the Quote being found by the Rule of Whole Numbers, all we have to account for, is the qualifying of the Quote and Remainder, and the additional Member for compleating the Quote. The Reafon of which will eafily appear, by comparing it with Multiplication.

We fhall firft fuppofe there is no Remainder, and then the Product of the Quote and Divifor is equal to the Dividend; but the decimal Places of any Product are equal to the Sum of the decimal Places in the Multiplier and Multiplicand: So the Number of decimal Places of the one Factor is the Difference of the Number in the Product, and in the other Factor, *i. e.* the Number of decimal Places in the Quote, muft be equal to the Difference of the Numbers in the Dividend and Divifor, when the Number in the Dividend is greater, or equal to the Number in the Divifor, (which accounts for thefe two

Cafes, fee *Examp.* 1, 2, 3.) *Again*, When the Number of decimal Places in the Divifor is greater than in the Dividend, then the Quote found (without any Qualification) being multiplied into the Divifor, there would be more decimal Places in the Product than in the Dividend; wherefore that is not the true Quote: But now let as many integral Places of o's as the Difference of the Number of Places in the Divifor and Dividend, be annex'd to the Quote; and this multiplied into the Divifor, there will be the fame Number of decimal Places in the Product as before. But as many of thefe Places next the Right, as the forefaid Difference of decimal Places in the Divifor and Dividend, being o's, becaufe of the o's annex'd to the Quote, they don't increafe the decimal Part; and therefore being cut away from the Product, they leave no more decimal Places in the Product than in the Dividend, the Product being the very fame Rank of Figures: therefore the Quote is truly qualify'd. So if 48 is divided by 24, the Quote is 2; but if the given Numbers are .024 and .48, the Quote muft be 20, for $.024 \times 2 = .048$, which makes one more decimal Place in the Product than in the Dividend; therefore that is not the true Quote; but reckoning this 20, the Product is $.480 = .48$

In the next place, fuppofe there is a Remainder; then, that the Quote and Remainder are duly qualify'd by the Rule, will eafily appear thus: The Product of the Quote and Divifor is equal to the Dividend, after the Remainder is fubtracted out of it, (taking them all as Whole Numbers;) therefore the Remainder added to that Product, makes the Dividend. But now the Quote being qualify'd, as in the Rule, (*i. e.* with regard to the number of decimal Places in the Dividend and Divifor) the Product muft neceffarily have as many decimal Places as the Dividend, otherways the Quote (which is qualify'd with a Regard to the decimal Places of the Dividend) would not be the true Quote out of that Product; which fhews the Reafon of the Rule for qualifying the Quote in this Cafe. Then for the Value of the Remainder, it's plain its Figures muft be of the fame Value with the Places of the Dividend of which it's the Remainder, which are the laft Places on the Right-hand: Or alfo thus; That the Remainder added to the Product of the Divifor and Quote (in their true Values) may make up the Dividend, it's evident it muft be of the fame Value with the Places of the Dividend to which it's added; and thefe are the laft Places on the Right-hand, which make the Num^r of a Fraction, whofe Den^r is that of the Dividend; wherefore the Remainder muft be fo alfo, (which, according to different Circumftances, will be a Whole Number, or a Fraction, or Mix'd.)

Another DEMONSTRATION.

The *Demonftration* of all that relates to this Rule, (*viz.* for both Members of the compleat Quote when there is a Remainder) may alfo be eafily deduced from the General Rule in *Chap.* 6. and fhewn to be the fame. *Thus*, When the Divifion is finifhed, or you have put a Stop to it, confider the Dividend and Divifor as Fractions, Proper or Improper, as they happen (reckoning always the o's, added to the Dividend and Remainders, to belong both to the Num^r and Den^r of the Dividend.) And let $\frac{a}{n}$ be the Divifor and $\frac{b}{m}$ the Dividend, (wherein the n and m are both decimal Den^{rs}, or the one of them fuch, and the other 1, as happens when there is no decimal Place in that Term, but all a Whole Number.) Then by the General Rule, the compleat Quote of $\frac{b}{m}$ divided by $\frac{a}{n}$, is $\frac{bn}{am}$ equal to $\frac{b}{a} \times \frac{n}{m}$. Now $\frac{b}{a}$ is the Quote of the two Num^{rs}, which if it's a Whole Number (there being no Remainder) call it q, and if there is a Remainder, let it be r, then is $\frac{b}{a} = q + \frac{r}{a}$ (the very thing found by the Rule of Divifion of Decimals:) But this

this must be multiplied by $\frac{n}{m}$, *i. e.* both Parts, q, and $\frac{r}{a}$, if there is a Remainder; which is the same very thing in effect that this Rule directs to be done, for qualifying the Quote first found, and then compleating it. For it's plain, if $n = m$ (*i. e.* if the decimal Places in the Divisor and Dividend are equal in number) the Quote found is not thereby alter'd, and therefore $q + \frac{r}{a}$ is the compleat Quote.

Again, Suppose n and m unequal, then by cutting away an equal Number of o's from n and m, they are equally divided. Now therefore if m (the Den^r of the Dividend) has more o's than n (the Den^r of the Divisor) the Fraction $\frac{n}{m}$ is equal to a Fraction whose Num^r is 1, and its Den^r a decimal one, having as many o's as the Difference of the Number of o's in n and m. For *Examp.* $\frac{n}{m} = \frac{10}{1000} = \frac{1}{100}$ But to multiply by such a Fraction, is to divide by its Den^r; *that is*, plainly to divide $q + \frac{r}{a}$ (the Number found by the Operation) by a decimal Den^r, having as many o's as the decimal Places of the Dividend exceed in Number those of the Divisor. And this Division is done by setting off so many decimal Places in the Part q; and in the Part $\frac{r}{a}$, by setting as many o's after the Den^r a, which is the 2*d* Case in the Rule.

Again, let n be greater than m, then is $\frac{n}{m}$ equal to a Whole Number consisting of 1, and after it as many o's as those in n exceed those in m. *Examp.* $\frac{n}{m} = \frac{1000}{10} = 100$. But $q + \frac{r}{a}$, is to be multiplied by this; which is done by setting as many integral Places of o's both after q, and after r, as there are in $\frac{n}{m}$ so reduced; *i. e.* as the decimal Places of the Divisor exceed in Number those of the Dividend. Which is the last Case of the Rule.

We have here demonstrated the Reason for qualifying the Quote, and compleating it; but have not considered the true Value of the Remainder by itself, which is that in which it must be added to the Product of the Divisor and Quote, taken in their true Values to make up the Dividend, (according to its true Value.) But this is done in the former Demonstration.

Of the Use *and* Application *of* DECIMAL FRACTIONS.

WE have already observed, that the great Benefit proposed by Decimal Fractions, is a more simple and easy Operation than what Vulgar Fractions, taken either in their proper Form, or as mix'd Integers, do require. We shall consider how the Application is made for answering that End, and how far it's a real Advantage.

In the first place, this is very evident, that if instead of the Subdivision of Coins, Weights and Measures, (and other kind of Quantities useful in Society) which now obtain, there were one standard superior Species, and all the Subdivisions were Decimals, whether the several Parts were also distinguished by Names, or only by their decimal Denominations, it were the same thing to the purpose; then the Common Operations would be as simple and easy as Whole Numbers. The Rules and Reasons of which are, I hope, compleatly explained in the preceding Part. But supposing this were so, yet either we could not entirely avoid the Consideration of Vulgar Fractions, or we must admit of some

Inaccuracies in Calculations, which are unavoidable with Decimals; and which will be of more or less Consequence in different Circumstances. For we have seen that Decimals will have Remainders (because every Number is not an Aliquot Part of every other) and then the Quote is not compleat without bringing in a Vulgar Fraction; and therefore if we take the Quote without this Correction, it's less than just according to the Value of the Remainder, or rather the Value of the Vulgar Fraction that's necessary to compleat it. Now, if the Number found by this Division is the final Answer of a Question, which is to be applied in no further Calculation, then if it is brought so low as to be less than any Quantity of that kind that is used, (for Example, the smallest real Coin or Weight, *&c.* that has any Name or distinct Being in Society) then the Defect is not to be complained of; because if you do compleat the Quote, the additional Part is of no use: But if a Quote is to be further employ'd in Calculation, especially if it's to be multiplied, the Defect may become considerable; and it will be the more so, as the Multiplier is greater, and also according to the Value of the Integer. Now the only Remedy for this, while we use none but decimal Fractions, is to bring the Division very low; *i. e.* carry it on till the Den^r be very large, and consequently what is deficient be very little: and this is to be regulated according to the Circumstances above-mentioned; for which you'll find more particular Rules afterwards. But then this Inconveniency will frequently happen, That by this means we shall have very large Numbers to work with, which will prove more troublesome than the Method of Vulgar Fractions. These things we shall find more particularly exemplify'd afterwards.

Again, Tho' decimal Subdivisions are not in common use, yet they may be applied by a Reduction of the common Species to Decimals, and these back again to the other: I shall therefore explain this Reduction, and then by particular *Examples* shew the Application, with such Remarks as may give a general View of the Conveniences and Inconveniences of Decimals, and consequently help to judge where they are preferable or not to the common Method.

PROBLEM I.

To reduce a Vulgar Fraction to a Decimal.

RULE. To the Num^r of the given Fraction add one or more o's, as Decimal Places, till it be greater than (or equal to) its Den^r; then divide by the Den^r, adding o's to the Remainders, and carrying on the Division (as directed in *Division of Decimals,*) till o remain, or as far as you please: then make the Quote a Num^r, and apply to it a Decimal Den^r, with as many o's as the Number of o's added to the Dividend and Remainders. This Decimal is exactly equal to the given Vulgar Fraction, if there is no Remainder in the Division; but if there is still a Remainder, that Decimal is deficient by a Compound Fraction, the one Member of which is a Simple Fraction whose Num^r is the Remainder, and its Den^r the Divisor; and the other Member is a Fraction whose Num^r is 1, and its Den^r is that of the Decimal already found. Or it is a Simple Fraction whose Num^r is the Remainder, and its Den^r the Divisor multiplied by the Den^r of the Decimal Fraction found; *i. e.* having as many o's prefix'd to it as belong to the Den^r; wherefore the more Places the Decimal found has, the less is the Defect.

The *Reason* of this Rule you have in the *Division of Decimals;* for the Dividend with the o's added is an Improper Decimal. Or you may take it from *Prob.* 7. *Chap.* 2. for the o's added to the Num^r and Remainder belong to the Decimal Denominator of the Fraction sought, by which the Num^r of the given Fraction is multiplied, and the Division made finds the correspondent Num^r according to that Rule, to which the Den^r is ap-

plied according to the Notation of Decimals; and what this Decimal Fraction is deficient, is also found according to the same Rule.

Ex. 1. $\frac{1}{2} = .5$ *Ex.* 2. $\frac{3}{4} = .75$ *Ex.* 3. $\frac{1}{16} = .0625$ *Ex.* 4. $\frac{7}{13} = .5384$, &c.

Operation.
2) 10 (5

Oper.
4) 30 (75
28
20
20

Oper.
16) 100 (625
96
40
32
80
80

Oper.
13) 70 (5384
65
50
39
110
104
60
52
Rem. 8

In the 4th *Examp.* there is a Remainder 8; and so the Quote wants $\frac{8}{130000}$ of the complete Value of the given Fraction; *i. e.* $\frac{7}{13} = .5384 + \frac{8}{130000}$; this being the true Value of the Remainder.

Observe, The Decimal may be found all at once in the Division, by first setting a Point in the Place of the Quote; then if one o does not make the Numr equal to the Denr, set o after the Point; and if another o makes it still less than the Denr, set another o in the Quote, and so on: *That is*, set as many o's after the Point in the Quote as the Number of o's, which being set on the Right-hand of the Numr, leaves it still less than the Denr; and then after these o's comes the Numr of the Decimal, found by setting one o more on the Right-hand of the Numr which gives the first significant Figure of the Quote, and by o's gradually annex'd to the Remainders, the Work is carried on, and the Numr and Denr of the Decimal sought are thus both together formed; the Reason of which is manifest from the way of finding and applying the Denr to the Numr. See this *Example*.

Ex. $\frac{1}{128} = .0078125$

Operation.
128) 1000 (.0078125
896
1040
1024
160
128
320
256
640
640

COROLLARY 1. In dividing any Whole Number by another, when there is a Remainder, instead of making a Vulgar Fraction of it, we may turn it into a Decimal equal or nearly equal to it, by carrying on the Division with o's added to the Remainders, (in the manner taught in *Division of Decimals*, or in the preceding *Problem*,) till o remain; then the Decimal is equal to the Vulgar Fraction: or till there be many Decimal Places, and then it is nearly equal to it. But how far it ought to be carried on, depends upon Circumstances of the Application (See the following *Scholium* 2.) And after the Integral Quote found, set a Point; so that all the Figures that come after, are Decimal Places: As in the following *Example*.

28) 7645 (273.0357
56
―――
204
196
―――
85
84
―――
100
84
―――
160
140
―――
200
196
―――
Rem. 4 whose
true Value is .0004.

CorOL. 2. When of a simple applicate Whole Number it is proposed to find a certain Part, instead of reducing the Remainders in the Division to lower known Species, we may carry on the Division decimally; and so all the Numbers of inferiour Species that would arise by reducing and dividing are thus turned into a decimal Fraction, (the Design and Use of which you will hear of more particularly afterwards.) If the Division does not soon come to an end, carry it on as Circumstances make it necessary. See the Rule in *Schol.* 2. following.

Scholium I.

That every Vulgar Fraction is not reducible to a determinate Decimal, (*i.e.* where there is no Remainder in the Division,) we know in fact by Examples; in which we find this certain Mark That the Division will never come to an end, *viz.* that there happens a Remainder which is the same with a former Remainder; in which Case it is not only certain that the Division will never have an end, but this we know also that the remaining Figures of the Quote must necessarily be a continual Repetition of the same Figures (in the same order) that stand already in the Quote from that one which proceeded from that former Remainder; as in these Examples.

Examp. 1.
3) 20 (66, *&c.*
18
―――
20
18
―――
2, *&c.*

To reduce $\frac{2}{3}$ to a Decimal, it is .666, *&c.* (the 6 being always repeated,) for it is manifest the same Figure 6 will always arise in the Quote, because it is the same Dividend 20.

Examp. 2. To reduce $\frac{211}{495}$, it is .42626, *&c.* the 26 being still repeated; for the Remainder at which the Operation is stop'd being the same as a former which was the very first Remainder, it is plain that carrying on the Work, we should have for the next two Figures in the Quote 26, and so on still 26 *in infinitum.* Therefore whenever this happens in any Case, we need proceed no further, but observing what Figures in the Quote would be repeated, take as many of them, or the whole of them, as many times as we think fit.

Operation of Ex. 2.
495) 2110 (.426, *&c.*
1980
―――
1300
990
―――
3100
2970
―――
Rem. 130

Such Decimals are very properly called Circulating Decimals, because of the continual Return of the same Figures; and may be called Indeterminate or Infinite Decimals, because they can never come to an end: as we also call those which are the Effect of a Reduction which has no Remainder, Finite or Determinate Decimals. *Observe* also, that these Infinite Decimals may be reckon'd as complete, because tho' they are composed of an infinite Series of Fractions, yet there is a certain and known Order in the Progression of the Series, from the constant Repetition of the same Figures, whereby

whereby they are capable of being managed in Operations so as nothing shall be wanting. But the Demonstration of the Theory and Rules of Operation with such Fractions requires other Principles than have yet been explained, and must therefore be referred to another place: (See *Book* 5. *Chap.* 4.) wherein you'll find it demonstrated, that every Vulgar Fraction will reduce to a Decimal, either finite or circulating.

SCHOLIUM II.

Tho' some Vulgar Fractions will become finite Decimals, yet if these have a great many Places, the Use of them will become very inconvenient and tedious in Practice. Also, tho', we have Rules for managing Circulating Decimals without any Defect, yet the same Inconveniency will arise in these, when they circulate upon many Figures, or when the Circulation begins at a great distance from the Point; therefore it is sufficient for common Use to carry the Reduction so far only, as that the Defect be inconsiderable; (for the further the Reduction is carried, the Defect is the less:) in order to which, I shall here shew you how

PROBLEM II.

To carry the Reduction of a Vulgar Fraction so far, that the Decimal found shall want less than any assigned Fraction.

RULE. Let the assigned Fraction be represented by $\frac{a}{b}$, if a Decimal is carried to so many Places after the Point, as are expressed by b; the defect of that Decimal cannot be equal to $\frac{a}{b}$: For it cannot want a Fraction whose Numr is 1, and its Denr that of the decimal Quote already found, which we may express $\frac{1}{100 \&c.}$ the Denr having as many o's as b has Figures. Since in that Case the Figure in the last Place found would necessarily be greater than it is by 1; or, because by the preceding Rule of this *Problem* the defect is only a compound Fraction, whereof one Member is this Fraction $\frac{1}{100 \&c.}$ which Defect is therefore less than $\frac{1}{100 \&c.}$ and this is evidently less than $\frac{a}{b}$; since 100, *&c.* having as many o's as b has Figures, must be a greater Number; and a is not less than 1: Wherefore $\frac{1}{100 \&c.}$ must be less than $\frac{a}{b}$.

To apply this more particularly: Consider to what Integer any Decimal refers; reduce that Integer to the lowest known Denomination; if the Decimal has as many Places after the Point, as that Number of the lowest Denomination which is equal to the Unit to which the Decimal refers, then the Decimal does not want the Value of an Unit of that lowest Denomination. And if the proposed Decimal were again to be multiplied by any Number, then to make it so that the Product shall not be deficient by an Unit of the lowest Denomination, make it have as many Places after the Point, as the Sum of the Number of Figures in the proposed Multiplier, and in the Number of Units of the lowest Denomination which makes an Unit of the Denomination to which the Decimal refers.

For *Examp.* If a Decimal of 1 *l.* has 3 Places after the Point, it does not want $\frac{1}{1000}$ of it, therefore does not want 1 Farthing, which is $\frac{1}{960}$ of it. And if the Decimal is again to be multiplied by a Number of 5 Places, let the Decimal have 8 (= 5 + 3) Places, and the Product shall not want 1 *f.* For being carried to 8 Places, it cannot want $\frac{1}{100000000}$ of 1 *l* which is $\frac{1}{100000}$ of $\frac{1}{1000}$ But $\frac{1}{1000}$ is less than 1 *f.* and

hence

hence $\frac{1}{100000}$ of $\frac{1}{1000}$ *l.* is leſs than $\frac{1}{100000}$ of 1 *f.* Conſequently if the Decimal carried to 8 Places is multiplied by a Number of 5 Places, (which is leſs than 100000) the Product cannot want 1 *f.* The Univerſality of this Reaſon for all Caſes is manifeſt.

PROBLEM III.

To reduce Integral Numbers of inferiour Denominations to the Decimal of a higher.

CASE I. *To reduce a Simple Number.*

RULE. Expreſs it firſt as a Vulgar Fraction of the higher, by making itſelf the Num^r^, and taking for the Den^r^ the Number of the inferiour Denomination that is equal to 1 of the higher. Then reduce this Vulgar to a Decimal Fraction, by the laſt *Problem.*

Examp. 1. To expreſs 5 *ſh.* in the Decimal of a Pound: Firſt, it is $\frac{5}{20}$ *l.* and this again is reduced to .25 *l.*

Examp. 2. 3 ℔ *Averdupoiſe* Weight the greater, to the Decimal of a hundred Weight. It is $\frac{3}{112}$ *C.* and this is again .02678 *C.* Here the *Diviſion* is imperfect, and we are to carry it on leſs or more as Circumſtances require, according to the preceding Directions.

Operation.

```
112) 300 ( .0267857
     224
     ---
      760
      672
      ---
       880
       784
       ---
        960
        896
        ---
         640
         560
         ---
          800
          784
          ---
   Rem.    16
```

Another Method. It will be in moſt Caſes eaſier to divide gradually from one Species to another, as in the Margin; where 3 ℔ is divided by 28, (at two Steps, *viz.* by 4 and 7.) to bring it to the Decimal of 1 *qr.* and this again by 4, which brings it to the Decimal of 1 *C.*

```
       ℔.
   ( 4 | 3.00
28 ( 7 |  .7500000
   ( 4 |  .1071428
       |  .0267857
```

CASE II. *To reduce a Mix'd Number.*

RULE. Reduce each of the Numbers by the firſt *Caſe*, and then add their Decimals together: Or reduce the mix'd Number to a ſimple Number of the loweſt Species, and then turn that into a Decimal.

Examp. To reduce 9 *ſh.* 6 *d.* to a Decimal of 1 *l.* it is .475 *l.* by adding .45, (the Decimal equal to 9 *ſh.*) and .025, (that equal to 6 *d.*) or by reducing 9 *ſh.* 6 *d. viz.* 114 *d.* equal to $\frac{114}{240}$ *l.* = .475 *l.*

Another Method to find the Decimal of a mix'd Number. Reduce the Number of the loweſt Species to the Decimal of the next above, (whether there be any Number of that Species in the Queſtion, or not,) add to it the Number of that Species in the Queſtion, (if

(if there is any) and reduce the Sum to the next higher Species; adding to the Number found, the Number of that Species given in the Question; and go on so till you come to the proposed Integer.

Examp. To reduce 4 *sh.* 7 *d.* 3 *f.* to the Decimal of a Pound: Makes .2322916 *l.*

4	3.00
12	7.750000
20	4.6458333, &c.
	.2322916, &c.

Thus, 3 *f.* is .75 *d.* to which add 7 *d.* Then is 7.75 equal to .6458333, &c. *sh.* to which add 4 *sh.* and then is 4.645833, &c. equal to .232291666, &c. *l.*

Scholium.

Concerning the Construction and Use of Decimal Tables.

In order to the Application of Decimals, we ought to have ready calculated the Decimal of any Integer of *Money*, *Weight*, *Measure*, &c. answering to every Number, Simple or Mix'd, of inferiour Denomination, and of less Value than that Integer; which Decimals being orderly collected and disposed, make what we call *Decimal Tables*, by which any Decimal required may be readily found, or also the Value of any given Decimal in known inferiour Species.

As to the Construction of these Tables, it would certainly be a very tedious Work to find every Decimal by a separate Application of this *Problem*, [tho' this is a complete general Rule.] There are other Methods to shorten and make that Construction easier; which I shall here explain as far as the Principles already taught do permit.

Rule. Find the Decimal equal to an Unit of the lowest Species; and if that is a determinate Decimal, then from it, as the Root, the Decimals of all the rest may be found accurately by *Addition*. Thus, double the Root, that gives the decimal Fraction for 2 of that lowest Species; then add the Decimal of 1 and 2, the Sum is the Decimal of 3, and so on by adding still the last Decimal to the first, till you come to a Number of that Species equal to an Unit of the next Species above; then make that the First or Root of all the Decimals of that Species, making them up the same way as the last, *i. e.* doubling the first for the Decimal of 2 (of that Species,) add the Decimal of 1 and 2 for 3, and so on in this manner go through all the Species till you come to the Integer itself; and if you add the Decimal of the Number next less than the Integer to the Root, the Sum will be Unity in the Place of Integers.

If the Root, carried to a certain Number of Places, is not determinate, then is it a deficient Decimal; and if we make up the Table from that Root, all the other Decimals in the Table are also deficient, wanting gradually more and more from the Root upwards, so that the Number that comes against the Integer will be a Decimal. But the more Places the Root is carried to, the less is the defect in that and every other Part of the Table. And that there may not be wanting in any of these Decimals an Unit of the lowest Specϵies, or any Part you please of such an Unit, follow the Directions already given.

Observe again, That as all Vulgar Fractions become Decimals, which are either determinate or circulate, so there is an easy way of making up the Tables by Addition from a Root which has one or more circulating Figures, (and that by using only the first Period of them) so as all these Decimals which would be found determinate by a separate Reduction, shall come out so in the Table, and all the rest circulate upon the same Number of Figures in the same Places as the Root does. But as the Reason of this Method de-

depends upon the particular Doctrine of circulating Decimals, and both the Method and Reaſon will be eaſily underſtood when you learn that Doctrine in *Book* 5. *Chap.* 4. I ſhall ſay no more of it here. And only *obſerve* theſe few things.

1. That to bring ſome Vulgar Fractions to a Decimal, determinate or circulating, will be ſuch a long Work, that it's more convenient to take it imperfect with a leſs number of Places.

2. As to the following Tables *obſerve*, That the Roots of ſome of them are determinate; whence all the other Decimals of ſuch Tables are alſo determinate; and are known by their wanting this Sign + after them. In others the Root circulates, and ſo the Table is made up not by the Method of Addition ſimply, but with a due regard to that Circulation; ſo that the Decimals in the Table are theſe which would be found by calculating each ſeparately, and carrying it to the ſame Number of Places.

3. But again *obſerve*, That the Roots of ſome of the Tables circulating upon a ſingle Figure, that Decimal is taken as far as the firſt Figure of the Circulation; and what Decimals in ſuch Tables do circulate, are marked by the Sign + after the circulating Figure; thoſe which want it being determinate.

4. In the laſt place *obſerve*, That for the Tables of *Averdupoiſe Weight the greater*, and of *Time*, the Root was carried on to a Circulation, and the Table made up with a regard to that: But theſe Decimals running to many Places, you have here only the firſt ſeven Places, which are enow for common Uſe. And for *Money*, I have made two different Tables; the Manner and Deſign of which I have explained with the Table.

D E-

DECIMAL TABLES of MONEY, WEIGHTS, *and* MEASURES.

TABLE I. MONEY.

The Integer 1 Pound.

Farth.	1	.0010416	+
	2	.002083	+
	3	.003125	
Penny	1	.00416	+
	2	.0083	+
	3	.0125	
	4	.016	+
	5	.02083	+
	6	.025	
	7	.02916	+
	8	.03	+
	9	.0375	
	10	.0416	+
	11	.04583	+
Shilling	1	.05	
	2	.1	
	3	.15	
	4	.2	
	5	.25	
	6	.3	
	7	.35	
	8	.4	
	9	.45	
	10	.5	
	11	.55	
	12	.6	
	13	.65	
	14	.7	
	15	.75	
	16	.8	
	17	.85	
	18	.9	
	19	.95	

Another TABLE of MONEY.

Farth.	1	.0010416666
	2	.0020833332
	3	.0031249998
Penny	1	.0041666664
	2	.0083333328
	3	.0124999992
	4	.0166666656
	5	.0208333320
	6	.0249999984
	7	.0291666648
	8	.0333333312
	9	.0374999976
	10	.0416666640
	11	.0458333304
Shilling	1	.0499999968
	2	.0999999936
	3	.1499999904
	4	.1999999872
	5	.2499999840
	6	.2999999808
	7	.3499999776
	8	.3999999744
	9	.4499999712
	10	.4999999680
	11	.5499999648
	12	.5999999616
	13	.6499999584
	14	.6999999552
	15	.7499999520
	16	.7999999488
	17	.8499999456
	18	.8999999424
	19	.9499999392
	20	.9999999360

Observe, This second *Table of Money* I have made for no other end but to shew, that if the Root of any Table is not determinate, yet being taken to many Places, the Error will be very little; for the Table being here carried to 20 *sh.* this instead of being equal to 1 *l.* [as it would be had the Root been determinate, or the Numbers calculated with a due regard to that, as in the first Table,] it is a Decimal of 1 *l.* but which wants less than $\frac{1}{10000000}$ Part; the rest wanting less as they stand nearer to 1 *f.* For the Error grows from 1 *f.* to 1 *l.*

TABLE II. *Troy Weight,*

The Integer		1 ounce.	
Grains	1	.002083	+
	2	.00416	+
	3	.00625	
	4	.0083	+
	5	.010416	+
	6	.0125	
	7	.014583	+
	8	.016	+
	9	.01875	
	10	.02083	+
	11	.022916	+
	12	.025	
	13	.027083	+
	14	.02916	+
	15	.03125	
	16	.03	+
	17	.035416	+
	18	.0375	
	19	.039583	+
	20	.0416	+
	21	.04375	
	22	.04583	+
	23	.047916	+
Penny Weight	1	.05	

The Table for Penny Weights is the same as that for Shillings with respect to 1 Pound.

TABLE V. *Of Liquid Measure.*

The Integer		1 gall.
q^r of a Pint	1	.03125
	2	.0625
	3	.09375
Pint	1	.125
	2	.25
	3	.375
	4	.5
	5	.625
	6	.75
	7	.875

TABLE III. *Averdupoise Weight the Lesser.*

The Integer		1 lb.	
q^r of a Dram	1	.00097656	+
	2	.00195313	+
	3	.0029296	+
Dram	1	.00390625	
	2	.0078125	
	3	.01171875	
	4	.015625	
	5	.01953125	
	6	.0234375	
	7	.02734375	
	8	.03125	
	9	.03515625	
	10	.0390625	
	11	.04296875	
	12	.046875	
	13	.05078115	
	14	.0546875	
	15	.05859375	
Ounce	1	.0625	
	2	.125	
	3	.1875	
	4	.25	
	5	.3125	
	6	.375	
	7	.4375	
	8	.5	
	9	.5625	
	10	.625	
	11	6875	
	12	.75	
	13	.8125	
	14	.875	
	15	.9375	

TABLE IV. *Averdupoise Weight the Greater.*

The Integer		1 Cwt.
q^r of an Ounce	1	.0001395
	2	.0002790
	3	.0004185
Ounce	1	.0005580
	2	.0011160
	3	.0016741
	4	.0022321
	5	.0027901
	6	.0033482
	7	.0039062
	8	.0044642
	9	.0050223
	10	.0055803
	11	.0061383
	12	.0066964
	13	.0072544
	14	.0078125
	15	.0083705
Pound	1	.0089285
	2	.0178571
	3	.0267857
	4	.0357142
	5	.0446428
	6	.0535714
	7	.0625, Exact.
	8	.0714285
	9	.0803571
	10	.0892857
	11	.0982142
	12	.1071428
	13	.1160714
	14	.125, Exact.
	15	.1339285
	16	.1428571
	17	.1517857
	18	.1607142
	19	.1696428
	20	.1785714
	21	.1875, Exact.
	22	.1964285
	23	.2053571
	24	.2142857
	25	.2232142
	26	.2321428
	27	.2410714

TABLE VI. *Of Dry Measure.*

The Integer		*1 qr Chald.*
Pint	1	.001953
	2	.003906
	3	.005859
qr of a Peck	1	.0078125
	2	.015625
	3	.0234375

For Pecks and Bushels, they have the same Decimals as qr of Pints, and Pints in Liquid Measure with respect to 1 Gallon.

TABLE VII. *Of Long Measure.*

The Integr		*1 yd.*
qr of Nails	1	.015625
	2	.03125
	3	.046875
Nails	1	.0625
	2	.125
	3	.1875
qr of Yard	1	.25
	2	.5
	3	.75

TABLE VIII. *Of Long Measure.*

The Integer		*1 foot.*	
qr of an Inch	1	.02083	+
	2	.0416	+
	3	.0625	
Inches	1	.083	+
	2	.16	+
	3	.25	
	4	.3	+
	5	.416	+
	6	.5	
	7	.583	+
	8	.6	+
	9	.75	
	10	.83	+
	11	.916	+

TABLE IX. *Of Time.*

The Integer		*1 Year.*
Days	1	.0027397
	2	.0054794
	3	.0082191
	4	.0109589
	5	.0136986
	6	.0164383
Weeks	1	.0191780
	2	.0383561
	3	.0575342
	4	.0767123
	5	.0958904
	6	.1150684
	7	.1342465
	8	.1534246
	9	.1726027
	10	.1917808
	11	.2109589
	12	.2301369
Quarter	1	.25
	2	.5
	3	.75

In this Table of Time, the Decimal for 1 Day is taken by the 365th Part of a Year; and so the Table is carried on to 12 Weeks, or 84 Days. Then as I reckon 13 Weeks to 1 Quarter of a Year, so after 12 Weeks comes next 1 Quarter. But that Decimal and the following are taken accurately, which would not happen if they were continued from the preceding. So that as the Decimals from 1 Day to 84, or 12 Weeks, are true to 7 Places; these for Quarters are accurate, tho' a Quarter is not a precise Number of Days, but 91 Days and $\frac{1}{4}$ of a Day; reckoning 365 Days to a Year: So that in applying this Table to the Calculations of Interest, for which chiefly it is designed, you must reckon 91 Days to a Quarter; by which means the Decimal will be a little greater than what corresponds to 91 Days. But if we continue the Table, then the Decimals for 13 Weeks, 26 W. 39 W. 52 W. (which are less than 1 Year by 1 Day,) are as in the Margin. And if we reckon 13 Weeks a Quarter, then these are the Decimals for 1, 2, 3, 4 Quarters; but deficient for the exact Quarter of a Year, *&c.* And perhaps it may be best to use these Numbers.

13 Weeks	.2493150
26	.4986301
39	.7479452
52	.9972602

So that if he time is within 13 Weeks, or 91 Days, take the Decimals in the 2 upper Parts of the Table; and if it exceeds 91 Days, take for 91 Days (= 13 W.) or 182 Days (= 26 W.) or 273 Days (= 39 W.) the Decimals in this last Part; taking Decimals for what Days are over any of these Numbers, and less than 91, in the former Part; and add all together. In short, Reduce the Number of Days to Quarters, Weeks, and Days, (at 91 Days to a Quarter,) and take their corresponding Decimals and add together. You shall see the Application particularly afterwards.

USE of the preceding TABLES.

I. To find the Decimal of any Integer (which is in the Tables) anſwering to any Number, ſimple or mix'd, leſs than that Integer.

Rule. (1.) If it is a ſimple Number, ſeek it in the Left Column of the Table relating to that Integer, and againſt it you have the Decimal ſought: So for 7*s.* (in the Table of Money) we find .35 *l.*

(2.) If it is a Mix'd Number, ſeek the Decimals anſwering to the ſeveral Members; their Sum is the Decimal ſought. *Examp.* The Decimal for 9*s.* 8*d.* is .483 (the 3 circulating for ever) which is the Sum of .45 and .0333*f.* the Decimals of 9*s.* and of 8*d.*

Obſerve. Decimal Tables would be more compleat, if they were made up for every Number, Mix'd as well as ſimple, leſs than the Integer; but as this would ſwell them to a great Bulk, ſo theſe for the ſimple Numbers are ſufficient; becauſe from them the others can be got eaſily as there is occaſion. Or if any body wants ſuch Tables, they are eaſily made, either by adding their Parts, or by the Method of *Caſe* 2. *Probl.* 3.

II. Having the Decimal of any Integer to find the correſponding Number, ſimple or mix'd, of known inferior Species.

Rule. (1.) Seek the given Decimal in the Table; if you find it there, againſt it ſtands the Number ſought: So againſt .75 in the Table of Money ſtands 15*s.*

(2.) If the given Decimal is not exactly found in the Table, take the next leſſer found there, the Number againſt it is Part of the Anſwer; then take the Difference betwixt that Decimal and the given one; and ſeek it or the next leſſer in the Table, and againſt it you have another part of the Anſwer in a lower Species than the preceding part. Go on thus as long as you can, and you'll find the Anſwer as near as poſſible in known Species.

Examp. 1. To find the Value of .6875 *l.* I ſeek this in the Table, but the neareſt to it (leſſer) which I can find is .65, to which anſwers 13*s.* then the Difference of .6875 and .65 is .0375, which I find in the Table, againſt 9*d.* therefore the Anſwer is 13*s.* 9*d.*

Examp. 2. For .4768 *l.* the neareſt leſſer Decimal is .45 againſt 9*s.* then .4768 leſs .45 is .0268, and the next leſs than this is .025 againſt 6*d.* then .0268 leſs .025 is .0018, the next leſs than which is .00104, &c. againſt 1 *farth.* So the Anſwer is 9*s.* 6*d.* 1*f.* with a Fraction of a Farthing.

But *obſerve*, That if any Decimal given is not found exactly in the Tables, the Value of it may be had in moſt caſes as eaſily, by Reduction (as in *Probl.* 12. Reduction of Vulgar Fractions.) And for any Integer in the preceding Tables, it will be ſufficient to take the firſt three Places after the Point. But the eaſieſt way to ſolve the Problem is by ſuch compleat Tables as are already mentioned.

As the *Decimals* of Money are of the greateſt uſe, ſo alſo there is an eaſy way of finding the Value of any Decimal of a Pound; or finding the Decimal for any Number leſs than 1 *l.* without Tables. Thus:

I. *To find the Value of any Decimal of* 1 *l. without Tables or Pen.*

Rule. Take the firſt three Figures after the Point, neglecting the reſt; then double that Number which ſtands in the firſt Place (after the Point) it is ſo many Shillings of the Anſwer. And if the Figure in the ſecond Place is 5 or greater, add 1 to the Shillings already found; then take what the Figure in the ſecond Place exceeds 5, with the Figure in the third Place, (and if there is no Figure in the third Place, ſuppoſe 0) conſider theſe two Figures, in order as they ſtand, as one Number. If they make a Number not exceeding 23, take ſo many Farthings (and reduce them to Pence) for the remaining Part of the

Answer: But if that Number exceeds 23, take 1 from it, and the Remainder is so many Farthings in the Answer. Thus you shall have the Answer exact in all its Value of known Species; or so, that the Error shall not be 1 Farthing.

Examp. 1.	.4 *l.* = 8 *s.*	*Examp* 5.	.089 *l.* = 1 *s.* 9 *d.* 2 *f.*
2.	.35 *l.* = 7 *s.*	6.	.67 *l.* = 13 *s.* 5 *d.*
3.	.248 *l.* = 4 *s.* 11 *d.* 3 *f.*	7.	.038 *l.* = 9 *d.* 1 *f.*
4.	.317 *l.* = 6 *s.* 4 *d.*	8.	.04 *l.* = 9 *d.* 3 *f.*

If you compare these *Examples* with the *Rule*, the manner of finding the Value will be plain, without any further Explication.

The *Reason* of this Rule is thus:

1. Since 1 Shilling is the $\frac{1}{20}$ Part of a Pound, and double any Number of 10th Parts, makes so many 20th Parts; (so $\frac{1}{10}=\frac{2}{20}$) therefore double the Figure in the 1*st* Place (whose Denr is 10th Parts) is equal to so many 20th Parts, or Shillings. Again,

2. Since $\frac{1}{20}=\frac{5}{100}$, therefore 5 in the 2d Place (whose Denr is 100th Parts) is 1 Shilling. Then,

3. The Figure in the 3d Place has 1000th Parts for its Denr, and this with the Number over 5 in the 2d Place, makes so many 1000th Parts; which is little less than so many Farthings; because 1 Farthing is $\frac{1}{960}$ Part of a Pound. But when we make up a Decimal Table for Farthings from 1 to 47, (which is 11 *d.* 3 *f.*) we find this true in Fact, That from 1 to 23 Farthings the Figures in the 2d and 3d Places of the Decimal are the same with the Number of Farthings: But from 24 to 47, the Figures in the 2d and 3d Places make a Number 1 more than the Number of Farthings. And tho' in all these Decimals (except that for 6*d.* or 24*f.*) there are Figures after the 3d Place, yet their Value is not 1 Farthing, because they do not make .001, which is less than 1 Farthing.

II. *To find the Decimal of a Pound, answering to any Number of Shillings, Pence, and Farthings, less than a Pound, without Tables or Pen.*

Rule 1. If the Number of Shillings is even, take its Half and set in the first Place after the Point, (*Ex.* 1.) If it's odd, set the Half of the next lesser even Number in the first Place, and 5 in the 2d, (*Ex.* 2.) then reduce the *d.* and *f.* to *f.* and if they are fewer than 24 (*i. e.* the *d.* fewer than 6=24*f.*) set that Number in the 2d and 3d Places (*i. e.* in the 3d Place if it's but one Figure, (*Ex.* 3.) and if it has two, add that which is in the Place of Tens to the Figure standing already in the 2d Place, if there is any, and set the other in the 3d Place, *Ex.* 4.) But if these Farthings exceed 23 (*i.e.* if the *d.* exceed 5) add 1 to them, and set that Number in the 2d and 3d Places as before, (*Ex.* 5, 6.) Thus you have the Decimal sought, true to three Places, which is sufficient for common use. But,

2. If you would compleat the Decimal, then if the Number of *f.* to which the *d.* and *f.* in the Question are equal, do not exceed 23, take that Number of *f.* or, if they exceed 23, take the Remainder after 24 is subtracted from them, and divide that Number or Remainder decimally, (*viz.* by prefixing 0's to it) by 24 (which is easily and readily done by 4 and 6) the Quote, which will either be determinate, or circulate on 6 or 3, being set after the Figures already found, the Decimal is compleated.

Examp. 1.	4 *s.* = .8 *l.*	*Examp.* 6.	9 *s.* 9 *d.* = .4875
2.	13 *s.* = .65	7.	7 *d.* 3 *f.* = .0322916 &c.
3.	6 *s.* 2 *f.* = .308	8.	4 *s.* 8 *d.* 3 *f.* = .2364583 &c.
4.	8 *s.* 3 *d.* = .4125	9.	5 *s.* 4 *d.* 1 *f.* = .2677083 &c.
5.	7 *s.* 6 *d.* = .375		

The

The *Reason* of this *Rule* is this:

1. For the Shillings: The half of any Number of 20^{th} Parts (*i. e.* of any Number of Shillings) makes so many 10^{th} Parts; and if there is an odd Shilling, it is equal to 5 in the 2^d Place, or $\frac{5}{100}$ Parts, because $\frac{5}{100}$ is $\frac{1}{20}$.

2. For the Pence and Farthings: If we make a Table of Decimals for any Number of Farthings from 1 to 47 (equal to 11 *d.* 3 *f.*) then for any Number less that 24 (or 6 *d.*) the Decimal has that Number in the 3^d, or 2^d and 3^d Places, (*i. e.* it will have so many 1000^{th} Parts. And if these Farthings exceed 23, the Decimal has 1 more 1000^{th} Parts. Again, For any Number of Farthings less than 24 (or 6 *d.*) consider, that because 1 Farthing is $\frac{1}{960}$ Part of a Pound, which is greater than $\frac{1}{1000}$ Part; therefore besides so many 1000^{th} Parts, there must be added such a Decimal as is equal to the Difference of so many 960^{th} Parts and 1000^{th} Parts. Now, if we subtract $\frac{1}{1000}$ from $\frac{1}{960}$ the Remainder is $\frac{1}{24000}$; wherefore any Number less than 24, of 24000^{th} Parts, will make a Decimal, whose first Place will fall in the fourth Place after the Point: Consequently this Decimal which remains to compleat the Decimal sought, falls after the 3 Places already found.

Lastly, If the Number of Farthings exceeds 23, it is either 24; and then there is nothing to be added to the Number of 1000^{th} Parts already set down: Or it's more than 24. And what's more, being less than 23; the Decimal to be added for that, comes under the same Rule as the last Article.

QUESTIONS, *shewing the* Application *of* Decimals *in Multiplication and Division.*

Quest. 1. There is 14 *l* : 8 *s* : 6 *d* in each of 6 Bags: How much is in the whole? *Answer,* 86 *l* : 11 *s.* Thus, 14 *l* : 8 *s* : 6 *d* is 14.425 *l.* which multiplied by 6 produceth 86.55 equal to 86 *l* : 11 *s.*

Quest. 2. If 1 Yard of Cloth cost 14 *s* : 8 *d.* what is the Value of 24 *yds* : 3 *qr* : 1 *na* *Ans.* 18 *l* : 3 *s* : 2 *f.* nearest. Thus, 14 *s* : 8 *d* is .7333 *&c. l.* and 24 *yds* : 3 *qr* : 1 *na* is 24.7625 *yds*: Which multiplied by .7333 produces 18.158 *&c. l.* which is 18 *l* : 3 *s* : 2 *f* nearly.

Quest. 3. If 30 *l.* buy 124 *yds* : 1 *qr* : 2 *na.* of Cloth, How much will 1 *l.* buy? *Answ.* 4 *yds* : 2 *na.* and .32 nearly. Thus, 124 *yds* : 1 *qr* : 2 *na.* is 124.375, which divided by 30, quotes 4.145 *&c.* which is 4 *yds* : 2 *na.* and .32 nearly.

Quest. 4. If 8 *l* : 9 *sh* : 4 *d.* buy 3 hundred Weight : 1 qr : and 18 Pound of Sugar, What may be bought for 1 *l*? *Answ.* 1 qr : 17 ℔ and .1 nearly. Thus, 8 *l* : 9 *sh* : 4 *d.* is 8.4666 *&c.* and 3 C : 1 *qr.* 18 ℔. is 3.410714, which divided by the other, quotes .4028 *&c.* which is 1 qr : 17 ℔ and .1 nearly.

Observe, The Use and Application of Decimals will more fully appear in the remaining Parts of this Work; especially by applying them in *Book* 6. As to which Application, this in general only needs to be further said here, That any Integer being consider'd as the highest Denomination, all Numbers or Quantitys less than that are to be expressed decimally by taking the Decimal of that Integer answering to that lesser Quantity; and in the same Question using Decimals of the same Integer for all Numbers of the same kind, (*i. e.* for all Numbers of Money, use the same Integer as 1 *l.*) Then multiply and divide by these Numbers according to the Rules of Decimals. And in Multiplication you may use the manner of Contraction explained in *Ch.* 8. §. 5.

I must *observe* in the last place, that most Questions in common Business are sooner done without Decimals, by the common Methods of Reduction; but when to use Decimals, or the common Methods, must be left to every body's own choice: and indeed a good deal of Practice will be necessary to enable one to chuse judiciously. And particularly as to the preceding Examples, *observe,* That the first is easier done by the common Method; because it can be done without Reduction, the Multiplier being small. The second and fourth cannot be done by any Method hitherto explain'd, (except by Decimals;) because by the Method of Reduction they require both Multiplication and Division; as you will afterwards understand in the *Rule of Three,* (*Book* 6.) The third may be easily done, the common way.

BOOK

BOOK III.

Of the Powers and Roots of Numbers.

CHAP. I.

Containing the THEORY.

DEFINITIONS.

I. A *Power* is a Number, which is the Product of a certain Number of equal Factors, *i.e.* of the fame Number multiply'd into itfelf continually a certain Number of Times.

II. A *Root* is a Number, by whofe continual Multiplication into itfelf another Number (which is called the Power) is produced.

Example. Let any Number 2 be multiplied into itfelf, the Products are 4 $(=2\times 2)$ 8 $(=4\times 2)$ 16 $(=8\times 2)$ *&c.* Then is 2 called the Root of thefe Products, which are called the Powers of that Root.

Hence it is plain, that Power and Root are relative things. Every Power is the Power of fome Root, and every Root is the Root of fome Power: So that by calling one Number the Power or Root of another, we mean that it is the Number produced, or the Number producing that other by continual Multiplication.

But the feveral Powers, and the Root in relation to thefe, are alfo diftinguifhed by particular Names, which fhall next be explained.

III. The firft Product (*viz.* that of the Root multiplied by itfelf) is called the Square of the Number multiplied; which in refpect of the other is called the Square Root. So 4 is the Square of 2, and 2 the Square Root of 4, becaufe $2\times 2=4$.

The fecond Product (*viz.* of the Square by the Root) is called the Cube, and the Root in refpect of it is called the Cube Root: So 8 is the Cube of 2, and 2 is the Cube Root of 8; for $2\times 2\times 2$, or $4\times 2=8$.

Others of the Powers had alfo particular Names among the Ancients; but they are very complex and burthenfome to the Memory, and tend no way to the Improvement or Eafinefs of the Science: Whereas it is obvious that we have no more to do, but diftinguifh them by their Order in the Series of Products, calling the firft Product the firft Power, the fecond Product the fecond Power, and fo on; whereby thefe Names do of themfelves in a very fimple and eafy manner diftinguifh the feveral Powers, in confequence of the general Definition of a Power: for they exprefs the Number of Multiplications of the

Root

Root in the Production of each Power; which the ancient Names do not. For the Names Square and Cube, of which the reſt were compounded, are Names of geometrical Quantities applied to Numbers, only from this Conſideration, that the Meaſures of theſe Quantities are found by ſuch an Application of Numbers, as do produce the Numbers, which are hence called *Square* and *Cube*.

But obſerve again, that tho' in conſequence of the preceding Definitions of Power and Root, theſe Terms ought always to be contradiſtinguiſhed, ſo that the Products only can be called Powers; yet for the ſake of a particular Conveniency, which we ſhall preſently underſtand, the Root is called the firſt Power, and the Products in order are called the ſecond, third, *&c.* Power, as here:

2.	2 × 2.	4 × 2.	8 × 2.	16 × 2.	*&c.*
2.	4	8	16	32	*&c.*
Root or firſt Power.	Square or 2d Power.	Cube or 3d Power.	4th Power.	5th Power.	

In which Method the Root is the ſame with the firſt Power, and contradiſtinguiſhed only from the ſuperior Powers, with reſpect to which we call it the ſecond or third, *&c.* Root; tho' more commonly we uſe the Names Square and Cube, and Square Root, Cube Root; uſing the Names fourth, fifth, *&c.* Power and Root, for the degrees above the Cube or third Degree.

Of the univerſal Notation of Powers *and* Roots.

I. *Of* POWERS.

TAKE any Number A for a Root, and the Series of its Powers according to the Definition will be thus:

A.	AA.	AAA.	AAAA.	*&c.*
Root, or 1ſt Power.	2d Power.	3d Power.	4th Power.	

Each of theſe Terms expreſſing the continual Product of A, taken ſo oft as it is placed in each of them, which being once more at every Step gradually from the Root, we have alſo this more convenient Method of expreſſing them, *viz.* by writing only the Root with a Number over it, to ſignify how oft the Root is to be taken, or placed as a Factor in producing that Power. Thus the 4*th* Power of A is AAAA, to be written, according to this other Method, thus, A^4; and ſo of others, the whole Series of the Powers being repreſented thus:

$$A^1,\ A^2,\ A^3,\ A^4,\ A^5,\ A^6,\ \&c.$$

When a Number A has no Figure or Mark of Power, it's ſuppoſed to be the firſt, ſo that A^1 or A are equivalent.

Theſe Figures we call *Indexes* or *Exponents* of the Powers; becauſe by ſhewing the Number of Factors, they ſhew what Power is ſignified by that Expreſſion, or what Term in order of the Series; for the Numbers of Factors increaſe gradually in the Series, the Root

Root ſtanding alone in the *1ſt* Term, twice in the *2d*, and ſo on. And ſince the Denominations of the Powers are taken from their Places in the Series, they do alſo expreſs the Number of equal Factors, or the Number of times the Root is placed by Multiplication in every Power, and conſequently the Index is the Denomination; ſo if $A=2$ then $A^3=AAA=2\times2\times2=8$.

Again: By this Method any Power indefinitely may be expreſſed by a general or indefinite Index thus, A^n; which is any Power of A, according to the Value we put upon the Index ^{n}.

Hence any Series of Powers decreaſing from a given one down to the Root may be expreſſed thus:

$$A^n,\ A^{n-1},\ A^{n-2},\ A^{n-3},\ A^{n-4},\ \&c.$$

Still ſubſtracting one more from the Index till it become equal to 1, and then you have the Root.

II. *For* Roots.

The Root of any Number conſidered as a Power may alſo be conveniently expreſſed by that Number with an Index; thus, over the Number which is the Denomination of the Root, ſet ^{1}, in form of a Fraction; this is the Index of the Root: For Example; The Square Root of A is $A^{\frac{1}{2}}$, the Cube Root $A^{\frac{1}{3}}$, the 4th Root $A^{\frac{1}{4}}$, and ſo on; ſo that if $A=4$, then $A^{\frac{1}{2}}=2$: Or if $A=8$, then $A^{\frac{1}{3}}=2$; And an indefinite Root thus, $A^{\frac{1}{n}}$.

There is alſo another way of marking Roots by this Mark $\surd$, ſetting the Power before it, and the Index above it: Thus the Square Root of A is $\sqrt[2]{A}$, the ^{n} Root is $\sqrt[n]{A}$.

And now, to underſtand the Conveniency of diſtinguiſhing the Powers by their Order in the Series, *i.e* by the Number of Factors or Indexes, Conſider that the various Powers of the ſame Root differ only by theſe Indexes, or Numbers of Factors; and the Rules for their mutual Application to one another by Multiplication and Diviſion, (by which chiefly their different Properties are diſcovered,) depending upon the Conſideration of theſe different Numbers of Factors, it is a more ſimple and eaſy Method to make the ſame Number expreſs both the Number of Factors, and give a Denomination to the Power; which would not be, if we ſhould begin the Numeration of the Powers at the firſt Product, calling AA, or A^2 the Firſt Power. It is true indeed, that by this Method the Denomination would always be one leſs than the Index or Number of Factors, and ſo would be a certain regular Method of ſhewing that Number; but ſtill the other is more ſimple and eaſy: Which the Applications to be made afterwards will make appear more evidently.

There is one thing more you may obſerve upon this Method of denominating Powers, *viz.* That tho' the Root is not a Product of itſelf multiplied into itſelf, and ſo is not a Power according to the general Definition; yet we may always contra-diſtinguiſh Root and Power, underſtanding them according to the general Definition, and at the ſame time take the Denominations of Powers from the Indexes or Numbers of Factors; provided we underſtand theſe Denominations or Indexes to expreſs no other thing but the Number of Factors, *i.e.* a Power compoſed of ſo many Factors as the Index expreſſes, and not as ſignifying the Degree and Order of the ſeveral Powers from the firſt Product, which, according to the general Definition, is the firſt Power, tho' the Index is 2; ſo for Example, A^4 is called the Fourth Power, not as being the fourth Term in the order of Products, for it is only the 3d Product, but as being compoſed of four Factors; *viz.* the Root ſtated as a Factor four times; ſo $A^4=A\times A\times A\times A$.

But now after all, it's to the ſame Purpoſe in which of theſe Views you take the Denomination; for the whole Conveniency lies in having the Number of Factors expreſſed, which is done either way. Others again conſider 1 as a Factor in every Power, and then

they make the Index expreſs the Number of Multiplications by which a Power is produced: Thus $A^2 = 1 \times A \times A$; in which are two Multiplications, 1ſt $1 \times A$, 2d $1 \times A$ by A.

DEFIN. IV. *Powers* or *Roots* are called *Like* or *Similar* to one another, whoſe Denominations or Indexes are the ſame; ſo A^2, B^2, or A^n, B^n, are ſimilar Powers, and theſe ſimilar Roots, $A^{\frac{1}{n}}$, $B^{\frac{1}{n}}$. Such are alſo ſaid to be Powers or Roots of the ſame Degree or Order.

And when the Indexes are different or unequal, they are called *unlike* or of a different Order: As A^n, B^m; alſo $A^{\frac{1}{n}}$, $B^{\frac{1}{m}}$.

V. The finding any Power of a Number is called *Raiſing* that Number to ſuch a Power; as finding the 4th Power of A is called Raiſing A to the 4th Power: And this is alſo call'd in general, *Involving*, or the *Involution* of that Number, according to the Index of the Power.

VI. The finding any propoſed Root of a Number, is called the *Extracting* of ſuch a Root from that Number; as finding the Cube Root of A is call'd the extracting the Cube Root of A; and this we call in general *Evolving*, or the *Evolution* of that Number, according to the Index of the Root.

VII. As any Number may be made a Root, and involved to any Power, ſo if a Number C is a Power of another B, which is again a Power of another A, then may C be called a *Compound* Power of A, *i. e.* a Power of a Power of A, (as with reſpect to B it's a ſimple Power,) and may be generally expreſſed thus: $\overline{A^m}|^n$, that is, the ^{n} Power of A^m. Example: 64 is the Square of the Cube of 2, for it is the Square of 8, which is the Cube of 2. The Compoſition may alſo conſiſt of more than two Members, as the ^{n} Power of the ^{m} Power of A^o.

VIII. If any Number A is a certain Root of another B, which is alſo a certain Root of another C, then may A be called a compound Root of C (as with reſpect to B it is a ſimple Root) and may be expreſſed thus, $\overline{C^{\frac{1}{m}}}|^{\frac{1}{n}}$, that is, the ^{n} Root of $C^{\frac{1}{m}}$. Example: 2 is the Cube Root of the Square Root of 64.

SCHOLIUM. That theſe Compound Powers and Roots muſt be equal to ſome immediate or ſimple Power or Root of the Number to which they are referred, will eaſily be underſtood from the Nature of ſuch Nunbers; that is, the ^{n} Power of the ^{m} Power of A is ſome immediate ſimple Power of A, as A^o; and ſo of Roots: How ſuch ſimple Expreſſions are found, ſhall be explained in its place.

IX. A Number which is firſt conſidered as a certain Root of another, as the ^{n} Root, may be itſelf involved according to ſome other Index ^{m}, and this Power being referred to the ſame Number to which the preceeding Root was referred, may be called a *mix'd* Power of that Number, ſo if $B = A^{\frac{1}{n}}$, then the ^{m} Power of B, that is, the ^{m} Power of the ^{n} Root of A is a mix'd Power of A (which referred to B is a ſimple Power) and may be expreſſed $\overline{A^{\frac{1}{n}}}|^m$. Example: 9 is the Square of tho Cube Root of 27, for it is the Square of 3, whoſe Cube is 27.

In the ſame manner, a Number being conſidered as a certain Root of a certain Power (whoſe Index is different from that of the Root) of a Number, it may be called a mix'd Root, as the ^{m} Root of the ^{n} Power of A, repreſented thus: $\overline{A^n}|^{\frac{1}{m}}$. Example: 9 is the Cube Root of the Square of 27, for the Square of 27 is 729, which is alſo the Cube of 9.

Obſerve, For either of theſe Kinds, *viz.* a mix'd Power, or mix'd Root, we may inſtitute this manner of Repreſentation $A^{\frac{n}{m}}$, which may ſignify either the ^{m} Root of A^n, or the ^{n} Power of $A^{\frac{1}{m}}$. But then obſerve, that we can't make it repreſent either of theſe indifferently, till we have firſt demonſtrated that they are equal; which ſhall be afterwards done; and till then, I ſhall only uſe it for the ^{m} Root of the ^{n} Power.

SCHOLIUM. Every Number is a Root of any Order whatever, becauſe it may be *involved* to any *Power*; but every Number is not a Power of any Order; ſome being Powers of no Order but the firſt, which is only being a Root; *i. e.* there are ſome Numbers which cannot be produced by the continual Multiplication of any Number whatever; and ſuch are 3, 5, 6, 7, and an infinite Number of others. Some again are Powers of one particular Order only; as 4, which is only a Square; and 8, which is only a Cube. Some, in the laſt place, are Powers of more than one Order, but limited to a certain Number of different Orders; as 64 is both a Square and a Cube; its Square Root being 8, and the Cube Root 4; for $8\times8=4\times4\times4=64$: The Demonſtration of theſe things you'll learn afterwards; to mention them in general is enough here, which was only neceſſary for the ſake of the following Definition.

DEFIN. X. When a Number A is propoſed as a Power of any Order ^{n}, and yet is not a Power of that Order, *i.e.* if it has no determinate Root of that Order, or there is no Number which involved as the Index ^{n} directs, will produce that Number; yet it has what we may call an indeterminate Root, (as ſhall be afterwards explain'd) and this imagin'd Root, under the Notion of a true and compleat Root, is called a *Surd* (*i.e.* inexpreſſible) Root, and is repreſented in the general Form $A^{\frac{1}{n}}$; and ſuch Roots as are real, are, in Diſtinction from Surds, called *Rational Roots.* *Exam.* 8 is not a Square; for there is no Number, which multiplied into itſelf, will produce 8. No Integer will, ſince $2\times2=4$, and $3\times3=9$; and that no mix'd Number betwixt 2 and 3 can do it, will be afterwards demonſtrated.

But now as to Surds, don't miſtake, as if ſuch Roots or Repreſentations were nothing at all, or ſo merely imaginary as to be of no Uſe in Arithmetick; for though there be no ſuch determinate or aſſignable Number, whoſe ^{n} Power is equal to A; yet we can find Numbers mix'd of Integers and Fractions, that ſhall approach nearer and nearer to the Condition required, *in infinitum*; *i. e.* we can find a Series of Numbers decreaſing continually, whoſe Sum taken at every Step is a Number, the Power of which approaches nearer and nearer to the given Number; and this Series conſider'd in its infinite Nature, as going on by the ſame Tenor and Law without end, and thereby approaching infinitely near to the Condition of a true Root, is truly and properly what we call a Surd; which, 'tis plain, is ſomething real in itſelf, though we can't expreſs the whole Value of it by any definite Number; for that is contrary to its Nature: So we find that the Surd Roots of different Numbers have certain Connections and relative Properties the ſame way as rational Numbers have; (all which things ſhall be demonſtrated in their proper place.)

Therefore we conceive Surds as Quantities compleat of their own kind, and ſo uſe the ſame general Notation for Surds and rational Roots: And hence the following Theory relating to Roots are to be underſtood generally, whether they are Surds, or rational Roots; concerning the Reaſon and Application of which to Surds, you'll learn more Particulars afterwards in *Chap.* 3.

XI. The *Powers* and *Roots* of *Fractions* are to be underſtood the ſame way as of whole Numbers; that is, any Fraction being continually multiplied into itſelf, is a *Root* or *firſt Power,* with reſpect to the Products which are the ſuperior Powers.

Exam-

Example. $\frac{2}{3}\times\frac{2}{3}=\frac{4}{9}$, and $\frac{4}{9}$ is therefore the Square of $\frac{2}{3}$, and may be univerſally expreſſed thus, $\overline{\frac{a}{b}}|^{n}$; or thus, $\frac{a^{n}}{b^{n}}$; and the Root thus, $\overline{\frac{a}{b}}|^{\frac{1}{n}}$; or thus, $\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}$.

Obſerve, That only is the proper and immediate Power of a Fraction, whoſe Terms are the Powers of the Terms of that Fraction; yet as the ſame Fraction may be expreſſed in various Terms, ſo all equivalent Fractions may be taken for the Power or Root of the ſame Fraction, becauſe they have the ſame Effect in all Operations, if any one of them is ſo, according to the Definition. Thus, becauſe $\frac{2}{3}\times\frac{2}{3}=\frac{4}{9}$ and becauſe $\frac{4}{6}=\frac{2}{3}$ therefore $\frac{4}{9}$, which is the Square of $\frac{2}{3}$, may be alſo called the Square of $\frac{4}{6}$, alſo $\frac{8}{18}$ $(=\frac{4}{9})$ may be called the Square of $\frac{2}{3}$ or $\frac{4}{6}$.

SCHOLIUM. When a complex literal Expreſſion is conſider'd as a Root, to expreſs any Power of it we draw a Line over the whole Expreſſion, and then annex the Index; thus, $\overline{A+B}^{n}$ is the ^{n} Power of $\overline{A+B}$, and $\overline{AB}^{n}$ is the ^{n} Power of the Product $A\times B$: But if there is no Line over, then the Index is applied only to the Member over which it is immediately ſet, as $A+B^{n}$ is only the Sum of A and B^{n}, and $A\,B^{n}$ the Product of $A\times B^{n}$.

AXIOMS.

Iſt, LIKE *Powers* or *Roots of equal Numbers* (however differently expreſſed) are equal. Thus: If $A=B$, then $A^{n}=B^{n}$, and $A^{\frac{1}{n}}=B^{\frac{1}{n}}$. But unequal Numbers have their ſimilar Powers and Roots unequal; the greater having the greater Number for its Power or Root. Hence,

COROL. If the different Powers of unequal Numbers are equal, the Power of the leſſer Root has the greater Index. Thus: If $A^{n}=B^{m}$, and A be leſs than B; then is n greater than m: For if $n=m$, then is A^{n} leſs than B^{m}; and much more is it ſo, if n is leſs than m. *Exam.* $8^{2}=4^{3}=64$.

II*d*, If a Number is involved to any Power, and from this Power a Root of the ſame Denomination is extracted, this Root is the ſame Number which was firſt involved. So if A is involved to the n Power, and of this Power, *viz.* A^{n}, we extract the ^{n} Root, it is equal to A, *i. e.* $A=\overline{A^{n}}|^{\frac{1}{n}}$. And reverſly extract any Root of a Number, and then involve that Root to a Power of the ſame Denomination; this Power is the ſame Number from which the Root was firſt extracted; ſo the ^{n} Power of $A^{\frac{1}{n}}$, or $\overline{A^{\frac{1}{n}}}|^{n}=A$. Hence,

COROL. 1. An Expreſſion of this Sort $A^{\frac{n}{n}}$, where the Numerator and Denominator of the Index are equal, whether it is underſtood as the ^{n} Root of the ^{n} Power; or the ^{n} Power of the ^{n} Root of A, is no other thing in Effect and Value but A. Hence again;

2. Involution and Evolution are directly oppoſite, the one undoing the Effect of the other; whereby they are mutual Proofs one of the other.

THEOREM I.

IF two ſimilar Powers of different Numbers or Roots are multiplied together, the Product is the like Power of the Product of theſe Numbers or Roots. Thus, the Product of two Squares is the Square of the Product of their Roots. *Univerſally* $A^{n}\times B^{n}=\overline{AB}^{n}$.

Demonſtr.

$A=2$, $A^2=4$
$B=5$, $B^2=25$
$AB=10$. $A^2\times B^2=$
$100=\overline{AB}^2$.

Demonstr. In the Product $A^n\times B^n$, A and B are applied as Factors equal times; so that in the whole Product there is a Number of Factors equal to $2n$; and consequently the Product AB, consider'd as one Number or Factor, is applied in that Product n times [for in whatever Order the Factors are taken, the Product is still the same]; so that the Product given is AB raised to the n Power, or $\overline{AB}^n$. Thus particularly, $A^3\times B^3=AAA\times BBB=AB\times AB\times AB=\overline{AB}^3$.

Corol. Hence we learn, how the Product of a Number expressed as a Power, and another not expressed as such (but supposed to be one) may be reduced to such an Expression. Thus, $A^n\times B=\overline{A\times B^{\frac{1}{n}}}\rceil^n$; for the n Power of A is A^n, and the n Power of $B^{\frac{1}{n}}$ is B, therefore $A^n\times B=\overline{A\times B^{\frac{1}{n}}}\rceil^n$.

Schol. This Theorem is true also, when the Roots are the same; as $A^n\times A^n=\overline{AA}^n$. But I have not taken this Case into the *Theorem*, because it falls within another (see *Theor.* 6.) where the Product is expressed in a more simple and convenient way. You are to understand the same of the *Theorems* 2, 3, and 4.

Theorem II.

If two similar Powers of different Numbers or Roots are divided one by the other, the Quote is the like Power of the Quote of the given Roots. Thus the Quote of two Squares is the Square of the Quote of their Roots *universally* $B^n\div A^n=\overline{B\div A}\rceil^n$.

$A=3$. $A^3=27$
$B=6$ $B^3=216$
$B\div A=2$. $B^3\div A^3=8$
$\overline{B\div A}\rceil^3=8$.

Demonstr. 1. Suppose A, B are Integers, then $B\div A$ expressed fractionally is $\frac{B}{A}$, whose n Power, according to *Defin.* 11. is $\frac{B^n}{A^n}=B^n\div A^n$.

2. Suppose B and A fractional; thus, $B=\frac{c}{o}$ and $A=\frac{d}{a}$, then is $B^n=\frac{c^n}{o^n}$ and $A^n=\frac{d^n}{a^n}$.

Also $B^n\div A^n=\frac{c^n}{o^n}\div\frac{d^n}{a^n}=\frac{c^n a^n}{o^n d^n}=\frac{\overline{ca}^n}{\overline{od}^n}$ (*Theor.* 1.) But $\frac{ac}{od}=\frac{c}{o}\div\frac{d}{a}=B\div A$, and its n Power is $\frac{\overline{ac}^n}{\overline{od}^n}$, which is $\overline{B\div A}^n$.

Scholium. I have not here considered, whether $B^n\div A^n$ is a whole Number or a Fraction; for which so ever of them it be, you see plainly that it is equal to $\overline{B\div A}^n$. In another place you'll find it demonstrated, that according as $B^n\div A^n$ is integral or fractional, so is $B\div A$; and reversly.

The following Corollaries 2, 3, 4, *and* 5, *are deduced from this and the first* Theorem *jointly considered.*

Coroll. 1. The Quote of two Numbers, whereof one is expressed as a Power and the other not, (tho' supposed to be one,) may be reduced to such an Expression thus; $A^n\div B=\overline{A\div B^{\frac{1}{n}}}\rceil^n$; for A^n is the ^{n} Power of A, and B is the ^{n} Power of $B^{\frac{1}{n}}$.

2. If any Product and one of the Factors are similar Powers, the other is also a similar Power, and its Root is the Quote of the Roots of the other two Terms. Thus, if

A^n

$A^n \times B = D^n$, then is $B = D^n \div A^n = \overline{D \div A}|^n$ whose ^{n} Root is $D \div A$. In another Place it ſhall be demonſtrated that $D \div A$ muſt be integral if $D^n \div A^n$ or B is ſo.

III. If the Dividend and either of the other two Terms, *viz.* Diviſor and Quote, are ſimilar Powers, the other of thoſe Terms is alſo a ſimilar Power, whoſe Root is the Product of the Roots of the Dividend and the former of the other two Terms. Thus if $B^n \div A = D^n$, then $A = B^n \times D^n = \overline{BD}|^n$.

4. The Product of two Numbers being a Power, the Factors are either both like Powers with the Product, or neither of them is ſo. Alſo, if one Factor is a Power, and the other not a like Power, the Product is not a like Power. Let $AB = p^n$, if A is a Power of the Order ^{n}, ſo is B, for if $A = a^n$ then is $a^n \times B = p^n$, and conſequently B is of the Order ^{n} (by the 2d) whence alſo if the one A or B is not a Power of the Order ^{n}, neither is the other; for one being ſo, the other would be ſo too. Again, if $A^n \times B = D$, and B not a Power of the Order ^{n}, neither is D; for if D were ſo, B would be ſo too.

5. If the Quote of two Numbers is a Power, theſe Numbers are either both Powers like the Quote, or neither of them is ſo: This is the Reverſe of the laſt.

SCHOLIUM. In theſe Corollaries you are to underſtand Rational Powers; for otherwiſe any Number may be repreſented as a Power of any Order.

THEOREM III.

If two ſimilar Roots of different Numbers are multiplied, the Product is the like Root of the Product of theſe Numbers. Thus two Cube Roots produce a Number which is the Cube Root of the Product of their Cubes. Univerſally, $A^{\frac{1}{n}} \times B^{\frac{1}{n}} = \overline{AB}|^{\frac{1}{n}}$.

Example:

$A = 27.\ A^{\frac{1}{3}} = 3$

$B = 8.\ B^{\frac{1}{3}} = 2$

$AB = 216.\ A^{\frac{1}{3}} \times B^{\frac{1}{3}} = 6$

$\overline{AB}|^{\frac{1}{3}} = 6$

Demonſtration This is but the Reverſe of the 1ſt *Theorem*, and follows eaſily from it. Thus, Let A B be any ſimilar Powers, then is $A \times B = \overline{A^{\frac{1}{n}} \times B^{\frac{1}{n}}}|^n$ by *Theor.* I. and by *Ax.* 1. $\overline{A \times B}|^{\frac{1}{n}} = A^{\frac{1}{n}} \times B^{\frac{1}{n}}$, which is the ^{n} Root of $\overline{A^{\frac{1}{n}} \times B^{\frac{1}{n}}}|^n$.

COROLL. The Product of two Numbers, whereof one is expreſſed as a Root and the other not, may be reduced to ſuch an Expreſſion thus; $A^{\frac{1}{n}} \times B = \overline{A \times B^n}|^{\frac{1}{n}}$; for A is the ^{n} Power of $A^{\frac{1}{n}}$, and B^n of B; therefore, by this Theorem $A^{\frac{1}{n}} \times B = \overline{A \times B^n}|^{\frac{1}{n}}$. This is alſo the Reverſe of *Coroll. Theor.* 1.

THEOREM IV.

If two ſimilar Roots of different Numbers are divided, one by the other, the Quote is the like Root of the Quote of the one Number divided by the other. Thus, two Cube Roots give for a Quote the Cube Root of the Quote of the Cubes. Univerſally, $D^{\frac{1}{n}} \div A^{\frac{1}{n}} = \overline{D \div A}|^{\frac{1}{n}}$.

Example:

$D = 216.\ D^{\frac{1}{3}} = 6$

$A = 27.\ A^{\frac{1}{3}} = 3$

$D \div A = 8.\ D^{\frac{1}{3}} \div A^{\frac{1}{3}} = 2$

$\overline{D \div A}|^{\frac{1}{3}} = 2$

Demonſtr. 1. Suppoſe $D^{\frac{1}{n}}, A^{\frac{1}{n}}$ are both Integers, then are D, A alſo Integers, (*viz.* the Powers, or Products of integral Factors,) therefore $D^{\frac{1}{n}} \div A^{\frac{1}{n}}$ or $\frac{D^{\frac{1}{n}}}{A^{\frac{1}{n}}}$ is a Fraction in Terms whoſe ^{n} Power is $\frac{D}{A}$; *i. e.* the ^{n} Root of $\frac{D}{A}$ is $\frac{D^{\frac{1}{n}}}{A^{\frac{1}{n}}}$ or $D^{\frac{1}{n}} \div A^{\frac{1}{n}}$.

2. If

2. If $D^{\frac{1}{n}}, A^{\frac{1}{n}}$ are fractional, then are D, A also fractional, (according to a former Observation.) Suppose $D^{\frac{1}{n}} = \frac{b}{a}$, and $A^{\frac{1}{n}} = \frac{d}{c}$, then is $D = \frac{b^n}{a^n}$, and $A = \frac{d^n}{c^n}$ (*Ax.* 1.) Also

$D^{\frac{1}{n}} \div A^{\frac{1}{n}} = \frac{b}{a} \div \frac{d}{c} = \frac{bc}{ad}$; and $D \div A = \frac{b^n}{a^n} \div \frac{d^n}{c^n} = \frac{b^n c^n}{a^n d^n} = \overline{\frac{bc}{ad}}^{\,n}$; therefore $\overline{D \div A}|^{\frac{1}{n}} = \frac{bc}{ad} = D^{\frac{1}{n}} \div A^{\frac{1}{n}}$.

Coroll. The Quote of two Numbers whereof one is expressed as a Root, and the other not, may be reduced to such, thus; $D^{\frac{1}{n}} \div A = \overline{D \div A^n}|^{\frac{1}{n}}$. For $D^{\frac{1}{n}}$ is the ^{n} Root of D, as A is of A^n.

Scholium. From the two last Theorems jointly consider'd, we have four *Corollaries* after the same manner as the *2d, 3d, 4th, 5th Corollaries,* deduced from *Theorem* II*d* and I*st*.

Theorem V.

The Product or Quote of any like Mixt Power (or Root) of two different Numbers, is the like Mixt Power of the Product or Quote of these two Numbers: Thus $A^{\frac{n}{m}} \times B^{\frac{n}{m}} = \overline{AB}^{\frac{n}{m}}$ and $A^{\frac{n}{m}} \div B^{\frac{n}{m}} = \overline{A \div B}^{\frac{n}{m}}$.

Demonstr. 1. $A^n \times B^n = \overline{AB}^n$ (*Theor.* I.) and $\overline{A^n}|^{\frac{1}{m}} \times \overline{B^n}|^{\frac{1}{m}} = \overline{A^n \times B^n}|^{\frac{1}{m}}$ (*Theor.* III.) But $\overline{A^n}|^{\frac{1}{m}} = A^{\frac{n}{m}}$ and $\overline{B^n}|^{\frac{1}{m}} = B^{\frac{n}{m}}$ by the Notation; and since $A^n \times B^n = \overline{AB}^n$, hence $A^{\frac{n}{m}} \times B^{\frac{n}{m}} = \overline{AB}^{\frac{n}{m}}$.

2. $A^n \div B^n = \overline{A \div B}^n$ (*Theor.* II.) And $\overline{A^n}|^{\frac{1}{m}} \div \overline{B^n}|^{\frac{1}{m}} = \overline{A^n \div B^n}|^{\frac{1}{m}}$, (*Theor.* IV.) $= \overline{A \div B}^{\frac{n}{m}}$, (*Ax.* I.) That is, $A^{\frac{n}{m}} \div B^{\frac{n}{m}} = \overline{A \div B}^{\frac{n}{m}}$.

Example. The Square Root of 16 is 4, and the Cube of 4 is 64, therefore the Cube of the Square Root of 16, or $16^{\frac{3}{2}}$ is $= 64$. In like manner, the Cube of the Square Root of 81, or $81^{\frac{3}{2}}$ is $= 729$. Then $81 \times 16 = 1296$, whose Square Root is 36, and the Cube of this is $46656 = 64 \times 729$; that is, $16^{\frac{3}{2}} \times 81^{\frac{3}{2}} = \overline{16 \times 81}|^{\frac{3}{2}}$.

Theorem VI.

If two Powers of the same Root are multiplied, the Product is such a Power of the same Root, whose Index is the Sum of the Indexes of the Factors. Thus, The Product of the *2d* and *3d* Powers of any Number, is the *5th* Power of the same Number. Universally, $A^n \times A^m = A^{n+m}$.

Example.

$A = 3$. $A^2 = 9$
$A^3 = 27$. $A^2 \times A^3 = 243$
$A^5 = 243$.

Demonstr. A^n and A^m being each a Product of A continually multiplied by itself, their Product must be a Product of A continually by it self; *i. e.* a Power of A.

 Also

Alſo it's plain, that in $A^n \times A^m$, the Root A is applied as a Factor as oft as the Sum $n+m$, ſo that $A^n \times A^m = A^{n+m}$. Particularly, ſuppoſe $A^2 \times A^3$, the Product is $AA \times AAA = AAAAA = A^5$.

SCHOLIUM. What's here proved for two Factors, holds equally for three or more: Thus $A^n \times A^m \times A^r = A^{n+m+r}$.

COROLL. Hence we learn how to find any Power of a given Root, without finding all the intermediate Powers; *viz.* by multiplying together two or more Powers of that Root, the Sum of whoſe Indexes is the Index of the Power ſought: Thus, having the 3*d* and 4*th* Powers, their Product is the 7*th* Power.

THEOREM VII.

IF one Power is divided by another Power of the ſame Root; the Quote is equal either to another Power of the ſame Root, or to a fractional Power whoſe Numerator is 1, and the Denominator ſome Power of the ſame Root. Thus, particularly, If the Dividend is greater than the Diviſor, the Quote is a Power of the ſame Root whoſe Index is the Difference of the Indexes of the propoſed Powers; and if the Dividend is leſs, the Quote is a fractional Power whoſe Numerator is 1, and the Denominator is ſuch a Power of the given Root whoſe Index is the Difference of the given Indexes. Thus $A^n \div A^m$ is either A^{n-m}, or $\frac{1}{A^{m-n}}$.

Example 1.
$A=3 \quad A^4 = 81$
$A^2 = 9 \quad A^4 \div A^2 = 9$
$A^{4-2} = A^2 = 9$

Example 2.
$A^2 = 9 \quad A^5 = 243$
$A^2 \div A^5 = 9 \div 243 = \frac{1}{A^{5-2}} = \frac{1}{A^3} = \frac{1}{81}$

Demonſtr. This is the Reverſe of the laſt *Theorem*, and the Reaſon of it contain'd in that, from the reciprocal Nature of Multiplication and Diviſion, with that of Addition and Subſtraction. Or, this ſubſtracting of the one Index from the other, is only the taking out equal Factors from the Diviſor and Dividend, *i.e.* dividing them equally; which makes the Quotes ſtill the ſame: Thus, $A^5 \div A^2 = A^3 \div 1 = A^3$, and $A^2 \div A^5 = 1 \div A^3$.

THEOREM VIII.

EVERY *Compound* Power (or Power of a Power) of any Root is equal to ſuch a Simple Power of the ſame Root whoſe Index is the Product of the given Indexes: Thus the 3*d* Power of the 2*d* Power is the 6*th* Power. Univerſally, the ^{n} Power of the ^{m} Power of A, or $\overline{A^m}|^n$ is $= A^{nm}$.

Example.
$A=2.\ A^2=4$
$A^6 = 64 = \overline{A^2}|^3$

Demonſtr $A^m \times A^m = A^{2m}$ (*Theor. 6.*) and involving or multiplying A^m by itſelf once more, the Index of the Product contains m once more; ſo that however oft A^m is employed as a Factor, as n times, the Index of the Product will be ſo many times m, or $m \times n$: But A^m employed as a Factor n times makes the n Power of A^m; which is therefore equal to the mn Power of A, or A^{mn}.

SCHOLIUMS.

1. The ſame Reaſoning is good, when the Compoſition conſiſts of more than two Steps: Thus the n Power of the m Power of the r Power, is the mnr Power of A, or A^{nmr}.

2. If the Index of any Power is the Product of two Numbers, it may be conſidered as a Compound Power; or if it is the Product of more than two Factors, it may be reduced to a Compound of two, by taking any one of theſe Factors for one Member, and the Product of all the reſt for the other. Thus, A^{nmr} is the r Power of A^{nm}, or the n Power of A^{mr}, or the m Power of A^{nr}. In ſhort, if the Index of a Power is the Product

of

of other Numbers, whatever Variety is in the Composition of that Product, there is the same Variety in the Composition of that Power.

COROLLARIES.

1. Here we learn to find a Number, which is a Power of as many different Orders as can be proposed, *viz.* by multiplying the Indexes of all these proposed Orders continually into one another, and raising any Number to a Power, whose Index is that Product. Thus: To find a Number which is both a Square and Cube, raise any Number to the sixth Power. Universally, to find a Number which is a Power of the Orders n, m, r, raise any Number A to the nmr Power, and A^{nmr} is the Number sought: for it is the r Power of A^{nm}, the n Power of A^{mr}, the m Power of A^{nr}.

Observe also, That if one of the Indexes is an *aliquot* Part of another, we need not multiply that Index which is the *aliquot* Part. Thus: To find a Number which is both a second and fourth Power, we need only to take some fourth Power, since every fourth Power is also a second Power. For because $2\times 2=4$, therefore A^4 is the second Power of A^2. By this you'll understand any other Case.

2. If out of a certain Power of a given Number, a Root is to be extracted of a different Index, and if the Name of the Root is an *aliquot* Part of the Name or Index of the Power; by dividing the Name of the Power by that of the Root, and applying the Quote as an Index to the given Number, we have an Expression for the Root sought. Thus; the Square Root of A^6 is A^3, because $6\div 2=3$. And universally, let $n\div m=r$. Then is $A^{\frac{n}{m}}=A^r$, *i. e.* where there is a mixt or fractional Index (which expresses a certain Root named by the Denominator, of a certain Power named by the Numerator) if the Denominator is an *aliquot* Part of the Numerator; then dividing the Numerator by the Denominator, the Quote is an Index, which applied to the same Number, expresses the Value of that mixt Root.

THEOREM IX.

EVERY Compound Root (or Root of a Root) of any Number, is equal to such a simple Root of the same Number, whose Index is the Product of the proposed Indexes. Thus; the Square Root of the Cube Root is the sixth Root. Universally, the n Root of the m Root is the $n\,m$ Root, or $\overline{A^{\frac{1}{m}}}|^{\frac{1}{n}}=A^{\frac{1}{nm}}$.

Example:
$A=64$. $A^{\frac{1}{3}}=4$
$A^{\frac{1}{6}}=2=\overline{A^{\frac{1}{3}}}|^{\frac{1}{2}}$.

Demonstr. The Reason of this is contained in the preceding, because extracting of Roots is opposite to raising Powers. Or it may be demonstrated thus: Suppose $A^{\frac{1}{nm}}=B$, then $A=B^{nm}$ (*Ax.* 1, 2. for A is the $n\,m$ Power of its $n\,m$ Root) and $A^{\frac{1}{m}}=B^{\frac{nm}{m}}=B^n$ (*Cor.* 2. *Theor.* 8.) Again, $\overline{A^{\frac{1}{m}}}|^{\frac{1}{n}}=B$ (*Ax.* 1.) But $B=A^{\frac{1}{nm}}$; therefore $\overline{A^{\frac{1}{m}}}|^{\frac{1}{n}}=A^{\frac{1}{nm}}$.

THEOREM X.

ANY Power of any Root of a Number is equal to the same Root of the same Power of that Number. Thus, the Square Root of the Cube of any Number is the Cube of the Square Root of that Number. Universally, $\overline{A^{\frac{1}{m}}}|^{n}=\overline{A^{n}}|^{\frac{1}{m}}$.

Example:
$A=9$. $A^{\frac{1}{2}}=3$. $A^3=729$.
$\overline{A^3}|^{\frac{1}{2}}=27=\overline{A^{\frac{1}{2}}}|^{3}$.

Demonstr. Suppose $A^{\frac{1}{m}}=B$, then is $A=B^m$ (*Ax.* 1.) and $A^n=B^{mn}$ (*Ax.* 1. with *Theor.* 7.) Again, $\overline{A^n}|^{\frac{1}{m}}=B^{\frac{mn}{m}}$ (*Ax.* 1.) $=B^n$ (*Cor.* 2. *Theor.* 8.) But since $A^{\frac{1}{m}}=B$, therefore $\overline{A^{\frac{1}{m}}}|^{n}=B^n$; consequently $\overline{A^n}|^{\frac{1}{m}}=\overline{A^{\frac{1}{m}}}|^{n}$.

SCHOL.

SCHOLIUMS.

1. When $A^{\frac{1}{m}}$ is rational, so also is $\overline{A^n}|^{\frac{1}{m}}$, because this is equal to $\overline{A^{\frac{1}{m}}}|^{n}$, which is the Power of a rational Root: But tho' $\overline{A^n}|^{\frac{1}{m}}$ is rational, it does not follow that $A^{\frac{1}{m}}$ is so, as one Example shews. Thus: Let $A=3$, then $A^{\frac{12}{3}}=A^4=81$. But $A^{\frac{1}{3}}$ is Surd, for 3 has not a Cube Root.

2. Here we learn, that $A^{\frac{n}{m}}$ may indifferently be taken for the *n* Power of the *m* Root, or the *m* Root of the *n* Power of A, since these are equal. So that henceforth we shall take it either way, as shall be most useful. Hence also we have a Rule for expressing the *Involution* of a simple Root, or the *Evolution* of a simple Power.

THEOREM XI.

IF the Index of a mixt Power (*i. e.* which has a fractional Index) has both its Members, Numerator and Denominator, equally multiplied or divided, the Products or Quotes put in place of the others make an equivalent Expression, or express a mixt (or simple) Power or Root of the same Number equal in Value to the former. Thus $\frac{4}{6}=\frac{2}{3}$, and therefore $A^{\frac{4}{6}}=A^{\frac{2}{3}}$. Universally, let $n=ab$, $m=ad$. Then is $A^{\frac{n}{m}}$ (or $A^{\frac{ab}{ad}}$) $=A^{\frac{b}{d}}$.

Example:

$A=16777216$. $A^{\frac{1}{6}}=16$

$A^{\frac{4}{6}}=65536=16^4$

$A^{\frac{1}{3}}=256$. $A^{\frac{2}{3}}=65536$.

Demonstr. $A^{\frac{ab}{ad}}$ is the *ad* Root of the *ab* Power of A (by the Notation) which is the *d* Root of the *a* Root (*Theor.* 9.) of the *a* Power of the *b* Power (*Theor.* 8.) Now the *b* Power of A is A^b, and the *a* Power of this is A^{ab}, (*Theor.* 8.) whose *a* Root is A^b, (*Theor.* 8. *Cor.* 2.) and the *d* Root of this is $A^{\frac{b}{d}}$ (*per* Notation) *i. e.* $A^{\frac{ab}{ad}}=A^{\frac{b}{d}}$. Or the Demonstration will go on the same way by taking $A^{\frac{ab}{ad}}$ for the *ab* Power of the *ad* Root.

THEOREM XII.

THE Simple Power of a Mixt Power, or Mixt Power of a Simple, of any Number, is equal to a Mixt Power of the same Number, whose Numerator is the Product of the Simple Index by the Numerator of the Mixt one, and its Denominator that of the Mixt one; or, in short, whose Index is the Product of the two given Indexes. Thus, the *r* Power of $A^{\frac{n}{m}}$ is $A^{\frac{rn}{m}}$; which is also the $\frac{n}{m}$ Power of the r Power; *i. e.* $\overline{A^r}|^{\frac{n}{m}}$.

Example.

$A=4096$. $A^{\frac{1}{4}}=8$. $A^{\frac{3}{4}}=512$

Squ. of $A^{\frac{3}{4}}$ or $A^{\frac{6}{4}}=32768$

Demonstr. $A^{\frac{n}{m}}$ is the *n* Power of the *m* Root of A; therefore the *r* Power of $A^{\frac{n}{m}}$ is the *r* Power of the *n* Power of the *m* Root: But the *r* Power of the *n* Power is the *rn* Power, (*Theor.* VIII.) therefore the *r* Power of $A^{\frac{n}{m}}$ is the *rn* Power of the *m* Root, *i. e.* $A^{\frac{rn}{m}}$, (*per* Notation.) Again, This is also the $\frac{n}{m}$ Power of A^r: For the Index $\frac{n}{m}$ is the *m* Root of the *n* Power; wherefore $\overline{A^r}|^{\frac{n}{m}}$ is the *m* Root of the *n* Power of the *r* Power; *i. e.* the *m* Root of the *nr* Power, or *nr* Power of the *m* Root; *viz.* $A^{\frac{rn}{m}}$.

THEO-

THEOREM XIII.

THE Simple Root of a Mixt Power, or Mixt Power of a Simple Root of any Number, is such a Mixt Root of the same Number, whose Denominator is the Product of the Simple Index into the Denominator of the Mixt one, and the Numerator that of the Mixt one. In short, whose Index is the Quote of the Mixt one, by the Name of the Simple one, or the Product of the two Indexes, taking the Simple Root fractionally: Thus the r Root of $A^{\frac{n}{m}}$, or the $\frac{n}{m}$ Root of $A^{\frac{1}{r}}$, is $A^{\frac{n}{mr}}$: For $\frac{n}{m} \div r = \frac{n}{m} \times \frac{1}{r} = \frac{n}{mr}$.

Example:
$A = 65536$. $A^{\frac{1}{2}} = 256$.
$A^{\frac{3}{2}} = 16777216$. Sq. Root of $A^{\frac{3}{2}}$, or $A^{\frac{3}{4}} = 4096$.

Demonstr. $A^{\frac{n}{m}}$ is the m Root of the n Power of A; therefore the r Root of $A^{\frac{n}{m}}$ is the r Root of the m Root of A^n, *i. e.* the rm Root, (*Theor.* 9.) or $A^{\frac{n}{rm}}$. Again, this is also the $\frac{n}{m}$ Root or Power of $A^{\frac{1}{r}}$. For $\frac{n}{m}$ expresses the n Power of the m Root, and so the $\frac{n}{m}$ Root of $A^{\frac{1}{r}}$ is the n Power of the m Root of the r Root, *i. e.* the n Power of the mr Root, or $A^{\frac{n}{rm}}$.

THEOREM XIV.

THE *mixt* Power of a *mixt* Power is equal to a *mixt* Power whose Index is the Product of the given Indexes. Thus, the $\frac{r}{s}$ Power of the $\frac{n}{m}$ Power is the $\frac{rn}{sm}$ Power, *i. e.* $\overline{A^{\frac{n}{m}}}|^{\frac{r}{s}} = A^{\frac{nr}{ms}}$.

Demonstr. The r Power of $A^{\frac{n}{m}}$ is $A^{\frac{rn}{m}}$, (*Theor.* 12.) and the s Root of this is $A^{\frac{rn}{sm}}$ (*Theor.* 13.) which is therefore the s Root of the r Power, or the r Power of the s Root of $A^{\frac{n}{m}}$.

THEOREM XV.

THE Product of any two Roots Simple or Mixt, or of any Power and Root of the same Number, is equal to such a Power or Root of the same Number whose Index is the Sum of the given Indexes. Thus $A^{\frac{1}{n}} \times A^{\frac{1}{r}} = A^{\frac{1}{n}+\frac{1}{r}} = A^{\frac{r+n}{nr}}$; and $A^{\frac{n}{m}} \times A^{\frac{r}{s}} = A^{\frac{n}{m}+\frac{s}{r}} = A^{\frac{ns+mr}{ms}}$; Also, $A^n \times A^{\frac{r}{s}} = A^{n+\frac{s}{n}} = A^{\frac{ns+r}{s}}$.

Demonstr. The most universal Case or Expression is $A^{\frac{n}{m}} \times A^{\frac{r}{s}}$; for by supposing n or r, or each of them, equal to 1, you make them Simple Roots; and by making m or s equal to 1, you make that Term a Simple Power: And so all the Variety supposed in the Theorem will be demonstrated in this one Form; thus: The Thing to be demonstrated is, that $A^{\frac{n}{m}} \times A^{\frac{r}{s}} = A^{\frac{n}{m}+\frac{r}{s}} = A^{\frac{ns+mr}{ms}}$: In order to which, suppose $A^{ns} = B$, and $A^{mr} = D$; then is $B^{\frac{1}{ms}} = A^{\frac{ns}{ms}}$ (*Ax.* 1.) $= A^{\frac{n}{m}}$ (*Theor.* XI.) Also $D^{\frac{1}{ms}} = A^{\frac{mr}{ms}} = A^{\frac{r}{s}}$; and $BD = A^{ns} \times A^{mr} = A^{ns+mr}$ (*Theor.* 6.) Hence $A^{\frac{ns+mr}{ms}} = \overline{BD}^{\frac{1}{ms}} = B^{\frac{1}{ms}} \times D^{\frac{1}{ms}} = A^{\frac{n}{m}} \times A^{\frac{r}{s}}$.

THEO-

THEOREM XVI.

THE Quote of any two Roots, Simple or Mixt, or of any Power and Root of the ſame Number, is equal to ſuch a Power or Root of the ſame Number, whoſe Index is the Difference of the given Indexes, when the Index of the Dividend is greater than that of the Diviſor. But if the Index of the Diviſor is the greater, ſet 1 over that Number ſo found, or divide 1 by it, and this Fraction or Quote, is the Quote ſought. Thus $A^{\frac{n}{m}} \div A^{\frac{r}{s}} = A^{\frac{n}{m} - \frac{r}{s}} = A^{\frac{ns - mr}{ms}}$ when $\frac{n}{m}$ is greater than $\frac{r}{s}$; but it is $1 \div A^{\frac{mr - ns}{ms}}$ if $\frac{n}{m}$ is leſs than $\frac{r}{s}$. All other Caſes are contain'd in this Form.

Demonſtr. Suppoſe $A^{ns} = B$, and $A^{mr} = D$, then is $B^{\frac{1}{ms}} = A^{\frac{ns}{ms}} = A^{\frac{n}{m}}$, and $D^{\frac{1}{ms}} = A^{\frac{mr}{ms}} = A^{\frac{r}{s}}$; alſo $B \div D = A^{ns} \div A^{mr} = A^{ns - mr}$, (*Theor.* VII.) Hence $A^{\frac{ns - mr}{ms}} = \overline{B \div D}^{\frac{1}{ms}} = B^{\frac{1}{ms}} \div D^{\frac{1}{ms}} = A^{\frac{n}{m}} \div A^{\frac{r}{s}}$.

But if $\frac{n}{m}$ is leſs than $\frac{r}{s}$, then alſo is ns leſs than mr; and therefore $A^{ns} \div A^{mr}$ is a proper Fraction. Let us ſuppoſe $mr = ns + a$, or $mr - ns = a$, then is $A^{ns} \div A^{mr} = A^{ns} \div A^{ns + a}$; but $A^{ns + a} = A^{ns} \times A^{a}$, (*Theor.* VI.) wherefore reduce the Fraction $A^{ns} \div A^{mr}$, or $A^{ns} \div A^{ns + a}$ to lower Terms by dividing both by A^{ns}, the new and equivalent Fraction is $1 \div A^{a} = 1 \div A^{mr - ns}$; but $A^{ns} \div A^{mr} = B \div D$, which is therefore $= 1 \div A^{mr - ns}$: Hence $1 \div A^{\frac{mr - ns}{ms}} = \overline{B \div D}^{\frac{1}{ms}} = B^{\frac{1}{ms}} \div D^{\frac{1}{ms}} = A^{\frac{n}{m}} \div A^{\frac{r}{s}}$.

General SCHOLIUM *relating to the preceding* Theorems.

FROM the preceding Theorems we have Rules for the Multiplication, Diviſion, Involution and Evolution of Numbers expreſſed in Form of Powers or Roots of other Numbers, *i.e.* for a more ſimple and convenient Expreſſion of the Products and Quotes, Powers and Roots; and the Subſtance of the whole may be repreſented in four General Rules, which being particularly exemplified, will ſhew in one ſhort View all the preceding Theory. And *obſerve,* That in order to reduce it to ſo few Rules, any Expreſſion made of any Letter A, with any Index, Integral or Fractional, may be called a *Power* of A; for it's ſuch a Power as the Numerator expreſſes, of a certain Root expreſſed by the Denominator.

RULE I. Add the Indexes of any two Powers of the ſame Number A, and the Sum is the Index of a Power equal to the Product of the other two.

RULE II. The Difference of the Indexes of two Powers of A, is the Index of a Power equal to the Quote of the other two; minding, that if the Index of the Diviſor is greateſt, 1 is to be ſet over the Power found, and that fractional Expreſſion is the true Quote.

RULE III. If the Index of a given Power of A is multiplied by another Index, the Product is the Index of a Power which is equal to ſuch a Power of the given Power as that

that other Index denominates. And because the Word *Power* does here signify both what is in a more particular Definition call'd *Power* and *Root*, therefore this Rule comprehends both *Involution* and *Evolution*.

RULE IV. If to the Product or Quote of two Numbers any Index is applied, the Expression is equal to the Product or Quote of the same Powers of these two Numbers.

EXAMPLES *of these* Rules.

RULE 1. *Theor.*

$A^n \times A^r = A^{n+r}$ - - - - - - 6th.

$\left.\begin{array}{l} A^n \times A^{\frac{1}{r}} = A^{\frac{1+nr}{r}} \\ A^{\frac{1}{n}} \times A^{\frac{1}{r}} = A^{\frac{r+n}{nr}} \\ A^n \times A^{\frac{r}{s}} = A^{\frac{ns+r}{s}} \\ A^{\frac{n}{m}} \times A^{\frac{r}{s}} = A^{\frac{ns+mr}{ms}} \\ A^{\frac{1}{n}} \times A^{\frac{r}{s}} = A^{\frac{rn+s}{ns}} \end{array}\right\}$ 15th.

RULE 2.

$A^n \div A^m = A^{n-m}$, or $A^{\frac{1}{m-n}}$ 7th.

$\left.\begin{array}{l} A^n \div A^{\frac{1}{r}} = A^{\frac{nr-1}{r}} \\ A^{\frac{1}{n}} \div A^{\frac{1}{r}} = A^{\frac{r-n}{nr}}, \text{ or } 1 \div A^{\frac{n-r}{nr}} \\ A^n \div A^{\frac{r}{s}} = A^{\frac{ns-r}{s}}, \text{ or } 1 \div A^{\frac{r-ns}{s}} \\ A^{\frac{n}{m}} \div A^{\frac{r}{s}} = A^{\frac{ns-mr}{ms}}, \text{ or } 1 \div A^{\frac{mr-ns}{ms}} \\ A^{\frac{1}{n}} \div A^{\frac{r}{s}} = A^{\frac{s-nr}{ns}}, \text{ or } 1 \div A^{\frac{nr-s}{ns}} \end{array}\right\}$ 16th.

RULE 3. *Theor.*

$\overline{A^m}|^n = A^{mn}$ - - - - - - - - 8th.

$\overline{A^{\frac{1}{m}}}|^{\frac{1}{n}} = A^{\frac{1}{mn}}$ - - - - - - - - 9th.

$\left.\begin{array}{l} \overline{A^{\frac{1}{m}}}|^n \\ \overline{A^n}|^{\frac{1}{m}} \end{array}\right\} = A^{\frac{n}{m}}$ - - - - - - - 10th.

$\left.\begin{array}{l} \overline{A^{\frac{n}{m}}}|^r \\ \overline{A^r}|^{\frac{n}{m}} \end{array}\right\} = A^{\frac{nr}{m}}$ - - - - - - 12th.

$\left.\begin{array}{l} \overline{A^{\frac{n}{m}}}|^{\frac{1}{r}} \\ \overline{A^{\frac{1}{r}}}|^{\frac{n}{m}} \end{array}\right\} = A^{\frac{n}{rm}}$ - - - - - - 13th.

$\overline{A^{\frac{n}{m}}}|^{\frac{r}{s}} = A^{\frac{nr}{ms}}$ - - - - - - - 14th.

RULE 4.

$A^n \times B^n = \overline{AB}|^n$ - - - - - - 1st.

$A^n \div B^n = \overline{A \div B}|^n$ - - - - - 2d.

$A^{\frac{1}{n}} \times B^{\frac{1}{n}} = \overline{AB}|^{\frac{1}{n}}$ - - - - - - 3d.

$A^{\frac{1}{n}} \div B^{\frac{1}{n}} = \overline{A \div B}|^{\frac{1}{n}}$ - - - - - 4th.

$A^{\frac{n}{m}} \times B^{\frac{n}{m}} = \overline{AB}|^{\frac{n}{m}}$ - - - - - - 5th.

$A^{\frac{n}{m}} \div B^{\frac{n}{m}} = \overline{A \div B}|^{\frac{n}{m}}$ - - - - - 5th.

THEOREM XVII.

IF the n Power of one Number is equal to the m Power of another, then is the m Root of the first equal to the n Root of the second: Thus if $A^n = B^m$, then is $A^{\frac{1}{m}} = B^{\frac{1}{n}}$.

Demonstr. Since $A^n = B^m$, then is $A^{\frac{n}{m}} = B$, (*Ax.* 1.) and $B^{\frac{1}{n}} = A^{\frac{n}{mn}}$, (*Ax.* 1.) $= A^{\frac{1}{m}}$, (*Theor* XI.)

The *Reverse* is also true, *viz.* That if $A^{\frac{1}{m}} = B^{\frac{1}{n}}$, therefore $A^n = B^m$; for $A^{\frac{n}{m}} = B^{\frac{n}{n}} = B$ and therefore $A^n = B^m$.

SCHOLIUM.

SCHOLIUM. This Theorem may be made yet more univerſal by taking any Mixt Index, and making the Theorem thus: If $A^{\frac{n}{m}} = B^{\frac{r}{s}}$, then is $A^{ns} = B^{rm}$; and hence again, $A^{\frac{1}{rm}} = B^{\frac{1}{ns}}$: For the firſt, ſuppoſe $A^n = a$, and $B^r = b$, then is $A^{\frac{n}{m}} = a^{\frac{1}{m}}$ and $B^{\frac{r}{s}} = b^{\frac{1}{s}}$; wherefore $a^{\frac{1}{m}} = b^{\frac{1}{s}}$; and hence, (as is already ſhewn) $a^s = b^m$. But $a = A^n$, and $b = B^r$, therefore $a^s = A^{ns}$, and $b^m = B^{rm}$; that is, $A^{ns} = B^{rm}$; whence again, $A^{\frac{1}{rm}} = B^{\frac{1}{ns}}$.

COROLL. If $A^n = B^m$, and if $A^{\frac{1}{m}}$ is Rational, ſo alſo is $B^{\frac{1}{n}}$, ſince they are equal.

THEOREM XVIII.

IF $A^n = B^m$, call $m - n = d$. Then is A^d a Power of the Order m, and B^d a Power of the Order n; i. e. $A^{\frac{d}{m}}$, and $B^{\frac{d}{n}}$ are both Rational, tho' they are not always equal.

Demonſtr. Since $A^n = B^m = B^{n+d}$, (for $m = n + d$) multiply both by A^d; and then $A^{n+d} = B^{n+d} \times A^d$; therefore A^d is a Power of the Order $n + d$ or m, (by *Coroll* 2. *Theor.* II.) Divide both by B^d and it is $A^n \div B^d = B^n$; therefore B^d is a Power of the Order n, (*Coroll.* 3. *Theor.* II) i. e. $A^{\frac{d}{m}}$ and $B^{\frac{d}{n}}$ are both Rational.

COROLL. If $d = 1$, that is, $m = n + 1$, then A has an $n + 1$ Root, and B an n Root: *i. e.* $A^{\frac{1}{m}}$ and $B^{\frac{1}{n}}$ are both rational; as they are alſo equal by the Theorem.

LEMMA.

If any Fraction $\frac{a}{r}$ is in its leaſt Terms, or is not ſo, then, accordingly, any Power of this Fraction, as $\frac{a^n}{r^n}$, is alſo in its loweſt Terms, or is not ſo. And the *Converſe*, if $\frac{a^n}{r^n}$ is or is not in its loweſt Terms, accordingly $\frac{a}{r}$ is or is not ſo.

The *Demonſtration* of this Truth muſt be referred to another Place, becauſe it depends upon Principles not yet explained: You'll find it demonſtrated in *Book* V. *Ch.* I. *Theor.* 13. *Coroll.* In the mean time we muſt ſuppoſe it to be true, for the ſake of ſome things belonging to this *Book*, whoſe Demonſtration depends upon this *Lemma*, and which could not be ſo regularly referred.

THEOREM XIX.

IF any Integer A has not a propoſed Root in Integers, it can have no determinate Root of that Order; *i. e.* it is a *Surd* Power of that Order. Or thus: If an Integer A is not the Power of a certain Order of an integral Root, it cannot be ſo of a fractional.

Example. 7 has no Square or Cube Root in Integers, and therefore has no ſuch determinate Root in a Fraction.

Demonſtr. Let $\frac{a}{r}$ be any Fraction in its loweſt Terms, and ſuch as is not equal to any Integer, *i. e.* let r be greater than 1; then, by the *Lemma*, $\frac{a^n}{r^n}$ is alſo a Fraction in its loweſt Terms: and conſequently, r^n is not an aliquot Part of A^n; nor, conſequently, is $\frac{a^n}{r^n}$ an Integer; for in this Caſe $\frac{a^n}{r^n}$ would not be in its loweſt Terms: Hence again it's clear, that no

no Fraction (such as is not equal to an Integer) can be any Root to a Whole Number: For suppose $\frac{s}{m}$ to be the n Root of A, then is $\frac{s^n}{m^n}=A$; but if $\frac{s}{m}$ is in its least Terms, so is $\frac{s^n}{m^n}$; therefore $\frac{s^n}{m^n}$ not an Integer is equal to A an Integer, which is absurd. Again, if $\frac{s}{m}$ is not in its least Terms, let $\frac{a}{r}$ be its least Terms; then, because $\frac{s}{m}=\frac{a}{r}$ therefore $\frac{s^n}{m^n}=\frac{a^n}{r^n}=A$. But $\frac{a}{r}$ being in its least Terms, so is $\frac{a^n}{r^n}$; whence the same Absurdity as before.

CHAP. II.

Containing the PRACTICE *of* Involution *and* Evolution.

§. I. Probl. *Of* INVOLUTION, *or* Raising *of Powers.*

THE *General Rule* for the Practice of *Involution* is plainly contained in the Definition, and is nothing else but a continued Multiplication of the Root into it self, whereby, to come at any higher Power, we must make up the Series of all the interior Powers: Thus; the Series of the Powers of 3 is 3 : 9 : 27 : 81 : 243, *&c.* By this Operation $3\times 3\times 3\times 3\times 3$, *&c.*

But there is a particular Method whereby all Powers above the Cube or 3d Power may be found, without actually finding all the inferior Powers: Which Rule is this:

RULE. Find, by the general Rule, two or more such Powers of the given Root as that the Sum of their Indexes be equal to the Index of the Power required; then multiply these Powers continually into one another; the Product is the Power sought. Or find any one Power whose Index is an aliquot Part of the Index of the Power sought, and involve that Power to an Index equal to the Denominator of that aliquot Part.

Example 1. To find the 7th Power of 4, I find the 2d, 3d, and 4th Powers; *viz.* 16, 64, 256; then the 3d × 4th, or $64\times 256=16384$ the 7th Power; because $3+4=7$. Or instead of finding the 4th Power I find the 5th, by multiplying the 2d and 3d; *viz.* $16\times 64=1024$; then the 2d × 5th, or $16\times 1024=16384$ the 7th Power.

Example 2. To find the 12th Power of 3, I find the Square, *viz.* $3\times 3=9$; the Square of the Square, or $9\times 9=81$ is the 4th Power; again, the 2d × 4th, or $9\times 81=729$ the 6th Power; then 6th × 6th, or $729\times 729=531441$ the 12th Power.

I need insist no more on this Practice; the *Reason* of which is plainly contained in the preceding *Theor.* VI. But I must *observe,* That it does not in every Case give any Advantage either of Ease or Expeditiousness to the Work; yet as it will do so in many Cases, and in none can it make the Operation more tedious, it will always be a very convenient Method.

Of the Practice *in Universal Characters.*

As to the literal Practice, or Involution of Numbers represented by Letters and Indexes, this is also sufficiently explained in the preceding Definitions and Theorems.

 But

But *observe*, That when a Root is represented as a complex Quantity; for Example, If instead of S we put A + B, its Powers may be represented two different Ways; Thus, $\overline{A+B}^2$ $\overline{A+B}^3$ or $\overline{A+B}^n$; which Method is in some Cases sufficient; but in others it's necessary to have the Operation performed, and the Power expressed according to the Result of the Multiplication; so that the Index be applied only to the single Letters: Thus, $\overline{A+B}^2 = A^2 + 2AB + B^2$. The most considerable and important of all these Cases of complex Roots, with their Powers, is that wherein there are only two Members in the Root, as A + B, called hence a *Binomial* Root; or A — B, called a *Residual* Root; the Consideration of whose Powers, *i.e.* of their Composition by the various Powers and Multiples of the different Members of the Root, has been found of very great Use in Mathematicks, and in Arithmetick, especially for the Business of the Extraction of Roots; in order to which I shall here explain it.

Of the Composition *of the Powers of a* Binomial *and* Residual *Root.*

In the annex'd Operation you see the several Powers of A + B raised by Multiplying, according to the common Rules, each Member of the Root into each Member of the several Powers, which produces the next Powers; in which these Things are remarkable.

Observations *on the Table of* Powers.

I. In the Expression of each Power there are as many and no more different Members, (which contain different Powers of the Parts of the Root A and B) as the Index + 1 expresses: For tho' each Member of any of the Powers being multiplied by A and B separately, do make in all twice as many Products as the Terms in the Power multiplied; yet each of the Series of Products by A and B have all their Terms similar, except the first Product by A, and the last by B; for these two are A^n and B^n; *i.e.* the two Series of Products by A and B contain in their several Members the same Powers of A and B, except the first Term of the Line of Products by A, which is A^n, and the last Term of the Line by B, which is B^n; consequently, these similar Products are reducible to a more simple Expression by adding them together, (*i. e.* addding the Numbers by which they are multiplied, and joining the common or similar Letters with their Indexes) thus, $A^3B + 3A^3B = 4A^3B$; also $3A^2B^2 + 3A^2B^2 = 6A^2B^2$; which explains the Reason of placing the Lines of Products by A and B, as is here done, *viz.* in order to the Addition of similar Products. That this Observation will hold true however far the Powers are carried, is easily seen from the Nature of the Thing; which will yet farther appear from the next Observation, in which we have a joint Demonstration of this.

TABLE *of* Powers *raised from the Root* A + B.

Root.	$A + B$
	$A + B$
	$A^2 + AB$
	$+ AB + B^2$
Square.	$A^2 + 2AB + B^2$
	$A + B$
	$A^3 + 2A^2B + AB^2$
	$+ A^2B + 2AB^2 + B^3$
Cube.	$A^3 + 3A^2B + 3AB^2 + B^3$
	$A + B$
	$A^4 + 3A^3B + 3A^2B^2 + AB^3$
	$+ A^3B + 3A^2B^2 + 3AB^3 + B^4$
4th Power.	$A^4 + 4A^3B + 6A^2B^2 + 4AB^3 + B^4$
	&c.

II. Each Power of A + B contains a Series of gradually different Powers of A and of B: Thus, The Index of any Power of $\overline{A+B}$ being *n*, the first Term is simply A^n, and the last

laſt is B^n; the intermediate Terms containing each a different Power of both A and B, multiplied together; the Index of A decreaſing gradually by 1 in each Term from A^n to the Term preceding the laſt, or B^n, in which it is ſimply A; and the Indexes of B increaſing the ſame way from the Term next after A^n, in which it is only B, to the laſt, or B^n; ſo that in all the intermediate Terms there is ſome Power of A and of B; and the Sum of their Indexes is equal to n, the Index of the Power propoſed of $\overline{A+B}$. Therefore, omitting the other Numbers, which are Multipliers in the ſeveral Terms of any Power of a Binomial, whoſe Index is n, theſe Terms, in as far as they are compoſed of the Powers of A and B, may be repreſented thus;

$$A^n + A^{n-1} \times B + A^{n-2} \times B^2 + A^{n-3} \times B^3 + \text{\&c. } A^2 \times B^{n-2} + A \times B^{n-1} + B^n.$$

wherein there are as many Members as $n+1$, according to the firſt Obſervation; and the Index of B, or the Number taken from n in the Index of A, is the Number of Terms after A^n to any Term.

This *Obſervation* we ſee to be true ſo far as the Table of Powers is carried; and that it muſt be true for ever, is eaſy to perceive. Or it may be demonſtrated, *thus*: Suppoſe it's true in any one Caſe or Power of $\overline{A+B}$, as the n Power, it muſt be true in the next Caſe, or the $n+1$ Power: becauſe when each Term of the given Power is multiplied by A, the Products muſt have A once more involved in them than in the Term multiplied: And ſince the Indexes of A decreaſe gradually from A^n

$$\begin{array}{l} A^n + A^{n-1} \times B + A^{n-2} \times B^2 + A^{n-3} \times B^3 +, \text{\&c. } + A \times B^{n-1} \times B^n \\ A + B \\ \hline A^{n+1} + A^n \times B + A^{n-1} \times B^2 + A^{n-2} \times B^3 +, \text{\&c.} + A^2 \times B^{n-1} + A \times B^n \\ \quad + A^n \times B + A^{n-1} \times B^2 + A^{n-2} \times B^3 +, \text{\&c.} + A^2 \times B^{n-1} + A \times B^n + B^{n+1} \end{array}$$

in the Terms multiplied, conſequently they will decreaſe gradually from A^{n+1} in the Series of Products; the Powers of B continuing as they were. Again; The Series of Products made by B muſt have B once more involved in each; and conſequently increaſing gradually from B to B^{n+1}, leaving the Powers of A as they were: But again, Theſe Products made by B are all of them, (except the laſt B^{n+1}) ſimilar to the ſeveral Products, (after the firſt A^{n+1}) made by A; becauſe the Indexes of A in the given Power decreaſing from the firſt Term A^n, which has no Power of B multiplied into it, and thoſe of B increaſing from the ſecond Term $A^{n-1} \times B$ to the laſt Term B^n, it's plain that A multiplied into any Term except the firſt A^n, and B multiplied into the preceding, muſt make ſimilar Products; for A multiplied into any Term raiſes the Index of the Power of A by 1, which makes it equal to the Index of A in the preceding Term, without changing that of B; and B multiplied in the preceding Term, raiſes the Power of B in it to the Index of B in the following Term, without changing that of A; conſequently theſe Products; are ſimilar, which makes the thing obſerved manifeſtly true in any Caſe, in conſequence of its being true in the preceding: But it's true in the Root or 1ſt Power, and as far as we have carried the Powers, therefore it's univerſally true. And this alſo is manifeſt, that the Sum of the Indexes of A and B that are in any Term, is always equal to the Index of the Binomial Power, *viz.* n. Add alſo this Obſervation, that the Index of A or B is always 1 leſs than the Number of Terms from A^n or B^n, to that Term.

SCHOLIUM. If any one Member of a Binomial is 1, as $A+1$, then the Powers of 1 being all 1, the Powers of ſuch a Root will conſiſt only of the Series of the Powers of A, and 1 added; thus, $A^n + A^{n-1} + A^{n-2} + \text{\&c.} + 1$. Or thus, $1 + A + A^2 + \text{\&c.} + A^n$.

III. The Numbers which in every Power are multiplied into the ſeveral Terms are called *Coefficients* (*i e.* joint Multipliers or Factors) of theſe Terms; and from the Manner of raiſing

the Powers this is to be obſerved, That the Coefficients of the firſt and laſt Terms are 1, and thoſe of the intermediate Terms are each the Sum of the Coefficients of the correſponding and preceding Terms of the preceding Power; thus, The Coefficient of the third Term of the 4th Power is 6, equal to $3+3$, the Coefficients of the 3d and 2d Terms of the 3d Power, (ſee the preceding *Table of Powers.*) Now that you may perceive the Reaſon of this, and that it muſt continue ſo for ever in all Powers, conſider theſe two Articles:

(1.) The Products of the ſeveral Terms of any Power, made by A or by B, do not change the Coefficients of the Terms multiplied, becauſe A and B have no Coefficient but 1. Then

(2.) The ſimilar Products made by A and by B are theſe, *viz.* The Product of the 2d Term, (of the Power multiplied) by A, and the Product of the 1ſt Term by B; the 3d Term by A, and the 2d Term by B; and ſo on. Which ſimilar Products are added by the adding of their Coefficients, and annexing the ſimilar Parts or Powers of A and B.

Now from theſe two Things the univerſal Truth of the Obſervation is manifeſt; and the annex'd Table, ſo far as it is carried on by this Rule, ſhews the Series of Coefficients of any Power of a Binomial.

TABLE *of Coefficients of the Powers of a Binomial Root.*

Powers.	Coefficients.
1ſt	1 : 1
2d	1 : 2 : 1
3d	1 : 3 : 3 : 1
4th	1 : 4 : 6 : 4 : 1
5th	1 : 5 : 10 : 10 : 5 : 1
6th	1 : 6 : 15 : 20 : 15 : 6 : 1
7th	1 : 7 : 21 : 35 : 35 : 21 : 7 : 1
8th	1 : 8 : 28 : 56 : 70 : 56 : 28 : 8 : 1
9th	1 : 9 : 36 : 84 : 126 : 126 : 84 : 36 : 9 : 1.
	&c. &c.

COROLL. From the two laſt Obſervations we learn a new and eaſier Way than the common, for raiſing any Power of a Binomial Root. Thus: take the Series of Products of the Powers of A and B, according to the ſecond Obſervation; and to theſe apply the proper Coefficients, as they ſtand in this Table; and if you have no ſuch Table, you muſt raiſe one, as far as the propoſed Power; which being done by ſimple Addition, is much eaſier than the common Rule. Thus, for Example; The 4th Power of $A+B$ is $A^4+4A^3B+6A^2B^2+4AB^3+B^4$.

But, again, to make this yet eaſier, ſee the following Obſervation, and its Corollary.

IV. Any two Coefficients in the Series belonging to each Power are the ſame Numbers, if they are taken at equal Diſtances from the Extremes, (which have both 1) for the Coefficients increaſe from the one Extreme to the Middle Term, where there is one Middle Term, and decreaſe from that to the other Extreme by the ſame Series by which they increaſed; and if there are two Middle Terms, theſe are equal, and they decreaſe upon each hand by the ſame Numbers to the Extremes. The univerſal Truth of this is manifeſt from the way of conſtructing the Table: For being true in any Caſe, (as we ſee it is as far as the Table is carried) it muſt be true in the next Caſe or Power, and ſo on for ever. And hence again, if we call the Place of any Term from the one Extreme a, the other Term whoſe Coefficient is equal, is from the ſame Extreme in the Place expreſſed by $n-a+2$. (n being the Index of the Power) For the whole Number of Terms is $n+1$, by *Obſerv.* 1. And that Term which is in the $\bar{a}$ Place, from the one Extreme, muſt neceſſarily be in the $\overline{n+1}-a+1=n-a+2$ Place from the other Extreme. And ſince Coefficients at equal Diſtances from the two Extremes are equal; hence it is, that reckoning them both from the ſame Extreme, their Places are a and $n-a+2$. Again, If we call the Index $n=a+b-2$. (*i.e.* add 2 to the Index, and ſuppoſe the Sum $n+2=a+b$; whereby $n=a+b-2$) then are the Coefficients equal which are in the a and b Places from

from the same Extreme; for $b = n - a + 2$; and we have seen already that the Coefficients, in the a and $n - a + 2$ Places are equal.

In the last Place, take Notice, That the 2d Term from either Extreme has for its Coefficient the Index of the Power.

COROLL. Hence, in making up the Table of Coefficients for any Power, as $\overline{A+B}^n$, when we are come to that Series which has as many Terms as $\frac{n+1}{2}$ *i. e.* the half Number of Terms belonging to the proposed Power n, when that Number $n+1$ is an even Number; or that has as many Terms as $\frac{n+2}{2}$ when $n+1$ is an odd Number; we need raise the following Series of the Table to no more Terms, till we come to the proposed Power n; and the remaining Terms of it are the same with these already found, taken in a reverse Order, as above explained. Thus: To find the Coefficients of the 9th Power, which has ten Terms; when you have arrived, in making the Table, at the 4th Power, which has five Terms, you need raise no more Terms in the following Series till you come to the 10th, and then make the remaining five Terms of it the same with the preceding, in a reverse Order. And for the 8th Power, which has nine Terms, you must also have the Coefficients compleat to the 4th Power, which has five Terms; and when you come to the 8th, the remaining four are the same with the first four reversely.

V. The preceding Observations were all obvious: But the following most valuable Property of the Coefficients, in which we have a curious Rule for finding the Coefficients of any Power without regard to the preceding Powers, we owe to the happy Genius of the incomparable Sir ISAAC NEWTON; which is this, *viz.*

Rule. *The Coefficient of any Term is equal to the Product of the Coefficient of the preceding Term multiplied into the Index of* A *in that preceding Term, and divided by the Number of Terms from* A^n *to that Term:* And because the Coefficients of the first and second Terms are always known, which are 1 and n, by *Observ.* 4. therefore it is plain, that by this Rule the Series of Coefficients of any Power may be found independently of preceding Powers.

Exam. The Coefficient of the fourth Term of the eighth Power is 56, the Index of A in that Term is 5; then $56 \times 5 = 280$, and $280 \div 4 = 70$, which is the 5th Term.

In order to the *Demonstration* of this Rule, we shall first explain the *universal Expression* of it in Letters, which is this: Take the Index of the Power n, and make this Series of Factors, $1 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times$, &c. carrying it to a Number of Terms equal to $n+1$; and the first Term or 1 is the Coefficient of the first Term of the Power; $1 \times \frac{n}{1}$ or simply n is the second Coefficient; $1 \times \frac{n}{1} \times \frac{n-1}{2}$ is the third Coefficient, and so on, taking in always one Factor more at every Step, till you have all the Coefficients belonging to that Power, which are in Number $n+1$. But, as is before observ'd, having found them for the one half of the Terms, or to the middle Term, the other half is found also without any farther Operation.

Now that this is a true and just Expression of the preceding Rule, will be plain from these Considerations: 1. That the first and second Terms are in all Cases 1 and n. 2. That the Index of A decreases continually from A^n the first Term; A^{n-1} being the second, and so on; whereby it is manifest, that according to this Rule with that in *Observ.* 2. the n Power of $A+B$ is represented as in the following Series; which is called

The Binomial Theorem.

$$\overline{A+B}^n = 1 \times A^n + 1 \times n \times A^{n-1} \times B + 1 \times n \times \frac{n-1}{2} \times A^{n-2} \times B^2 + 1 \times n \times \frac{n-1}{2} \times \frac{n-2}{3} \times A^{n-3} \times B^3$$

$+ 1 \times n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times A^{n-4} \times B^4 +$, &c. which is carried to a Number of Terms equal

equal to $n+1$; and then the last Term will be B^n; the Index of A being o, whereby A is out of that Term; and the Coefficient is 1.

Before we come to the *Demonstration*, we must observe upon this Expression of the Rule for the Coefficients, that the Numbers taken from n in the Numerators are always 1 less than the Denominators; and these (which are also equal to the Index of B, or the Number taken from n in the Index of A) being in Arithmetical Progression increasing from 1, the Numerators are in Arithmetical Progression decreasing from n. Hence the Denominator of the last Factor in each Coefficient is the Number of Factors after 1; and is also the Number of Terms after A^n to that Term; wherefore if the Denominator of the last Factor of any Coefficient is called a, that Term is in the $a+1$ place from the beginning; or if it is in the a Place, that Denominator is $a-1$. Wherefore the Coefficient of the a Place of the n Power is $1 \times n \times \frac{n-1}{2}$, *&c.* carried on till the last Factor, is $\frac{n-a+2}{a-1}$; or, make a the Place of the Term after the first, *i. e.* the Number of Terms -1; then the Coefficient is $1 \times n \times \frac{n-1}{2}$, *&c.* to $\frac{n-a+1}{a}$. And if we take this Series of Factors backwards, it is $\frac{n-a+2}{a-1} \times \frac{n-a+3}{a-2}$, *&c.* to 1, when a is the Number of Terms; or $\frac{n-a+1}{a} \times \frac{n-a+2}{a-1}$, *&c.* to 1, when $a+1$ is the Number of Terms. We shall next demonstrate the universal Truth of this Rule for Coefficients. Thus:

Demonstration of the preceding Rule for COEFFICIENTS.

1. If the Rule is good in any one Case or Power of $A+B$, as the n Power, it must therefore be good in the next Power, or $n+1$. To prove this Connection, see the two following Series; whereof the first contains the Coefficients for the n Power, according to the Rule; and the other the Coefficients for the $n+1$ Power, according to the same Rule; and because 1 does not multiply, I have omitted it in all the Coefficients but the first, which is itself 1.

For the n Power, $1,\ \frac{n}{1},\ \frac{n}{1} \times \frac{n-1}{2},\ \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3},\ \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, *&c.*

For the $n+1$ Power, $1,\ \frac{n+1}{1},\ \frac{n+1}{1} \times \frac{n}{2},\ \frac{n+1}{1} \times \frac{n}{2} \times \frac{n-1}{3},\ \frac{n+1}{1} \times \frac{n}{2} \times \frac{n-1}{3} \times \frac{n-2}{4}$, *&c.*

By *Observ.* 3. the Coefficients of any Power of $A+B$ are each equal to the Sum of the Coefficients of the corresponding and preceding Terms of the preceding Power of $A+B$; wherefore the first Series being the true Coefficients of the n Powers, the second will be the true Coefficients of the $n+1$ Power; providing that they have this Connection with the former, *viz.* that any Term is the Sum of the corresponding and preceding Terms of that former; which is therefore the thing to be proved. Thus:

The first Coefficient in all Powers is 1; then the Sum of the first and second Coefficients of the n Power is $1+\frac{n}{1}=\frac{n+1}{1}$ the second Term of the second Series: Again, the Sum of the second and third Terms of the first Series is $\frac{n}{1}+\frac{n}{1} \times \frac{n-1}{2} = \frac{n}{1} \times \overline{1+\frac{n-1}{2}}$ $= \frac{n}{1} \times \frac{n+1}{2} = \frac{n+1}{1} \times \frac{n}{2}$ (by changing the Order of the Numerators, which does not change the Product) and this is the third Term of the second Series; then the Sum of the

the third and fourth Terms of the first Series is $\frac{n}{1}\times\frac{n-1}{2}+\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}=\frac{n}{1}\times\frac{n-1}{2}\times 1+\frac{n-2}{3}=\frac{n}{1}\times\frac{n-1}{2}\times\frac{n+1}{3}=\frac{n+1}{1}\times\frac{n}{2}\times\frac{n-1}{3}$ (by changing only the Order of the Numerators, which does not change the Product) and this is the fourth Term of the second Series. From the Nature of these Series, it's evident they must have, every-where, the same Connection; or we may also shew the Universality of it. Thus:

By what's shewn in the Observation made upon the Expression of this Rule, the Coefficient of any Term, as that in the a Place after the first or A^n, may be thus expressed, $1\times\frac{n}{1}\times\frac{n-1}{2}\times$, &c. $\times\frac{n-a+2}{a-1}\times\frac{n-a+1}{a}$, and the preceding Term will be $1\times\frac{n}{1}\times\frac{n-1}{2}\times$ &c. $\times\frac{n-a+2}{a-1}$, which contains all the Factors of the other except the last. The Sum of these two is therefore, $1\times\frac{n}{1}\times\frac{n-1}{2}\times$, &c. $\times\frac{n-a+2}{a-1}\times\overline{1+\frac{n-a+1}{a}}=1\times\frac{n}{1}\times\frac{n-1}{2}\times$, &c. $\times\frac{n-a+2}{a-1}\times\frac{n+1}{a}=1\times\frac{n+1}{1}\times\frac{n}{2}\times\frac{n-1}{3}\times$, &c. $\times\frac{n-a+2}{a}$, (by changing the Order of the Numerators) which is the a Term after the first in the $n+1$ Power; because the Denominators are the same Series, 1, 2, 3, &c. to a, which are the Denominators in all Powers; and the Numerators decrease gradually from the Index $n+1$; so that the Number subtracted from the Index $n+1$ in the last, is less by one than the Denominator, (as has been observed and explained upon this Expression of the Rule); for the last Numerator is here $n-a+2=\overline{n+1}-a+1=\overline{n+1}-\overline{a-1}$.

2. But this Rule is true when applied to the first Power, and to all the Powers as far as we have raised them in the preceding Table; therefore, by what's now shewn, it's true in the next Power above; and consequently in all above, *i, e.* in all the Powers whatever of $A+B$.

SCHOLIUMS.

1st, Different Authors have made different Demonstrations of this Rule; I have chosen what I think as easy as any of them, and fittest for this place. In *Book* V. *Chap.* 6. you'll find another Demonstration of it from Principles which have not, as I know, been applied to this purpose.

2. If instead of a Binomial $A+B$ we take a Residual $A-B$, it's manifest that all the Difference betwixt its Powers and these of $A+B$ will be, That whereas all the Members of the Binomial Powers are added together, these of the Residual Powers will be connected with the Signs of Addition and Substraction, alternately; but the Powers of A and B, with the Coefficients are the very same: Thus, $\overline{A-B}^2=A^2-2AB+B^2$, and $\overline{A-B}^3=A^3-3A^2B+3AB^2-B^3$; also $\overline{A-B}^4=A^4-4A^3B+6A^2B^2-4AB^3+B^4$, and so on.

3. In applying this Rule for finding any Coefficient, (either of a Binomial or Residual) observe to take its Place from the nearest Extreme, A^n or B^n, which will make the Operation shorter, and produce the same Number, since the Coefficients are the same Series of Numbers, from either Extreme. Thus, to find the Coefficients in the a Place (from either Extreme) of the n Power: Compare a, and $n-a+2$ (for, by *Observ.* 4. the Coefficients in the a, and $n-a+2$ Places, from the same or different Extreme, are equal.) Which ever of these Numbers is least, find the Coefficient for that Place. *Example:* To find the 7th Coefficient of the 10th Power; I find the 5th Coefficient, which is equal to the 7th; for if $n=10$. $7=a$, then is $n-a+2=5$.

4. Tho'

4. Tho' we had taken no notice of the Equality of Coefficients at equal Diſtances from the two Extremes, yet the Rule now demonſtrated would have ſhewn it of itſelf: Thus; The Theorem for Coefficients is $1 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times$, &c. $\frac{n-a+1}{a}$; which is the Coefficient in the $a+1$ Place, or the $\bar{a}$ Place after the firſt. Now the Numerators decreaſe from n to $n-a+1$, or $n-a-1$, by a conſtant Difference 1, and the Denominators increaſe from 1 to $\bar{a}$. Again; Since the Number of Terms in the n Power is $n+1$, and in every Coefficient there are as many Factors as the Number of Terms from the Beginning, or A^n; therefore, if we want the laſt Coefficient, or that in the $n+1$ Place, then is $a=n$; and conſequently, $a+1=n+1$, and $n-a+1=n-n+1=1$; ſo that the Numerators and Denominators are the very ſame Series of Numbers, only in different Order, which alters not the Product; and being equal, therefore the Product is $=1$. Let us now expreſs the Series thus; $1 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$ &c. $\frac{3}{n-2} \times \frac{2}{n-1} \times \frac{1}{n}$; it's plain the laſt Coefficient but one, is the Product of this Series, excluding $\frac{1}{n}$; which Product is equal to $1 \times \frac{n}{1}$, for all the other Factors upon each hand of the middle one, (whoſe Numerator and Denominator muſt be equal) are reciprocal to one another, and ſo make the Product of them all only 1; or if there are two middle ones they are Reciprocals. By the ſame Reaſon, the Coefficient in the laſt Place but two, is is $1 \times \frac{n}{1} \times \frac{n-1}{2}$: For excluding $\frac{2}{n-1} \times \frac{1}{n}$, the middle Terms after $\frac{n-1}{2}$ deſtroy one another's Effect, and make their total Product no more than 1: The ſame Reaſoning holds in every Place. And hence again obſerve, that if we apply the Rule to find a Coefficient ſtanding from the firſt Place further than the middle Place, or the laſt of two middle Places; then whenever in writing down the Factors, we come to one whoſe Numerator and Denominator are equal, or to two adjacent Factors that are Reciprocals, there we may ſtop; for what follows will deſtroy the Effect of as many of theſe preceding that one whoſe Numerator and Denominator are equal, or the firſt of theſe two adjacent Reciprocals, as the remaining Number to be yet ſet down; and therefore, by cutting off ſo many of the Factors (as leaves a Number equal to the Place of the Coefficient ſought, number'd from the neareſt Extreme) we have what's ſought: Thus; For the 8th Coefficient of the 10th Power, it is $1 \times \frac{10}{1} \times \frac{9}{2} \times \frac{8}{3} \times \frac{7}{4} \times \frac{6}{5} \times \frac{5}{6} \times \frac{4}{7}$; which is $= 1 \times \frac{10}{1} \times \frac{9}{2} \times \frac{8}{3}$, for the reſt deſtroy one another, being Reciprocals.

5. If we take the perpendicular Columns of the Table of Coefficients, it's plain theſe are Coefficients all in the ſame Place, or Diſtance from the Beginning in different Powers; and may be called *Similar* Coefficients of different Powers. Again; We have explained above, that if the Place of any Coefficient is a, the laſt Factor that compoſes it is $\frac{n-a+2}{a-1}$, and ſo that Coefficient will be $1 \times \frac{n}{1} \times \frac{n-1}{2} \times$, &c. $\times \frac{n-a+2}{a-1}$; then by changing the Value of n this will expreſs all the ſimilar Coefficients in the a Place of different Powers: obſerving this, That the loweſt Value we can put upon n is $a-1$; becauſe no Power below that of the Order $a-1$ can have a Number of Terms equal to a (by *Obſerv.* 1.); and if $n=a-1$, the Coefficient will be 1; for it is the a Coefficient of the $a-1$ Power, which being the laſt Coefficient, is therefore 1, and is conſequently the firſt Term of the Series of ſimilar Coefficients of different Powers, from that whoſe Index is $a-1$: So that by taking n ſucceſſively equal to $a-1$, a, $a+1$, &c. we ſhall have the Series of Coefficients of the $\bar{a}$ Place of thoſe different Powers whoſe Indexes are $a-1$, a, $a+1$, $a+2$, &c.

But

But we may expreſs this Rule alſo thus: Inſtead of n put $a+b-2$, and it is $1 \times \frac{a+b-2}{1} \times \frac{a+b-3}{2} \times$ &c. to $\frac{\overline{a+b-2}-\overline{a-2}}{a-1}$ or $\frac{b}{a-1}$; which, according to the general Rule of Coefficients, is the Coefficient of the $\bar{a}$ Term of the $a+b-2$ Power; and by taking b ſucceſſively equal to 1, 2, 3, &c. we ſhall have hereby the Series of Coefficients in the $\bar{a}$ Place of all Powers from the $a-1$ Power; for if $b=1$, then is $a+b-2=a-1$, and the Rule gives the firſt ſimilar Coefficient, which is always 1; if $b=2$, then $a+b-2=a$, and we have the ſecond ſimilar Coefficient; if $b=3$, then $a+b-2=a+1$, and we have the third ſimilar Coefficient, and ſo on: Or if we take $m=a+b$, the Rule is $1 \times \frac{m-2}{1} \times \frac{m-3}{2}$ &c. $\times \frac{m-a}{a-1}$.

Hence again we have this general Truth to obſerve, *viz.* That the a Coefficient of any Power whoſe Index is $a+b-2$, is the ſame as the b Term of the Series of ſimilar Coefficients which are in the $\bar{a}$ Place of different Powers. And this will eaſily be proved from theſe two Conſiderations: The 1ſt is what we have already explained, *viz.* That if the Index of any Power is $a+b-2$, then the a Coefficient of that Power is equal to its b Coefficient (See *Obſerv.* 4) The 2d is, That from any Term in the Table of Coefficients, (*i e.* any Coefficient of any Power) there ſtand as many Terms on the right hand, as there are Terms above it in the perpendicular Column of ſimilar Coefficients; and therefore that Term is in the ſame Place of the ſimilar Coefficients, and of the Line of Coefficients of that Power, numbering from the right hand: Wherefore it's plain, that the a Coefficient (reckoning from the left hand) of the $a+b-2$ Power, is the ſame as the b Term of the Column of ſimilar Coefficients in the a Place of different Powers, becauſe it's the ſame as the b Coefficient from the Right of the ſame Power.

COROLLARIES.

1. Theſe Expreſſions of Powers of a *Binomial* Root ſhew us how the Difference betwixt any two ſimilar Powers is compoſed of the various Powers and Multiples of any one of the Roots, and the Difference betwixt the Roots: Thus, A being one Root, and B the Difference of that and another A + B, or A − B, the Difference of their Squares is $2AB+B^2$. Hence

Having any Power of any Root, we can find another ſimilar Power whoſe Root ſhall differ from the given one by any Difference, and that without either knowing or enquiring what that other Root is.

For *Example*: 144 is the Square of 12; and if the Difference betwixt this Root and another is 9, hence the Square of that other is $144+2\times12\times9+81=144+216+81=441$, if 12 is the leſſer Root; but it is $144-216+81=9$, if 12 is the greater Root.

Obſerve, If the given Difference B is 1, then in all the Terms wherein there is any Power of B, we have nothing but the Powers of A, with the Coefficients; except the laſt Term B^n, which ſtands alone, and is 1; for $\overline{A+1}^2=A^2+2A+1$, and $\overline{A+1}^3=A^3+3A^2+3A+1$.

2. We have here alſo learnt another Way of Raiſing a given Number to any Power, by means of the ſimilar Powers of the Binomial: Thus;

Take all the ſignificant Figures of the given Number in their compleat Value, as ſo many different Members that compoſe it, by Addition; then take the two higheſt, calling them A and B: Raiſe this Binomial to the propoſed Power; then conſider the firſt two Members as one; call their Sum again A, and call the next Member B, and raiſe this new Binomial to the ſame Power; in doing of which, *obſerve*, that ſo much of the Work is already done, becauſe the n Power of the firſt Member of the preſent Binomial is the total Power of the preceding Binomial, which is already found; ſo that what remains is to

make up the other Members of the Power fought, according to the general Canon. In the fame manner confider the three higheft Members as one, and call it A, and join the next Member, calling it B, and raife this Binomial; and thus proceed till all the Members are taken in.

Example: To find the Square of $246 = 200 + 40 + 6$, the Operation is

$$\overline{200 + 40}^2 = 40000 + 16000 + 1600 = 57600$$
$$\overline{A + B}^2 = A^2 + 2AB + B^2$$

Then $\overline{240 + 6}^2 = 57600 + 2880 + 36 = 60516$ the Square fought.

$$\overline{A + B}^2 = A^2 + 2AB + B^2$$

If there are more Members, you muft go on the fame Way.

Example 2. To find the Cube of $235 = 200 + 30 + 5$. the Work is

$$\overline{200 + 30}^3 = 8000000 + 3600000 + 540000 + 27000 = 12167000$$
$$\overline{A + B}^3 = A^3 + 3A^2B + 3AB^2 + B^3$$

Then $\overline{230 + 5}^3 = 12167000 + 793500 + 17250 + 125 = 12977875$ the Cube fought.

$$\overline{A + B}^3 = A^3 + 3A^2B + 3AB^2 + B^3$$

Scholium. As to this Method of raifing Powers, it's more tedious than the common Way, and therefore not to be recommended for Practice; the Defign of confidering it here being only for the fake of a particular Illuftration to be made by it of the Rules of Extraction.

§. II. *Of* Evolution, *or* Extraction *of* *Roots.*

Part I. *Of* Whole Numbers.

Problem I. *To extract the* Square Root *of a* Whole Number.

Roots.	Squares.
1,	1
2,	4
3,	9
4,	16
5,	25
6,	36
7,	49
8,	64
9,	81

Rule I. In Order to the Solution of this Problem, we muft have a Table of fimple Squares, or Squares of Numbers from 1 to 9, as here in the Margin: Then

II. Beginning at the Right hand, diftinguifh the Figures of the given Number into Periods of two Figures as long as you can, by putting a Point over the firft Figure, and over every other Figure, *i. e.* paffing one, take the next. *Example:* 1849 is pointed thus, $1\dot{8}4\dot{9}$, and 34968 thus, $\dot{3}4\dot{9}6\dot{8}$; the pointed Figure being the firft of each Period, and that on its Left the other; tho' the laft Period may fometimes have but one Figure.

The given Number being thus pointed, the Number of Points or Periods fhews us how many Figures the Root confifts of; to find which we proceed thus:

III. Take the laft Period, (or that next the Left) and feek it, or the next leffer Number you can find, in the Table of fimple Squares, the Root of this is the firft Figure on the Left of the Root fought; which being written down to the Right of the given Number, as we do the Quote in Divifion, then fet down its Square under the laft Period, and take their Difference, to which prefix the next Period of the given Number: And all this taken for one Number

ber, as it ſtands, we call the Second Reſolvend, becauſe out of it we ſeek the next Figure of the Root, (the laſt Period being the firſt Reſolvend) thus:

IV. Conſider the Figure found as having o before it, and then multiply it by 2, (which is in effect, multiplying it by 20); make this Product a Diviſor, and find how oft it is contained in the Reſolvend; which, to the preſent purpoſe, muſt not be taken above nine times; tho' in ſome Caſes it may be oftner contained; and then alſo it muſt be under this Limitation, *viz.* that the Square of the Quote, or Figure now ſet in the Root, added to its Product by the Diviſor, the Sum do not exceed the Reſolvend: Or, which is the ſame thing, put the Quote, or ſuppoſe it put, in the Place of the o which ſtands in the Place of Units of the Diviſor; then multiplying the whole by the Quote, the Product muſt not exceed the Reſolvend: For if it do, the Figure taken is too great, and you muſt try a leſſer, till it anſwer. The Figure thus found is the next Figure of the Root ſought, which muſt be ſet on the Right of the laſt: And ſetting the Sum or Product mention'd under the Reſolvend, take their Difference, to which prefix the next Period of the given Number; and all this conſidered as one Number, is your next Reſolvend; out of which the next Figure is to be ſought thus:

V. Take both the Figures of the Root found, as they ſtand, for one Number; double it, and prefix o, (or prefix o, and then double, which is the ſame thing,) and this is your Diviſor: Find how oft it is contained in the Reſolvend, under the ſame Limitations as formerly; place the Figure found on the Right of theſe before found, and ſubſtracting the Product directed to be compared with the Reſolvend from it; to the Difference prefix the next Period, and you have the next Reſolvend; to which make a Diviſor out of the Figures of the Root already found, the ſame way as before; and thus proceed till all the Periods are employed, finding a new Figure of the Root for every Period: And if at any Step the Diviſor is greater than the Reſolvend, or if 1 added to the Diviſor makes the Sum greater than the Reſolvend; then place o in the Root, and prefix another Period, forming a new Diviſor by ſetting another o to the former Diviſor, and ſo go on.

EXAMPLES.

Ex. 1. To find the Square Root of 1369, it is 37, as found by this

Operation.

$$\begin{array}{rll} & \dot{1}3\dot{6}9 & \left(\begin{matrix} a & b \\ 3 & 7 \end{matrix}\right. \\ a^2 = & 9 & \\ \hline 2a=60 & 469. & \text{2d Reſolvend.} \\ (a=30) & 469 = 2ab + b^2 & \\ \hline & 000 & \\ \hline \end{array}$$

Explication.

The given Number being pointed, the laſt Period is 13, and the next Square to this is 9, whoſe Root is 3, which is the laſt Figure of the Root; and calling it $\bar{a}$, I take $a^2 = 9$ out of 13, and to the remaining 4 I prefix the next Period 69, which makes 469 the 2d Reſolvend: Then taking $a = 30$, I double it, and make $2a = 60$ a Diviſor; and ſeeking how oft it is contained in 469, under the Limitations of the Rule, I find it 7 times, which is therefore the other Figure of the Root; which is proved by this, that $2ab + b^2 = 469$, the Reſolvend; and becauſe there is no Remainder, the given Number 1369 is a true Square, whoſe Root is 37.

Example 2. To find the Square Root of 23097636, it is 4806.

Operation.

$$\begin{array}{rl|l}
 & 23097636 & \left(\begin{array}{l} abbb \\ 4806 \end{array}\right. \\
a^2 = & 16 & a \quad a \\
\hline
2a=80,\ (a=40) & 709. \text{ 2d Resolvend.} & \\
 & 704 = 2ab + b^2 \ (b=8) & \\
\hline
2a=960,\ (a=480) & 576. \text{ 3d Resolvend.} & \\
\hline
2a=9600,\ (a=4800) & 57636. \text{ 4th Resolvend.} & \\
\hline
 & 57636 = 2ab + b^2 \ (b=6) & \\
\hline
 & 00000 &
\end{array}$$

Explication.

The last Period being 23, the next Square lesser is 16, whose Root is 4, which I place in the Root, and calling it a, I take $a^2 = 16$ out of 23, and to the Remainder 7 I prefix the next Period 90, which makes 709 the 2d Resolvend: Then taking $a = 40$, according to the Rule, I double it, and $2 \times 40 = 80$ is the Divisor, which is contain'd in the Resolvend 8 times; which Number also answers the Limitations of the Rule: For 88×8 is $= 704 = 2ab + b^2$, or $\overline{2a+b} \times b$, b being 8, wherefore 8 is the next Figure of the Root sought; and substracting 704 from 709, to the Remainder 5 I prefix the next Period 76, and 576 is the 3d Resolvend; then taking $a = 480$, its Double, 960, is the Divisor; which being greater than the Resolvend, I set 0 in the Root, then prefixing the next Period 36, the 4th Resolvend is 57636, and the Divisor is 9600, (a being here 4800) which is the former Divisor with 0 prefix'd: Then I find 9600 contained in 57636, 6 times, which Number answering the Limitation of the Rule, I set 6 in the Root, and calling it b, I find $2ab + b^2 = 57636$, the Resolvend; so that nothing remains; And so the given Number 23067636 is a perfect Square, whose Root is 4806.

SCHOLIUMS.

1. If you begin your Guess or Trial for the Quote in any of the Steps after the first, at the greatest Number of times, not exceeding 9, that the Divisor is contain'd in the Resolvend: Then the Limitation of the Rule for the Number to be compared with the Resolvend is sufficient to determine when we have the true Figure; because if that Number is greater than the Resolvend the Quote is taken too big; and then we try the next lesser, till it answer the Rule: Yet *observe*, that if you should make trial at random, then tho' $2ab + b^2$ be less than the Resolvend, yet the Quote b may be too little, and you'll prove it by this Mark, *viz*, The Remainder, after taking $2ab + b^2$ out of the Resolvend, may be greater than the Divisor, but it must not exceed the Sum of the Divisor and double the Quote, *i. e.* $2a + 2b$, else the Quote is too little: *See below the* Explication *of* Exam. 3. And here I must *observe*, That some Authors think the forming of a Divisor an useless thing, and would have us left altogether to a random Guess for the Figure of the Quote at every Step after the first, tho' they prescribe the same necessary Limitation of the Figure guessed: But they have not considered this Conveniency of the Divisor, that the greatest Number of times it is contained in the Resolvend not exceeding 9, is a Limit to our guessing; for the Figure sought cannot exceed that, and so will in many Cases save the Trouble of guessing at Figures which cannot answer, Besides, the Divisor is of a necessary Consideration in the Demonstration of the Rule; and a further Use of it, see in the next Article.

2. If the Divisor is contained in the Resolvend oftner than 9 times in any Step after the 2d, the Figure sought is certainly 9: And also in the 2d Step it's 9, if the first Figure is at the same time, 5, 6, 7, 8, or 9. But if this is below 5, we must make trial; for sometimes it will be 9, and sometimes not. *See* Exam. 3, 4.

3. For

3. For forming the Divisor a little more easily, you have no more to do after the 2d Step but add the double of the Figure last found to the last Divisor, and then prefix o; as you may easily perceive in the Examples.

4. If there is a Remainder after all the Periods are employed, then the given Number is not a perfect Square; and the Root found is the Root of the greatest Integral Square contained in it. How to find a Mixt Root whose Square shall be within any assigned Difference from the given Number, shall be taught in its proper Place.

Example 3. To find the Square Root of 151426. By the *Operation* we find it's not a Square, but the Root of the greatest Integral Square contained in it is 380.

Operation.

$$
\begin{array}{rll}
 & \dot{1}5\dot{1}4\dot{2}6 & (389 \\
a^2 = & 9 & \\
\hline
2 \times \overset{a}{30} = 60) & 614 & \\
 & 544 = 2ab + b^2 & \\
\hline
2 \times \overset{a}{380} = 760) & 7026 & \\
 & 6921 = 2ab + b^2 & \\
\hline
 & 105. \text{ Remainder.} &
\end{array}
$$

Explication.

Here in the second Step 60 is contained in 614, 10 times; yet the true Quote, or Figure for the Root, is only 8: For 9 would make the Product 621, which is greater than 614: And had we taken 7, it would have been found too little, from the Mark given in *Schol.* 1.; for then the Product is 499, which taken from 614 leaves 145, which is greater than 60 + 16 (or $2a + 2b$) = 76. And because the Remainder of the whole Work is 105, the greatest Square contained in 151426 is 151321, whose Root is 389.

Example 4. To find the Square Root of 15437052: The Root of the greatest Square contained in it is 3929.

Operation.

$$
\begin{array}{rll}
 & \dot{1}5\dot{4}3\dot{7}0\dot{5}2 & (3929 \\
a^2 = & 9 & \\
\hline
2 \times \overset{a}{30} = 60) & 643. \text{ 2d Resolvend.} & \\
 & 621 = 2ab + b^2. & \\
\hline
2 \times \overset{a}{390} = 780) & 2270. \text{ 3d Resolvend.} & \\
 & 1564 = 2ab + b^2. & \\
\hline
2 \times \overset{a}{3920} = 7840) & 70652 \text{ 4th Resolvend.} & \\
 & 70641 = 2ab + b^2. & \\
\hline
 & 11 \text{ Remainder.} &
\end{array}
$$

Explication.

In the second Step, 60 is contained in 643 10 times, and the true Quote is 9. In the third Step, 780 is contained in 2270 only 2 times; and 7840 in 70652 9 times: The Remainder of the Operation being 11. So that the greatest Square contained in the given Number is 15437041, whose Root is 3929.

Demonstration of the preceding Rule.

In order to this Demonstration, the following *Lemma's* must be first demonstrated.

LEMMA I.

The Product of any two Numbers can have at most but as many Places of Figures as are in both the Factors; and at least but one Place fewer. *Exam.* $3 \times 4 = 12$, and $2 \times 16 = 32$.

Demonstr. 1. That the Product may have as many Places as both the Factors, one *Example* is enough to demonstrate. Thus, $46 \times 82 = 3772$; and that in no Case it can have more, I thus prove.

Let any two Numbers be A, B; then take D the least Number possible, which has one Place more than B; it's evident from the Notation of Numbers, that D will consist of 1, with as many o's as the Number of Places in B; and also D will be a greater Number than B; if we then multiply A by D, the Product AD will be equal to A, with as many o's before it as are in D, *i. e.* as the Number of Figures in B: therefore it has as many Places, and can have no more than are in both A and B. But again; since B is a lesser Number than D, therefore AB is a lesser Number than AD, and consequently cannot have more Places, *i. e.* more than are in A and B both.

$A = 4678$. $B = 37549$
$A \times B = 175654222$
$D = 100000$. $A \times D = 467800000$.

2. The Product may have fewer Places than are in A and B both, which one Example will shew. Thus, $23 \times 346 = 7958$: but it can in no Case have above one Place fewer, which is thus proved.

Take any two Numbers A, B; and take D consisting of 1, with as many o's before it as the Figures less than one in B; *i. e.* the least Number possible, which has as many Figures as B; then will the Product AD be equal to A, with as many o's before it as are in D, which are one fewer than the Figures in B; consequently AD has as many Places, and can have no fewer than the Sum of the Places in A, and one fewer than are in B; *i. e.* all the Places in AD can be but one fewer than the Sum of those in A and B. But since B is a greater Number than D, so will AB be greater than AD; and consequently cannot have fewer Places than AD, which can be but one fewer than in A and B both.

$A = 23$. $B = 346$
$AB = 7958$
$D = 100$. $AD = 2300$.

Corol. A Number being multiplied into itself, the Product or Square cannot have more Places than double the Places of the Root; and but one fewer at least than that double. Wherefore a Square being distributed into Periods, as the Rule directs, the Root has precisely as many Figures as the Square has Periods.

LEMMA II.

If any Number A is not a Square, yet being distributed into Periods, according to the Rule, the greatest Square contained in it, as N^2, will have precisely as many Periods as that Number A has.

Exam. 237694 is not a Square, and the greatest Square contained in it is 237169; both which have three Periods.

Demonstr. 1. N^2 cannot have more Periods than A; for then it will have more Figures, and consequently be a greater Number than A; contrary to Supposition.

2. Take 1 with as many o's before it as there are Figures standing before the last Period of A (or on the right Hand of it) call the Number arising B; then it is plain that B is a square Number, whose Root is 1, with half as many o's as are in B. For to square any Number expressed by 1 with a Number of o's before it, it's manifest, from the Nature of Multiplication, that the Square is 1, with double as many o's; wherefore B is a Square of as many Periods as A has, and being evidently contained in it, it follows, that the greatest Square contained in it cannot have fewer.

$A = \dot{2}37\dot{6}9\dot{4}$
$N^2 = \dot{2}37\dot{1}6\dot{9}$
$B = 10000$
$B^{\frac{1}{2}} = 100$.

Corol. The Root of the greatest Square, contained in any Number A which is not a Square, hath as many Figures as A has Periods; for it has as many as its own Square has

has Periods (by *Corol. Lemma* 1.) which are as many as A has, by the present *Theorem.*

LEMMA III.

Any Number being distributed into Periods, the greatest Square contained in the last Period on the left, considered as one Number by itself, is the Square of the last Figure of the Root of the given Number, if it is a perfect Square; or of the Root of the greatest Square contained in it, if it's not a Square. Again; the greatest Square contained in the two last Periods, taken as one Number by themselves, is the Square of the last two Figures of the Root of the given Number, or of the greatest Square contained in it; and the same thing is true, comparing the 3 or 4, *&c.* last Periods, with the Square of the 3 or 4, *&c.* last Figures of the Root of the given Number, or of the greatest Square contained in it.

Demonstr. Let A be any Number, and B the Square Root thereof, or of the greatest Square contained in it; also let D represent the last, or 2 last, *&c.* Periods of A, [as in the annex'd Example, take D=22 or 2273, or 227338] and let r represent the last, or 2 last, *&c.* Figures of the Root B, [as here 4 or 47, or 476]; so that r^2 is the Square of that last, or 2, *&c.* last Figures of the Root B.

These things being settled, the Truths to be proved are comprehended in one universal Case, which is this, *viz.* that r^2 in the greatest Square contained in D; which I shall demonstrate in two Articles: Thus,

A=22733824. B4768
D r

1. r^2 is contained in D; for since (by *Corol.* to *Lem.* 1 and 2) there are as many Periods in A, as there are Figures in B; consequently there are, in every Case, as many Periods standing before D in the total A, as there are Figures before r in the total B; so that taking D and r in their compleat Values, as they stand in their Totals, there will be as many o's before D, as the Number of Figures in the Periods of A, which stand before, or on the right Hand of D; and as many o's before r, as the half of those before D. [*Exam.* If D=22000000, then is r=4000; and if D=22730000, then is r=4700.] Then r^2 will have as many o's before it in its compleat Value, as double the Number of o's before the Root r in its compleat Value; and consequently as many as before D. We shall now express these Numbers in their compleat Values; *thus*, $r$oo, *&c.* r^2oo,oo, *&c.* Doooo, *&c.* and shew that r^2 is contained in D. For,

If r^2 is greater than D, (both taken without the o's) then is r^2oooo, *&c.* greater than Doooo, *&c.* (r^2 and D being here equally multiplied) by an equal Number of o's prefix'd.) But A is equal to D, with as many Figures before it as there are o's before D or r^2 taken in their compleat Values; (*i. e.* Doooo, *&c.* r^2oooo, *&c.*) Therefore r^2oooo, *&c.* is greater than A; [for any Figures whatever in the Places of the o's before D, cannot be equal to the Excess of r^2 above D, tho' that Excess were but 1] *i. e.* the Square of r oo, *&c.* which is but a Part of B, is greater than the Square of B; because A is at least equal to B^2: but this is absurd; therefore r^2 cannot be greater than D, and consequently must be contained in it.

2. r^2 is the greatest Square contained in D: For suppose N^2 a greater Number than r^2; then take N with as many o's before it, as are before r in its compleat Value, and express it thus, Noo, *&c.* so that its Square is Noooo, *&c.* having double as many o's as Noo, *&c.* the Root has; or as many as r^2oooo, *&c.* or Doooo, *&c.* has. Now because D contains N^2, by Supposition; therefore Doooo, *&c.* contains N^2oooo, *&c.* Also because N^2 is supposed greater than r^2; therefore N is greater than r; and Noo, *&c.* greater than $r$oo, *&c.* or than r with as many of any Figures before it; *i, e.* Noo, *&c.* is greater than B, (which is equal to r, with as many certain other Figures before it, as there are o's before

fore r in its compleat Value $r$oo, *&c.* or in Noo, *&c.*) ſo that Doooo, *&c.* a Part of A, contains N^2oooo, *&c.* the Square of Noo, *&c.* a Number which is greater than B, the Root of the greateſt Square contained in A, which is abſurd; therefore r^2 is the greateſt Square contained in D.

Corol. If we find the Root of the greateſt Square contained in the laſt Period of any Number, we have the laſt Figure of the Root ſought: And if we find the Root of the greateſt Square contained in the two laſt Periods, we have the two laſt Figures of the Root ſought, and ſo on; which ſo far explains the Inveſtigation of the Rule; what remains to compleat it, you have in the following

LEMMA IV.

Part 1. If the Root of any known Square is ſuppoſed to conſiſt of two Parts, or Members; then if one of theſe Members is known, we have a Rule for finding the other from the Conſideration of the Square of a Binomial Root. Thus: If the Root is $A+B$, the Square is $A^2+2AB+B^2$, *viz.* the Sum of the Squares of the two Parts, and twice the Product of theſe Parts; wherein it is evident, that if the Square of either Part, as A^2, is ſubtracted from the total Square $A^2+2AB+B^2$, the Remainder is the Sum of the Square of the other Member, and the double Product of the two Members, *viz.* $2AB+B^2$. Now ſuppoſe A to be known; if we take 2 A for a Diviſor, and find how oft it is contained in that Remainder; but under this Limitation, *viz.* that the Quote being added to the Diviſor, and the Sum multiplied by the Quote, the Product ſhall be equal to the Dividend $2AB+B^2$; then it is manifeſt, that the Quote can be no other Number than B, the other Member of the Root ſought. For ſince $\overline{2A+B}\times B=2AB+B^2$ the Dividend, therefore it's plain that no other Number but B added to 2 A, and the Sum multiplied by the ſame B, will produce $2AB+B^2$; ſince either a greater or leſſer Number added to 2A, makes a greater or leſſer Sum; which being multiplied by the ſame Number, produces ſtill a greater or leſſer Number.

Part 2. Tho' a Number is not a Square, yet having one Member of the Root of the greateſt Square contained in it, we can find the other Member by the ſame Method, as if it were a Square. Thus:

$$M=A^2+2AB+B^2+R$$
$$M-A^2=2AB+B^2+R=D$$

Let M be any Number not a Square, and $A+B$ the Root of the greateſt Square contained in it; the Square is therefore $A^2+2AB+B^2$. Alſo let R be the Number that's more than $\overline{A+B}^2$ in M, ſo that $M=A^2+2AB+B^2+R$. Now A being known, if we take A^2 from M, the Remainder is plainly $2AB+B^2+R$, which we may call D. And if we find how oft 2 A is contained in D under this Limitation, *viz.* that the Quote being added to the Diviſor, and the Sum multiplied by the ſame Quote, the Product ſhall ſtill be leſs than D: [For this is to be obſerved, that there is no Number which will make a Product equal to D; becauſe then M would be a Square; therefore any Number you can take, will make the Product either greater or leſſer than D.] Then, I ſay, the Quote is the other Member of the Root ſought, *viz.* B: For let us ſuppoſe the Quote is another Number N, then if N is leſs than B, it follows, contrary to Suppoſition, that N is not the greateſt Number qualified according to the Rule, *viz.* which added to the Diviſor, and the Sum multiplied by the ſame Number, makes a Product leſs than D; for B is greater than N, and yet is a Number ſo qualified, becauſe $D=2AB+B^2+R=\overline{2A+B}\times B+R$: Therefore N is not leſs than B. Nor, again, can it be greater; for by Suppoſition $\overline{2A+N}\times N$ ($=2AN+N^2$) is leſs than D ($=M-A^2$) and adding A^2 to both, then $A^2+2AN+N^2$ ($=\overline{A+N}^2$) is leſs than M, and is therefore contained in it. But again; ſince N is greater than

than B, $A+N$ is greater than $A+B$, and $\overline{A+N}^2$ ($=A^2+2AN+N^2$) greater than $\overline{A+B}^2$ ($=A^2+2AB+B^2$) and consequently this is not the greatest Square contained in M, as was supposed: Wherefore N is not greater than B; and if it's neither greater nor lesser, it must be equal.

COROLL. If a given Number M is not a Square, the Number R which is over the greatest Square contained in it, (and is necessarily the Remainder, which happens in the Operation after B the second Member of the Root is found) may be greater than 2A the Divisor; because we have not taken 2A out of the Dividend D as oft as possible; but it can never exceed double of the Root found, if that is the true Root of the greatest Square contained in M: For let the Root found be called N, if the Remainder exceeds 2N, it must be at least $2N+1$, and if this is added to N^2, the Sum N^2+2N+1, ($=\overline{N+1}^2$) being evidently contained in M, it follows that N is not the Root of the greatest Square contained in it, as was supposed.

APPLICATION *of the preceding* Lemma's *for demonstrating the* Extraction *of the* Square Root.

1. The first and second *Lemma's* are already applied; from whence are deduced, as *Corollaries*, the first Thing asserted in the *Rule*, viz. That the Root must have as many Figures as the given Number has Periods.

2. From *Lemma* III. we have the Reason why the given Number is pointed from the Right Hand to the Left; because, being done so, it is demonstrated that the last Figure of the Root sought, the two last, and so on, make the Root of the greatest Square contained in the last, the two last, *&c.* Periods of the given Number.

3. The remaining Part of the Rule is to find the Figures of the Root, one after another, out of these Periods; the Reason of which is contained in *Lemma* III. and IV. and its *Coroll.* and is deduced thus:

We first take the last Period, and the greatest Square contained in it we seek in the Table of simple Squares, [which must be found there; for since a Period has but two Figures at most, the Root of the greatest Square contained in it can be but one Figure; becauuse the Square of 10, the least Number of two Figures, is 100, which has three Figures]. The Root of this Square is, by *Lem.* 3. the last Figure of the Root sought. So in the preceding *Example* 3. the given Number is $15\dot{1}42\dot{6}$; the last Period is 15, and the greatest Square contained in it is 9, whose Root is 3, the last Figure of the Root sought. Now if we suppose the two last Periods $15\dot{1}4$ to be the given Number, then the Root of the greatest Square contained in it has but two Figures, whereof we have found the last, *viz.* 3, whose Real Value is 30; and to find the other, it's plain, from *Lemma* IV. that calling the first Member of the Root now found, *viz.* $30=a$, and calling the Member sought b, then the greatest Square contained in 1514 is $a^2+2ab+b^2$; but $a^2=9$, or rather 900 taken in its true Value; so that 9 from 15, and 14 prefix'd to the Remainder, (which is the Method of the Rule) is the same thing as 900 from 1514: The Remainder is 614, the second Resolvend, which is equal to $2ab+b^2$ at least, with some Remainder over perhaps; we shall therefore call the Remainder $2ab+b^2+r$: What remains then, is to find this second Member of the Root b; and according to *Lem.* IV. if we make 22 the Divisor, and find how oft it is contained in the Resolvend $2ab+b^2+r$, so that calling the Quote b, this Quote added to the Divisor, and the Sum $(2a+b)$ multiplied by b, the Product $(2ab+b^2)$ shall not exceed the Resolvend $(2ab+b^2+r)$, then it's shewn that the Quote is truly the second Member of the Root: But it's manifest that this is the very Method of the Rule; wherefore it's just and

good when the Root sought has but two Figures. Again; The Number given having three Periods, as if it were 151426, then having found 38 the Root of the greatest Square contained in the two first Periods 1514, (as already shewn); these are the two last Figures of the Root of 151426, (by *Lem.* 3.) And if we take 38 in its true Value it is 380, because there is another Figure on its Right in the Root sought; then 380 being considered as one Member of the Root sought, we call it also a; and by *Lemma* IV. we are to subtract its Square, *viz.* $\overline{300+80}^2 = \overline{300}^2 + 2 \times 300 \times 80 + \overline{80}^2$ out of the given Number 151426: But this is already done, because we have taken first the Square of 3, (which was in the former Step called a) *viz.* 9, out of 15, which is equivalent to taking the Square of 300, (which is now a) *viz.* 90000, out of 151426, which leaves 61426; then b being 8, and $a=30$, we have taken $2ab+b^2=544$ out of 614, the former Remainder; to the Remainder 70 we have prefix'd 26, the first Period, which makes the whole 7026; and this is the same Number which remains, if taking $b=80$, and $a=300$, we take $2ab+b^2=48000+6400=54400$ out of the former Remainder 61426. Now the Square of 380 being taken out of the given Number 151426, the Remainder 7026 is the next Resolvend; and for a Divisor we have made $2\times a = 2\times 380 = 760$, and the Member sought we have found the same way as before, which is both according to *Lemma* IV. and the Rule for Extraction; which is therefore good for any Root of three Figures.

If there are more than three Figures in any Root, the Reasons of the Rule from one Step to another for ever are manifestly the same, and need not be further insisted on. I shall only illustrate this Application by one Example of a perfect Square, whose Involution by the Method shewn in the preceding Section, and its Evolution by the present Rule, will illustrate one another; and you'll evidently perceive, that as by knowing the true Place of every Figure found in the Root, we may take it in its compleat Value, and perform the Work that way, as in the following Operation; yet we save the trouble of many superfluous Figures by the Method of the Rule, which produces the same Effect.

Involution *of* 389 *to its Square makes* 151321.

Thus:

Root.

$$\begin{aligned}
&\underline{389 = 300 + 80 + 9}\\
&90000 = a^2.\ (a = 300)\\
&48000 = 2ab\ \ (b = 80)\\
&6400 = b^2\\
&\overline{144400} = \overline{a+b}^2 = \overline{380}^2 = A^2\\
&6840 = 2AB,\ (B = 9)\\
&81 = B^2\\
&\overline{151321} = \overline{A+B}^2 = \overline{380+9}^2 = \overline{389}^2
\end{aligned}$$

Evolution *of* 151321 *to its Square Root makes* 389.

Thus:

$$\begin{array}{rl}
 & \text{Square.}\quad \overbrace{\ a \qquad b\ }^{A} \qquad B \\
 & 151321\ (300 + 80 + 9 \\
a^2 = & 90000 \\
\text{Divisor. } 2\times \overset{a}{300}) & 61321 \text{ Resolvend.} \\
 & 48000 = 2ab \\
 & 6400 = b^2 \\
 & 54400 = 2ab + b^2 \\
\text{Divisor. } 2\times \overset{A}{380}) & 6921 \text{ Resolvend.} \\
 & 6840 = AB \\
 & 81 = B^2 \\
 & 6921 = 2AB + B^2 \\
 & 0000
\end{array}$$

These

These Operations are reverse of one another; and as to the Evolution, it differs from the Method of the Rule in this only, that the several Members of the Root are here written in their compleat Value, which occasions the writing down many Figures unnecessarily, which we avoid the other Way.

There remain yet two things to be demonstrated, which are delivered in *Schol.* 1. and 2.

1*st* The Remainder, after every Figure of the Root is found, cannot exceed the Sum of the Divisor and double the Quote: The *Reason* of this is contained in *Corol. Lemma* IV. where it's shewn, that what's over the greatest Square contained in any Number cannot exceed double the Root of the greatest Square; which is plainly the Sum of the Divisor at every Step, and double the Quote; for the Divisor is double of all the preceding Figures taken in their compleat Value, which therefore added to double the Quote, makes double all the Root. Thus, if any Root is $a+b$, the Divisor for finding b is $2a$; and when $2ab+b^2$ is taken out of the Resolvend, call the Remainder r; and in the *Coroll.* to *Lemma* IV. it's shewn that r cannot exceed $2a+2b$.

2. If the Divisor is contained oftner than 9 times in the Resolvend, after the second Step, or after the second Figure of the Root is found, the Figure sought is 9. For since the Resolvend contains the Divisor ($2a$) at least 10 times, it may be represented thus; $2a\times 9+2a+\text{R}$. Now taking 9 for the Quote, the Product according to the Rule is $2a\times 9+9\times 9$, which cannot exceed the Resolvend, because 9×9 cannot exceed $2a$, which in this Case exceeds 100, since there being two Figures found in the Root, and 0 prefix'd to them in order to form the Divisor $2a$, then is $2a$ a Number of at least three Places, which is therefore greater than $9\times 9=81$. Lastly, since it is demonstrated that in every Step, the Quote, under the Limitation of the Rule, is but one Figure; and 9, which is the greatest Number of one Figure, makes a Product not exceeding the Resolvend; therefore 9 is the Number sought.

In the second Step, if the Divisor is oftner than 9 times contained in the Resolvend, then it's plain, that if 81 is less than $2a$, [as it will certainly be when a is 5, 6, 7, 8, or 9, *i.e.* 50, 60, 70, 80, or 90, as they are taken in forming the Divisor; for then the Doubles, or $2a$, are 100, 120, 140, 160, 180.] then 9 is the Figure sought; because $2a\times 9+81$ must be less than $2a\times 9+2a+\text{R}$, the Resolvend, since 81 is less than $2a$. But if the first Figure is 1, 2, 3, or 4, that is, if $2a$ is 2×10, 2×20, 2×30, or 2×40, *i. e.* 20, 40, 60, or 80, which are less than 81, then the Figure sought will be less than 9, if $2a+\text{R}$ is less than 81; and it will be 9, if $2a+\text{R}$ is equal to or greater than 81: Because the Resolvend being $2a\times 9+2a+\text{R}$, if the Figure sought is made 9, then the thing to be subtracted from the Resolvend is $2a\times 9+81$, so that $2a+\text{R}$ must be at least equal to 81; and if it is not, we must therefore take a less Figure for the Quote, so that the Resolvend be at least equal to the Number to be subtracted.

Of the Proof *of the Square Root.*

As Extraction is opposite to the Raising of Powers, so the one is the Proof of the other: Thus; To prove the Square Root, multiply it by itself, and if the Product is equal to the given Square, or to the given Number after the Remainder of the Extraction is taken out of it, then the Extraction is right done.

But this may be also proved by casting out of 9's: Thus; Cast the 9's out of the given Number, if there is no Remainder in the Extraction; or out of the Difference of that Number and Remainder; then cast the 9's out of the Root found, and multiply the Excess (or what it wants of 9) by itself, and cast the 9's out of the Product; if the Excess, or what it wants of 9, is equal to the preceding Excess of 9's, the Extraction is right.

Example 1. The Square Root 256 is 16; proved thus; the Excess of 9's in 256 is 4. In 16 it is 7, and this multiplied by itself is 49, in which the Excess of 9's is also 4.

Example 2. The greatest Square Root contained in 230 is 15, and 6 remains: For $230 - 6 = 225$, in which there is 0 over 9's; then in 15 there is 6 over 9; and $6 \times 6 = 36$, in which there is also 0 over 9's.

The *Reason* of this Practice is evident from what is demonstrated of it in Multiplication.

Problem II. *To Extract the* Cube Root *of a* Whole Number.

Roots.		Cubes.
1	:	1
2	:	8
3	:	27
4	:	64
5	:	125
6	:	216
7	:	343
8	:	512
9	:	729

Rule I. Make a Table of simple Cubes, as in the Margin; then

II. Distribute the given Number into Periods of three Figures, beginning at the Right Hand: The Number of Periods shews the Number of Figures in the Root.

III. Begin at the last Period, which is the first Resolvend, and seek it or the next Cube Number less than it in the Table of simple Cubes, the Root of that is the last Figure (or that in the highest Place) of the Root sought; which being set down, subtract its Cube out of the last Period; to the Remainder prefix the next Period, and you have the next Resolvend.

IV. Consider the Figure found as in the Place of 10's, or with 0 prefix'd, and under that Value take the Triple of it, and also the Triple of its Square, making the Sum of these the Divisor; [which being composed of two Parts, it will be convenient to distinguish them by calling the Triple Square the first Part, and the other the second Part.] Then find how oft this Divisor is contained in the Resolvend last formed; which must never be taken above 9, (tho' it may be oftner contained) and then also it must be under this Limitation, *viz.* That having multiplied the first Part of the Divisor by the Quote, (now found) and the second Part by the Square of that Quote; and, lastly, to the Sum of these two Products adding the Cube of the Figure found; this Sum shall not exceed the Resolvend: which Sum being therefore subtracted out of the Resolvend, and the next Period prefix'd to the Remainder, you have the next Resolvend.

V. Take both the Figures of the Root already found, and considering them as so many 10's, *i. e.* place 0 before them, and under that Value take the Triple of that Number, and also the Triple of its Square; whose Sum is your next Divisor, distinguished into first and second Part, as before: Then find how oft this Divisor is contained in the Resolvend last formed, under the same Limitations as before; place the Figure found on the Right of these already found in the Root; and subtracting from the Resolvend, as formerly directed, to the Remainder prefix the next Period for a new Resolvend.

VI. Make a new Divisor from the Figures of the Root found in the same manner as in the preceding Steps, and divide, under the same Limitation; and thus proceed till all the Periods are taken in; finding at every Step a new Figure of the Root: which will in some Cases be 0, as when the Divisor is greater than the Resolvend, or when 1 added to the Divisor makes the Sum greater than the Resolvend; in which Case, after the 0 is set in the Root, prefix the next Period to the same Resolvend; and go on, forming a new Divisor by prefixing another 0 to the last Divisor.

SCHO-

SCHOLIUMS.

These *particular* OBSERVATIONS may be usefully added to this Rule (tho' it's a compleat *general* Rule by itself):

1. If you begin your Trials for the Quote at the greatest Number of Times (not exceeding 9) that the Divisor is contained in the Resolvend; then the Limitation of the Rule, for the Number to be compared with the Resolvend, is sufficient to determine when you have the true Figure. Yet it will be useful to *observe*, That if you should begin at a lesser Figure than the Remainder, tho' it may be greater than the Divisor, yet it must never exceed the Sum of these two Numbers, *viz.* Triple all the Figures of the Root already found, (taken as one Number) and triple its Square.

2. If the Divisor is contained in the Resolvend oftner than 9 times, at the second Step, or when you seek the second Figure of the Root; and if, at the same time, the first Figure is 8 or 9, then the Figure sought is certainly 9: But if the first Figure is less than 8, you must make Trials. Again; If in any Step after the second, the Divisor is oftner than 9 times contained in the Resolvend, the Figure sought is certainly 9.

3. If there is a Remainder after all the Periods are employ'd, the given Number is not a Cube; and the Root found is that of the greatest integral Cube contained in it. How to find a Mixt Root whose Cube shall be within any assigned Difference from the given Number, shall be taught in its proper Place.

EXAMPLES.

Example 1. The Cube Root of 614125 is 85.

Operation.

	$\dot{6}14\dot{1}25$	($a.b$ / 85
	$a^3 = 512$	
$3a^2 + 3a = 19440$ (wherein $a = 80$) ($a^2 = 6400$)	102125.	2d Resolvend.
	$96000 = 3a^2 \times b$	
	$6000 = 3a \times b^2$	
	$125 = b^3$	
	102125.	Sum.
	000000.	Remainder.

Explication for *Examp.* 1.

The given Number being pointed, the last Period is 614, and the next Cube to that is 512, whose Root is 8, which I call a, and subtracting $a^3 = 512$ from 614, the Remainder is 102, to which the next Period prefix'd makes 102125, the 2d Resolv. Then for a Divisor I take $a = 80$, and so $a^2 = 6400$, and $3a^2 = 19200$; then $3a = 240$, and $3a^2 + 3a = 19200 + 240 = 19440$, the Divisor, which is contained in the Resolvend, under the Limitation of the Rule, 5 times, the 2d Figure of the Root, which calling b, then $3a^2 \times b = 96000$, $3a \times b^2 = 6000$, $b^3 = 125$, and the Sum of these is 102125, equal to the Resolvend; so that the given Number is a true Cube, whose Root is 85.

Example

Example 2. The Cube Root of 41421736 is 346.

Operation.

$$
\begin{array}{rl}
 & 41421736 \;\left(\begin{array}{c} ab \\ 346 \\ \mathrm{A\,B} \end{array}\right. \\
a^3 = 27 & \\
3a^2+3a=2790 \\ (a=30) \\ (a^2=900) & 14421.\ \text{2d Reſolvend.} \\
 & 10800 = 3a^2 \times b \\
 & 1440 = 3a \times b^2 \\
 & 64 = b^3 \\
 & 12304\ \text{Sum.} \\
3\mathrm{A}^2+3\mathrm{A}=347820 \\ (\mathrm{A}=340) \\ (\mathrm{A}^2=115600) & 2117736.\ \text{3d Reſolvend.} \\
 & 2080800 = 3\mathrm{A}^2 \times \mathrm{B} \\
 & 36720 = 3\mathrm{A} \times \mathrm{B}^2 \\
 & 216 = \mathrm{B}^3 \\
 & 2117736\ \text{Sum.} \\
 & 0000000\ \text{Remainder.}
\end{array}
$$

Explication for *Exam.* 2.

The firſt Period is 41, and the next Cube is 27, whoſe Root is $3 = a$. The ſecond Reſolvend is 14421: Then for the Diviſor I take $a = 30$, whereby $3a^2 = 2700$, $3a = 90$, and $3a^2 + 3a = 2790$, the Diviſor, by which the ſecond Figure of the Root is 4; which is proved by the *Operation*: for $3a^2b + 3ab^2 + b^3 = 12304$, (as in the Work) which ſubtracted from the Reſolvend leaves 2117; which is a Number not exceeding three times $34 + 3 \times$ Square of 34, as *Schol.* 1. preſcribes; for $34 \times 34 = 115600$, whoſe Triple is 346800, and $3 \times 34 = 102$, then $346800 + 102 = 346902$. Or ſhould we have begun to gueſs for the Figure ſought, the greateſt Quote is 5, which is too great for the Limitation of the Rule, and 4 the next Number not being too great, muſt be the true Number. The third Reſolvend is 2117736; and to form the Diviſor, I take $\mathrm{A} = 340$, whence $\mathrm{A}^2 = 115600$; and the Diviſor is 347820, whence the laſt Figure of the Root is 6, as the Work ſhews.

Example 3. If we ſeek the Cube Root of 705919947284, we find it's not a perfect Cube, but the Root of the greateſt Cube contained in it is 8904; whoſe Cube is 705919947264, and the Remainder is 20, as you ſee in the Operation.

Operation.

$$
\begin{array}{rl}
8^3 = & 705919947284 \;(8904 \\
 & 512 \\
\text{Diviſor} = 3a^2+3a = 19440 \\ (a=80,\ a^2=6400) & 193919.\ \text{2d Reſolvend.} \\
 & 172800 = 3a^2 b \\
 & 19440 = 3ab^2 \\
 & 729 = b^3 \\
 & 192969\ \text{Sum.} \\
(a=890)\ 3a^2+3a=2378970 & 950947.\ \text{3d Reſolvend.} \\
a=8900)\ 3a^2+3a=237656700 & 950147284.\ \text{4th Reſolvend.} \\
 & 950520000 = 3a^2 b \\
 & 387200 = 3ab^2 \\
 & 64 = b^3 \\
 & 950907264.\ \text{Sum.} \\
 & 20.\ \text{Remainder.}
\end{array}
$$

Here the 3d Reſolvend being leſs than the Diviſor, I put 0 in the Root, and form a new Reſolvend by the next Period. The reſt of the Work is obvious. DE-

DEMONSTRATION *of the preceding Rule.*

LEMMA I.

If any three Numbers are multiplied into one another, the Product can have at moſt but as many Figures as are in all the three Factors, and at leaſt but two fewer. Example: $3\times4\times9=108$.

Demonſtr. This is a plain Conſequence of *Lem.* 1. for the Square Root: Becauſe the Product of two Factors cannot have more Places than are in both Factors, or but one fewer at leaſt; which Product being conſidered as one Factor, and multiplied by a third Factor, the ſame is true of this new Product; which makes the Truth propoſed manifeſt.

COROLL. The Cube of any Number can have at moſt but as many Figures as triple the number of Figures in the Root, and but two fewer at leaſt. Wherefore, again; Any Cube being diſtributed into Periods of three Places, the Number of Periods, and the Number of Figures in the Root muſt neceſſarily be equal; and the laſt Period may, in ſome Caſes, conſiſt only of one or two Figures.

LEMMA II.

If any Number A is not a Cube, yet being diſtributed into Periods, according to the preceding Rule, the greateſt Cube contained in it, as N^3, will have preciſely as many Periods as that Number A has.

Example: 35987 is not a Cube; but being pointed has two Periods, *viz.* as many as the greateſt Cube contained in it, 35937.

$A = \dot{3}59\dot{8}7$
$N^3 = 35937$

$B = 1000$
$B^{\frac{1}{3}} = 10$

Demonſtration 1. N^3 cannot have more Periods than A, for then it will have more Figures, and conſequently be a greater Number than A, contrary to Suppoſition.

2. Take 1 with as many o's before it as there are Figures ſtanding before the laſt Period of A; (*i.e.* on the right Hand of it) call the Number ariſing B: Then it's plain that is a Cube Number, whoſe Root is 1, with as many o's before it as $\frac{1}{3}$ Part of the Number of o's before the 1 in the Cube B; for to cube any Number expreſſed by 1 with o's, it's manifeſt from the Nature of Multiplication, that the Cube is 1 with three times as many o's; wherefore B is a Cube of as many Periods as A has; and being evidently contained in it, it follows, that the greateſt Cube contained in it cannot have fewer.

COROLL. The Root of the greateſt Cube contained in any Number A, which is not a perfect Cube, hath as many Figures as A hath Periods: For it hath as many as its own Cube hath Periods, (by *Coroll. Lemma* I.) which are as many as A has, by the preſent *Theorem*.

LEMMA III.

Any Number being diſtributed into Periods according to the Rule, the greateſt Cube contained in the laſt Period on the left, conſider'd as one Number by itſelf, is the Cube of the laſt Figure of the Root of the given Number, if it's a perfect Cube; or of the Root of the greateſt Cube contained in it, if it's not a Cube. Again; the greateſt Cube contained in the two laſt Periods, taken as one Number by themſelves, is the Cube of the laſt two Figures of the Root of the given Number, or of the greateſt Cube contained in it. And the ſame thing is true, comparing the 3 laſt or 4 laſt Periods, and ſo on, with the

Cube of the 3 or 4, *&c.* last Figures of the Root of the given Number, or of the greatest Cube contained in it.

Demonstr. Let A be any Number, and B the Cube Root thereof, or of the greatest Cube contained in it. Also let D be the last, or 2 last, or 3 last, *&c.* Periods of A; [as in the annex'd Example, take D = 41, or 41421] and let r represent the last, or two last, or three last, *&c.* Figures of the Root B, (as here 3 or 34) so that r^3 is the Cube of that last or two last, *&c.* Figures of the Root. These things being settled, the Truths to be proved are comprehended in one universal Case, thus; *viz.* r^3 is the greatest Cube contained in D; which I shall demonstrate in two Articles, thus:

$$A = 41421736 \text{ (D) } \qquad B = 346 \text{ (} r \text{)}$$

1. r^3 is contained in D: for since (by *Cor.* to *Lem.* 1, and 2.) there are as many Periods in A, as there are Figures in B; consequently there are, in every Case, as many Periods standing before D in the total A, as there are Figures before r in the total B; so that taking D and r in their compleat Value, as they stand in their Totals, there will be as many o's before D, as the Number of Figures of the Periods in A, which stand before, or on the right Hand of D; and as many o's before r, as the third Part of these before D. [For Example: If D = 41,000,000, then is r = 300. If D = 41421,000, then is r = 340.] Then r^3 will have as many o's before it, in its compleat Value, as triple the Number of o's before the Root r in its compleat Value, and consequently as many as are before D. We shall now express these Numbers in their compleat Values, thus, $r$oo, *&c.* r^3oooooo, *&c.* Dooo,ooo, *&c.*

Again; If r^3 is greater than D, then is r^3ooo,ooo, *&c.* greater than Dooo,ooo, *&c.* (being equally multiplied by the equal Number of o's prefix'd.) But A is equal to D, with as many certain Figures before it as there are o's before D, or r^3 in their compleat Values, *i. e.* Dooo,ooo, *&c.* or r^3ooo,ooo, *&c.* therefore r^3ooo,ooo, *&c.* is greater than A; [for any Figures whatever in the Places of the o's before D, cannot be equal to the Excess of r^3 above D, tho' that Excess were but 1]; *i. e.* the Cube of $r$oo, *&c.* which is but a Part of B, is greater than the Cube of B, because A is at least equal to B^3. But this is absurd; therefore r^3 cannot be greater than D, and consequently must be contained in it.

2. r^3 is the greatest Cube contained in D. For suppose N^3 a greater Number than r^3; then take N with as many o's before it, as are before r in its compleat Value, and express it thus, Noo, *&c.* so that its Cube is Nooo,ooo, *&c.* having triple as many o's as Noo, *&c.* the Root has; or as many as r^3ooo,ooo, *&c.* or Dooo,ooo, *&c.* has. Now because D contains N^3 (by Supposition) therefore Dooo,ooo, *&c.* contains N^3ooo,ooo, *&c.* Also, because N^3 is supposed greater than r^3, therefore N is greater than r, and Noo, *&c.* greater than $r$oo, *&c.* or than r with as many of any Figures before it; *i. e.* Noo, *&c.* is greater than B, (which is equal to r with as many certain other Figures before it, as there are o's before r in its compleat Value $r$oo, *&c.* or in Noo, *&c.*) so that Dooo,ooo, *&c.* a Part of A contains N^3ooo,ooo, *&c.* the Cube of Noo, *&c.* a Number which is greater than B, the Root of the greatest Cube contained in A, which is absurd; therefore r^3 is the greatest Cube contained in D.

Corol. If we find the Root of the greatest Cube contained in the last Period of any Number, we have the last Figure of the Root sought; and if we find the Root of the greatest Cube contained in the two last Periods, we have the two last Figures of the Root sought, and so on. Which so far explains the Investigation of the Rule; what remains to compleat it, you have in the following

LEMMA IV.

Part I. If the Root of any known Cube is supposed to consist of two Parts; then if one of these Parts is known, we can find the other by means of the Cube of a Binomial Root. Thus:

A + B

$\overline{A+B}^3 = A^3 + 3A^2B + 3AB^2 + B^3$; wherein it's evident, that if the Cube of the known Part A, *viz.* A^3, is ſubtracted from the Cube of the whole, the Remainder is $3A^2B + 3AB^2 + B^3$. Now ſince A is known, ſo alſo is $3A^2 + 3A$; and if we ſeek how oft this is contained in the preceding Remainder, under this Limitation, that the firſt Member of the Diviſor, $3A^2$, being multiplied by the Quote, and the ſecond Member 3A being multiplied by the Square of the Quote, and to theſe two Products the Cube of the Quote be added, the Sum ſhall be equal to the Dividend; then the Quote ſhall be equal to B the Number ſought; becauſe no other Number but B can anſwer to this Condition: For if you call the Quote D, then muſt $3A^2D + 3AD^2 + D^3$ be equal to $3A^2B + 3AB^2 + B^3$; which is manifeſtly impoſſible, unleſs D be $=$ B; ſince otherwiſe the reſpective Members of the one will be leſſer or greater than thoſe of the other, and conſequently the Wholes will not be equal.

Part 2. Tho' a Number is not a Cube, yet having one Member of the Root of the greateſt Cube contained in it, we can find the other by the ſame Method, as if it were a Cube. Which will eaſily appear. *Thus*:

$M = A^3 + 3A^2B + 3AB^2 + B^3 + R$
$M - A^3 = 3A^2B + 3AB^2 + B^3 + R = D.$

Let M be a Number, not a Cube, and A + B the Root of the greateſt Cube contained in it; which Cube is therefore $A^3 + 3A^2B + 3AB^2 + B^3$. Again; let R be the Number that's more than that Cube in M; ſo that $M = A^3 + 3A^2B + 3AB^2 + B^3 + R$. Now A being known, take A^3 from M, the Remainder is $3A^2B + 3AB^2 + B^3 + R$; which we may call D: And then if we find how oft $3A^2 + 3A$ is contained in D, under theſe Limitations, *viz.* that the Quote being multiplied into $3A^2$, and the Square of the Quote multiplied into 3A, and to theſe Products the Cube of the Quote be added, the Sum ſhall ſtill be leſs than D. [For *obſerve*, that whatever Number you chuſe for the Quote, it will make this Sum either greater or leſſer than D, and never equal; becauſe were it equal, then M would be a Cube, contrary to Suppoſition.] Then, I ſay, the Quote is equal to B, the other Member of the Root ſought. Becauſe, if it can be different, ſuppoſe it to be N; which is either leſſer or greater than D: But it cannot be leſſer; for then it would follow, that, contrary to Suppoſition, N is not the greateſt Number qualified according to the Rule, *viz.* ſo that $3A^2N + 3AN^2 + N^3$ is leſs than D; for B is greater than N, and yet is ſo qualified, ſince $D = 3A^2B + 3AB^2 + B^3 + R$: Wherefore N cannot be leſs than B; nor can it be greater, becauſe, by Suppoſition, $3A^2N + 3AN^2 + N^3$ is leſs than D $(= M - A^3)$; and adding A^3 to both, then $A^3 + 3AN^2 + 3A^2N + N^3$ $(= \overline{A+N}^3)$ is leſs than M, and therefore is contained in it. But again; A + N is greater than A + B, and $\overline{A+N}^3$ greater than $\overline{A+B}^3$, conſequently $\overline{A+B}^3$ is not the greateſt Cube contained in M, contrary to Suppoſition; ſo that N cannot be greater than B: Wherefore, laſtly, ſince N cannot be either leſſer or greater than B, it muſt be equal to it.

COROL. If a Number M is not a Cube, the Number R, which is over the greateſt Cube contained in it, (which is neceſſarily the Remainder after the ſecond Member B is found) can never exceed the Sum of triple the Root found, and triple its Square: For if the Root found is N, then if the Remainder exceed $3N^2 + 3N$, it muſt be at leaſt $3N^2 + 3N + 1$; which added to N^3 makes $N^3 + 3N^2 + 3N + 1 = \overline{N+1}^3$. And ſince this Cube is manifeſtly contained in M, (for it's the Sum of the greateſt Cube N^3 contained in M, and the Remainder $3N^2 + 3N + 1$ added); it follows, contrary to Suppoſition, that N is not the Root of the greateſt Cube contained in M, becauſe N + 1 is greater than N; and $\overline{N+1}^3$ is contained in M, if R is greater than $3N^2 + 3N$; therefore this cannot be.

Application *of the preceding* Lemma's *for demonstrating the Extracting of the* Cube Root.

1. The first and second *Lemma's* are already applied; from whence are deduced as *Corollaries*, the first thing asserted in the Rule, *viz.* That the Root must have as many Figures as the given Number has Periods.

2. From *Lem.* 3. we have the Reason why the given Number is pointed from the right Hand, *viz.* because, being done so, it is demonstrated, that the last Figure of the Root sought, (*i. e.* the Figure in the highest Place) the two last, &c. make the Root of the greatest Cube contained in the last, or two last, &c. Periods of the given Number.

3. The remaining Part of the Rule is to find the Figures of the Root, one after another, out of these Periods; the Reason of which is contained in *Lem.* 3 and 4, and its *Corol.* and is deduced thus:

We first take the last Period; and in the Table of simple Cubes, we seek that Number, or the next Letter, whose Root is, by *Lem.* 3. the highest Figure of the Root sought. So in the preceding *Example*, the given Number is 41421736, which we shall here call N. The last Period is 41, and the next Cube to this is 27, whose Root is 3, the last Figure of the Root sought. Now if we suppose the two last Periods 41421 to be the whole of the given Number, then the Root of the greatest Cube contained in it has but two Figures, whereof we have now found the last; and to find the other (which is the next Figure of the Root sought, by *Lem.* 3.) we proceed thus: Calling the Figure found a, we subtract its Cube $a^3 = 27$, from 41 the last Period; and to the Remainder 14 prefixing the next Period 421, the whole 14421 is the 2d Resolvend. And observe, that as the 41 is really 41000, in respect of the total 41421; so the Figure found is really 30, in respect of the next to be found; and in that Value we do actually take it by subtracting it from 41, considering where this stands, and which the prefixing the next Period to the Remainder does farther clear: For this is the same thing as if we had written 27000 the Cube of 30, and taken that from 41421; wherefore this is the same Operation as that explained in *Lem.* 4. *i. e.* having found 30 the first Member of the Root of 41421, we take its Cube out of the whole; and out of the Remainder 14421, we seek the next Member of the Root, which we know cannot exceed 9, because it's the second Member of a Root consisting of two Figures; whereof we have found that belonging to the highest Place, which consider'd in its compleat Value is the first Member. Now to find the Figure sought, we form a Divisor according to the Rule (demonstrated in *Lem* 4.) thus: Taking $30 = a$, the Divisor is $3a^2 + 3a = 3 \times 900 + 3 \times 30 = 2700 + 90 = 2790$. And this we find contained in the Resolvend 14421, 5 times; but under the Limitation of the Rule we can take it at most 4 times; and 4 is the Figure sought; which calling b, the Proof of its being the true Figure is this: We take $3a^2b + 3ab^2 + b^3 = 12304$, which is less than the Resolvend 14421; and 4 is therefore the right Figure, because 5 would have made $3a^2b + 3ab^2 + b^3$ greater than 14421. Or had we at a guess taken $3 = b$, then would it be $3a^2b + 3ab^2 + b^3 = 8937$; which taken from 14421 leaves 5484; which is greater than $3 \times \overline{33}^2 + 3 \times 33 = 3 \times 1089 + 3 \times 33 = 3267 + 99 = 3366$; and therefore 3 is too little, as is shewn in *Schol.* 1. added to the Rule. Thus we have found 34, the Root of the greatest Cube contained in 41421, (the Remainder, or what is over, being 2117) and have shewn that the Rule is just and good for a Root of two Figures. Again; For a Number of three Periods, as 41421736, whose Root has three Figures; having found the two Figures in the highest Places; and taking these with o prefix'd, which makes the true Value; and calling this again a, or A, the first Member of the Root, the second Member, which is a single Figure, is found the same way as before explained; which is according to the Rule. But now in this there is so much of the Work already done; for the Cube of this

this first Member, or A^3, is already subtracted from the Total 41421736, because A is now equal to the former $a+b$; and it's evident from the Work that we have subtracted a^3, and then $3a^2b+3ab^2+b^3$ to make the Cube of $\overline{a+b}$. It's true, we have taken $a=30$ and $b=4$; whereas $A=380$, so that a should be 300 and $b=80$: But by the Places in which we have set a^3, and $3a^2b+3ab^2+b^3$, we have in effect taken them, as if it had been $a=300$ and $b=80$; and so we have duly subtracted A^3, or the Cube of 380, from the Total 41421736; the Remainder whereof is 2117, to which the next Period 736 is prefix'd, making 2117736 the Resolvend for finding the next Figure; which we find to be 6, by the same Rule and Reason as we found the last Figure.

If there are more than three Figures in any Root, the Reasons of the Rule are manifestly the same from one Step to another *in infinitum*. I shall add for an Illustration one Example, wherein each Figure of the Root is taken in its compleat Value.

Involution *of* 346 *to its Cube, makes* 41421736

Thus:

Root

$346=300+40+6$

$27000000=a^3.\ (a=300.)$
$10800000=3a^2b.\ (b=40.)$
$1440000=3b^2a.$
$64000=b^3.$

$39304000=\overline{a+b}^3=A^3.$
$2080800=3A^2B.\ (A=340.\ B=6.)$
$36720=3AB^2.$
$216=B^3.$

$41421736=\overline{A+B}^3.$

Evolution *of* 41421736 *to its Cube Root, makes* 346.

Thus:

$$41421736\left(\overbrace{300+40}^{A}+\overbrace{6}^{B}\right.$$

(with $a = 300$, $b = 40$)

$a^3=27000000$

$3a^2+3a=270900)\ 14421736$ Resolvend.
$10800000=3a^2b.$
$1440000=3ab^2.$
$64000=b^3.$
12304000 Sum.

$3A^2+3A=347820)\ 2117736$ Resolvend.
$2080800=3A^2B.$
$36720=3AB^2.$
$216=B^3.$
2117736 Sum.
000000

There remain yet two things to be demonstrated, which are deliver'd in *Schol.* 1 and 2. *viz.*

1. The Remainder can never exceed the Sum of these two Numbers, *viz.* triple all the Figures of the Root already found (taken as one Number) and triple the Square of the same; the Reason of which you have plainly in *Cor. Lem.* 4.

2. If the Divisor $3a^2+3a$ is contained oftner than 9 times in the Resolvend, then if it's so after the second Figure is found, 9 is the Figure sought; for in this Case the Resolvend may be thus represented, $3a^2\times10+3a\times10+R=3a^2 9+3a^2+3a\times10+R$, and the Sum upon which the Limitation depends, being $3a^2b+3ab^2+b^3$, if b is 9, then this Sum is $3a^2\times9+3a\times81+729$: Compare this with the Resolvend, they have this Part in common, *viz.* $3a^2\times9$: Set this aside, and compare the Remainders in both, *viz.* $3a^2+3a\times10+R$, and $3a\times81+729$; this last is less than the former; for, after two Figures of the Root are found, $\bar{a}$ consists of three Figures, in its compleat Value, and so must be at least 100: Therefore $3a\times81$ is less than $3a^2$, and 729 is less than $3a\times10$, which is at least $300\times10=3000$. Hence it is plain, that the Resolvend is greater than the Number to be compared with it in the Limitation of the Quote; and the greater that

$\bar{a}$ is,

$\bar{a}$ is, as it will be always greater at every Step after the second, so much will the Resolvend exceed that other Number.

Again; In the second Step, the Quote is certainly 9, if the first Figure found is either 8 or 9; *i. e.* if A = 80 or 90; which you'll find by comparing as before $3\,A^2 + 3\,A \times 10 + R$ with $3\,a \times 81 + 729$; for putting $a = 80$ or 90, you'll find $3\,A \times 81 + 729$ less than $3\,A^2 + 3\,A \times 10$: But when A is supposed 70, or 60, *&c.* it will be greater, and therefore the Quote must be less than 9, unless the Number R, which belongs to the Resolvend, is greater than the Excess of $3\,A \times 81 + 729$ above $3\,A^2 + 3\,A \times 10$, as in some Cases it will, and in some it will not.

SCHOLIUM, *concerning a different Method of Practice in the Extraction of a* Cube Root.

The preceding Rule is nearly according to the most common Method, that it might be accommodated to the Principles from which the Reason and Demonstration of it might be most easily deduced: But there is another Method, differing a little in one of the principal Steps, which is this:

Having pointed the given Number, and found the first Figure of the Root; then in all the succeeding Steps form the Divisor as before, and find the Quote under this Limitation, *viz.* That being added to the Divisor, and the Sum multiplied by the same, the Product shall be less than the Resolvend; which is so far like what we do for the Square Root: But, again; the Remainder must not exceed the Product of these Numbers, *viz.* the Sum of the Quote and the second Member of the Divisor multiplied into the Difference betwixt the Quote and its Square; *i. e.* add together these two Products, and their Sum must not exceed the Resolvend, and what remains here belongs to the next Resolvend.

You may also form your Divisor thus; Take the Figures already found, and to them prefix 1 (or take them with 0 prefix'd, and then add 1, which will fall in the Place of the 0); multiply this Sum by triple the Number to which the 1 was added: The Product is the Divisor. See this *Example* wrought after this Mannner.

$$614125\ \left(\begin{matrix} a.\,b \\ 8\ 5 \end{matrix}\right.$$
$$512 = a^3$$

Divisor.
$$\left.\begin{matrix} 3\,a^2 + 3\,a = 19440 \\ (= 3\,a \times \underline{a+1} = \\ 3 \times 80 \times 81) \end{matrix}\right)\ 102125.\ \text{2d Resolvend.}$$

$$97225 = \underline{3\,a^2 + 3\,a + b} \times b$$
$$4900 = \underline{3\,a + b} \times \underline{b^2 - b}$$

102125. Sum.

000000

The Letters and Operations shew the Application of this Method; and what is to be demonstrated is only this, that the Number compared to the Resolvend is equal to $3\,a^2 b + 3\,a\,b^2 + b^3$, which is the Number compared in the former Rule; and the Truth of this you'll find by performing the Operation of these two Products, and adding them thus, $\underline{3\,a^2 + 3\,a + b} \times b = 3\,a^2 b + 3\,a\,b + b^2$, then $\underline{3\,a + b} \times \underline{b^2 - b} = 3\,a\,b^2 + b^3 - 3\,ab - b^2$: which added to the former makes $3\,a^2 b + 3\,a\,b^2 + b^3$.

What I have further to observe is, That this Method will in many Cases be of Advantage, by helping us to discover more easily that some Figures are too great for the Quote, without the Trouble of making out the total Number, which is here to be compared with the Resolvend: For if the first Part of it (*viz.* the Product of the Quote by the Sum of the Divisor and Quote, or $\underline{3\,a^2 + 3\,a + b} \times b$) is equal to the Resolvend, or greater,

greater, that Figure is certainly too big to anſwer the Rule. But tho' that firſt Part is leſs than the Reſolvend, we cannot conclude that we have the true Figure, till we add the other Part alſo, and find that the Sum is not greater than the Reſolvend.

Obſerve alſo, That if the Product of the Quote b, and firſt Member of the Diviſor, *viz.* $3a^2$, is equal to the Reſolvend or greater, then certainly that Quote is too big, and ſo we might have the ſame kind of Advantage by the common Method; yet the Product of the Quote into the Sum of the Quote and Diviſor, being always a greater Number than the Product of the Quote and firſt Member of the Diviſor, the laſt Method will diſcover ſome Figures to be too great, which would not appear ſo without Trial by the other Method.

Of the Proof *of the Cube Root.*

Involve the Root found to the Cube, and compare it with the given Cube, or the Difference betwixt the given Number and the Remainder of the Extraction.

Or, By caſting out 9's thus: Caſt the 9's out of the given Number, if there is no Remainder in the Extraction; or out of the Difference of that Number and the Remainder of the Extraction: Then caſt the 9's out of the Root found, and ſquare the Exceſs, out of which caſt the 9's, and multiply this Exceſs by the preceding, and out of this Product caſt the 9's; the Exceſs or Defect of 9 muſt be equal to that found in the given Number.

Example: The Cube Root of 2744 is 14; thus proved: The Exceſs of 9's in 2744 is 8. in 14 it is 5; then $5\times5=25$, in which the Exceſs of 9's is 7, which, multiplied by the preceding Exceſs 5, the Product is 35, in which the Exceſs of 9's is 8.

$14\times14\times14=2744$

The Reaſon of this Practice is alſo obvious from what is ſhewn in Multiplication: For taking 14×14 as one Factor, and 14 as another, we firſt caſt out 9's out of 14×14, and then out of 14; and, multiplying theſe two Exceſſes together, we compare the Exceſs of 9's in the Product with that in 2744, which is $=\overline{14\times14}\times14$.

Problem III. *To Extract the* Root *of any Power above the* Cube.

General RULE.

WHatever Root is propoſed to be extracted, as in general the n Root, diſtribute the given Number into Periods, taking as many Figures to each Period as the Number of Units in the Index n; then make a Table of the ſimilar Powers (*i. e.* the n Powers) of all the Digits, as far at leaſt till you find one which is equal to, or exceeds the firſt Period on the Left of the given Number, taken by itſelf; the Root of that Power is the firſt Figure on the Left of the Root ſought, which call A; then ſubtract A^n from the ſaid firſt Period, and to the Remainder prefix the next Period for a Reſolvend; and to find the next Figure of the Root, form a Diviſor thus: take a Binomial $A+B$, and involve it to the n Power, as has been explained; your Diviſor is the Sum of all the Products of the ſeveral Powers of A, except the higheſt A^n, multiplied by the proper Coefficients of the Terms in which they ſtand in the Power $\overline{A+B}^n$: Thus, for the 4th Root the Diviſor is $4A^3+6A^2+4A$; for the 5th Root it is $5A^4+10A^3+10A^2+5A$, as you'll find from the *Table of Binomial Powers and Coefficients.* And *univerſally*, the Diviſor will be $nA^{n-1}+a\times A^{n-2}+b\times A^{n-3}+\&c.+n\times A$, where I have ſimply expreſſed the Coefficients by ſingle Letters, which you muſt underſtand as repreſenting the true Coefficients. Alſo remember, that the firſt Figure of the Root found, which A repreſents, muſt be mul-

tiplied

tiplied by 10, or 0 prefix'd to it, because that is its true Value with respect to the next Figure to be found; and in this Value you are to use it in forming the Divisor; then find the Quote B, (which can never exceed 9) limited so that the several Members of the Divisor being multiplied by the several Powers of B, which you find multiplied into them in the several Terms of the Binomial Power $\overline{A+B}^n$; and B^n added to the Sum of all these Products, the total shall not exceed the Resolvend. Thus in the 4th Power the Number to be compared with the Resolvend is this, $4A^3B+6A^2B^2+4AB^3+B^3$; in the 5th Power it is $5A^4B+10A^3B^2+10A^2B^3+5AB^4+B^5$. Universally, it is $nA^{n-1}\times B+aA^{n-2}\times B^2+bA^{n-3}\times B^3+$ &c. $+nAB^{n-1}+B^n$, which is the whole Binomial Power except the Term A^n.

In the next place take the two Figures found, and prefixing 0, call this Number A, and form a new Divisor as before, of the several Powers of this new Number A, multiplied by their Coefficients in $\overline{A+B}^n$, and by this find the 3d Figure of the Root, which call again B, under the same Limitation as before; and so proceed to the End.

SCHOLIUMS.

1. If you begin your Guess for the Quote (*i.e.* for any Figure of the Root after the first) at the greatest number of times (not exceeding 9) that the Divisor is contained in the Resolvend, then the Limitation of the Rule for the Number to be compared with the Resolvend is sufficient to determine when you have the true Figure. But if you chuse at a Guess, then you are to mind this Mark of a Figure too little, *viz.* That if you take all the Root found, taking in the Figure now put in the Root, and call it A; then take the Sum of the Products of the several Powers of it (except A^n) which belong to the new Divisor; the Remainder must not exceed this, else the Figure last found is too little.

2 If there is a Remainder after all the Periods are employed, the given Number is not a Power of that Order, and the Root found is only that of the greatest Power contained in it.

Roots.	5th Powers.
1 :	1
2 :	32
3 :	243
4 :	1024

I shall illustrate this Rule as far as is necessary by an Example. Suppose the 5th Root of this Number 74560898 is required. Having pointed it, the last Period is 745; and raising the 5th Powers of the Numbers from 1 to 4, whose 5th Power is the first which exceeds 745, I find 243 the greatest 5th Power contained in 745; and the 5th Root of this being 3, I put 3 as the last Figure of the Root sought.

$$\begin{array}{rl} & 74560898 \quad \left(\begin{matrix}A\,B\\3\,7\end{matrix}\right. \\ A^5 = & 243 \\ \hline 4329150) & 50260898.\ \text{2d Resolvend.} \\ \hline & 28350000 = 5A^4\times B \quad \left(\begin{matrix}A=30\\B=7\end{matrix}\right) \\ & 13230000 = 10A^3\times B^2 \\ & 3087000 = 10A^2\times B^3 \\ & 360150 = 5A\times B^4 \\ & 16807 = B^5 \\ \hline & 45043957.\ \text{Sum.} \\ \hline & 5216941.\ \text{Remainder.} \end{array}$$

Then raising $\overline{A+B}^5$, it is $A^5+5A^4B+10A^3B^2+10A^2B^3+5AB^4+B^5$; and taking $A=30$, the Divisor is $5A^4+10A^3+10A^2+5A=4329150$, which is found in the Resolvend, under the Limitation of the Rule 7 times; the Remainder being 5216941.

The

The Divisor being formed thus:

$$\begin{aligned} 5\,A^4 &= 4050000 \\ 10\,A^3 &= 270000 \\ 10\,A^2 &= 9000 \\ 5\,A &= 150 \\ \hline & \ 4329150 \end{aligned}$$

DEMONSTR. The Demonstration of this general Rule depends upon the same kind of Principles as those for the Square and Cube: And whoever understands these thoroughly will be able to extend them to this universal Rule with great Ease: For if we put n in the Place of 2 or 3 in the preceding *Lemma's*, they will become universal for all Cases.

SCHOLIUM. What a tedious thing it is to form the Divisors, and the Numbers to be compared with the Resolvend in high Powers, and indeed in all above the Cube, it's easy to perceive. All that can be said in favour of this general Rule is only this, That it is exceedingly preferable to our being left to a pure blind Guess, with no other Help than raising the Power of the Root guessed, and comparing it with the proposed Number. Yet the great Labour of this Rule has excited the Mathematicians to the Invention of other Methods; the explaining of which comes not within the Limits I have prescribed my self in this Work, except that Method which is by the help of *Logarithms*, as you'll find afterwards explained. In the mean time *observe*, that as Square and Cube Roots are the things only useful in the common Affairs of Life, so the Rules for them are tolerably easy, especially the Square. But there is also

Another General RULE *for* Compound *Roots* (i. e. *whose Index is the Product of two or more Numbers*).

Take any two or more Indexes whose Product is the given *Index*, and extract out of the given Number a Root answering to any of these lesser Indexes; and then out of this Root extract a Root answering to another of these lesser Indexes, and so on, till you go thro' them all: The last Root found is the Root sought.

Example 1. To find the 4th Root of 625, I find the Square Root 25; then the Square Root of this, which is 5, is the Root sought.

Example 2. To find the 6th Root of 4096: It is 4; which I find thus: $6=2\times 3$, therefore I find the Square Root of 4096, which is 64, and then the 3d Root of 64 is 4.

Demonstr. The *Reason* of this Rule is obvious, being only the Reverse of what's done and demonstrated for involving a Number to a compound Power; or you have the Reason of it in *Theor*. IX. §. 1. where it's shewn that $A^{\frac{1}{nm}} = A^{\frac{1}{n}\frac{1}{m}}$.

Observe, It's best to begin with the Root of the lowest Index.

Also, If the given Number is not a Power of the Order you first try, neither can it be a Power of the Order proposed; and to find the Root of the greatest like Power contained in it, other Methods do better.

Of the Proof *of all Roots of Integers universally.*

It is done either by the opposite Involution, or by casting out the 9's, thus:

Cast the 9's out of the given Number, or the Difference of it and the Remainder of the Extraction, and mark the Excess: Then cast the 9's out of the Root (and take the Excess, or the Root itself if less than 9); multiply it by itself, and cast out the 9's from the Product; then multiply the Excess by the Excess in the Root, and cast the 9's out of the Product; this last Excess multiply by the Excess in the Root, and cast the 9's out of the Product, and go on so till the Excess of 9's in the Root is employ'd as a Multiplier, as oft

oft as the Index of the Power expresses: The last Excess must be equal to that in the given Number.

§. II. Part II.

Probl. 4. *Of the* Extraction *of the Roots of* Fractions.

A Fractional Power is to be considered in two different Views: 1. As being an immediate Power, *i. e.* the immediate Effect of the continual Multiplication of some Fraction into itself, as $\frac{4}{9}=\frac{2}{3}\times\frac{2}{3}$; and $\frac{8}{27}=\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}$. Or, 2. As being only equivalent to some immediate Power, but not itself such a one, as $\frac{8}{18}=\frac{4}{9}$.

Now if a Fraction is immediately a Power, it's manifest from the Definitions, that if we extract the Root proposed from the Numerator and Denominator separately, these are the Numerator and Denominator of the fractional Root sought. *Example*: The Square Root of $\frac{64}{81}$ is $\frac{8}{9}$; for $8=\overline{64}|^{\frac{1}{2}}$, and $9=\overline{81}|^{\frac{1}{2}}$. But if the given Fraction is only equivalent to some immediate Power, the Root (*viz.* of that Power; which is also in another Sense, the Root of the given Fraction) cannot be discovered by this Method; for the Numerator and Denominator have not both in this Case, and perhaps neither of them has a perfect Root; and so we cannot determine by this Method, whether the Root sought is rational or surd: yet by other Methods we can discover this, and find the Root where there is one. For which take this

General Rule.

Reduce the given Fraction to its lowest Terms, and then extract the proposed Root from Numerator and Denominator separately; and these Roots are the Numerator and Denominator of the Fractional Root sought; which is also in its lowest Terms. But if both Numerator and Denominator have not such a perfect Root, the given Number is not a Power of the Order proposed, either immediately or equivalently.

Example 1. To find the Square Root of $\frac{27}{75}$, I find its least Terms $\frac{9}{25}$, whose immediate Root is $\frac{3}{5}$.

Example 2. To find the Square Root of $\frac{24}{69}$, I find its least Terms $\frac{8}{23}$. But neither 8 nor 23 are Squares, and therefore $\frac{24}{69}$ is not a Square in any Sense.

Demon. 1. If the lowest (or any) Terms of the given Fraction are Powers of the Order proposed, it's plain that their Roots make a Fraction, which is the Root of the given Fraction; by the Definition. And,

2. If the least Terms of a Fraction are not Powers of the given Order, no Terms of it are so; or the given Fraction is not a Power in any Sense. For let $\frac{A}{B}$ be a Fraction in its least Terms, and suppose $\frac{M^n}{N^n}=\frac{A}{B}$ (*i. e.* some other equivalent Terms of the Fraction to be an immediate Power.) Then because $\frac{A}{B}$ is in least Terms, $\frac{M^n}{N^n}$ is not so, because it consists of different Terms by Supposition. Consequently $\frac{M}{N}$, its n Root, is not

 in

in its leaſt Terms, (by *Lem.* preceding the *Theor.* Chap. 1.) Take $\frac{r}{s}$ in its leaſt Terms, and $=\frac{M}{N}$; then is $\frac{r^n}{s^n}$ in its leaſt Terms, (by the ſame *Lem.*) And ſince $\frac{r}{s}=\frac{M}{N}$, therefore $\frac{r^n}{s^n}=\frac{M^n}{S^n}=\frac{A}{B}$; wherefore $\frac{r^n}{s^n}$ and $\frac{A}{B}$ are both in the leaſt Terms, which is abſurd; or $\frac{r^n}{s^n}$, $\frac{A}{B}$ are the ſame Terms, which is alſo contrary to Suppoſition.

SCHOLIUMS.

1. A Fraction made of the greateſt Integral Root of the Numerator and Denominator may in one Senſe be called the Root of the greateſt Fractional Power contained in the given Fraction; which Root will, in ſome Caſes, be a deficient, and in ſome an exceſſive Root, *i.e.* whoſe Power wants of, or exceeds the given Fraction. *Example*: $\frac{5}{13}$, the greateſt Square Fraction contained in it, in this Senſe, is $\frac{4}{9}$, whoſe Root is $\frac{2}{3}$; which is an exceſſive Root to $\frac{5}{13}$, becauſe $\frac{4}{9}$ is a greater Fraction than $\frac{5}{13}$. But in $\frac{8}{15}$ the greateſt Square is $\frac{1}{9}$, which is leſs than $\frac{3}{15}$; therefore its Root $\frac{1}{3}$ is a deficient Root to $\frac{3}{15}$.

Again: In another Senſe, *i. e.* if we ask what is the greateſt Fraction which is an immediate Power, and is leſs than a given Fraction which is not a Power in any Senſe, then there is no ſuch thing as a greateſt; the Reaſon of which you'll find afterwards. (See *Corol. Prob.* 5.)

2. The preceding General Rule requires two Extractions, *viz.* both from the Numerator and Denominator; but I ſhall give you other particular Rules, whereby the Root is found by one Extraction; and ſuch as are accommodated to the Methods of Approximation, afterwards explain'd.

Particular RULES *for the Roots of Fractions.*

1. *For the Square Root.*

Multiply the Numerator and Denominator together, and extract the Square Root of the Product; which is always a compleat Square, if the given Fraction is ſo in any Senſe. Make this Root the Numerator to the given Denominator, and this Fraction is the Root ſought; or ſet the given Numerator fractionally over the Root found; and this alſo is the Root ſought, tho' neither of them is in the leaſt Terms. But if the Product is not a compleat Square, neither is the given Fraction: And having found the Root of the greateſt Integral Square contained in it, uſe that as directed; and you ſhall have a Root wanting of a juſt Root to the given Fraction, if the Root extracted is made Numerator; but exceeding, if it's made Denominator.

Example 1. To find the Square Root of $\frac{4}{9}$, I take $4\times 9=36$, whoſe Root is 6; and ſo the Root ſought is $\frac{6}{9}=\frac{2}{3}$, or $\frac{4}{6}=\frac{2}{3}$; for $\frac{2}{3}\times\frac{2}{3}=\frac{4}{6}$.

Example 2. To find the Square Root of $\frac{12}{147}$, I take $12\times 147=1764$, whoſe Root is 42; and ſo $\frac{42}{147}=\frac{12}{42}=\frac{2}{7}$ is the Root ſought: for $\frac{2}{7}\times\frac{2}{7}=\frac{4}{49}=\frac{12}{147}$.

Example 3. For the Square Root of $\frac{7}{12}$, I take $7\times 12=84$, which is not a Square; therefore $\frac{7}{12}$ is not ſo: But the greateſt Integral Root in 84 is 9, therefore $\frac{9}{12}=\frac{3}{4}$ is a Root wanting of a true Root to $\frac{7}{12}$; for $\frac{3}{4}\times\frac{3}{4}=\frac{9}{16}$, which is leſs than $\frac{7}{12}$, becauſe 9×12 is leſs than 7×16. And $\frac{7}{9}$ is an exceſſive Root; for $\frac{7}{9}\times\frac{7}{9}=\frac{49}{81}$ greater than $\frac{7}{12}$, becauſe 49×12 is greater than 7×81.

SCHOLIUM. Tho' the Fraction given be in its leaſt Terms, yet the Root found by this Method will not be in its leaſt Terms in every Caſe where the given Fraction is not a perfect Power, as the preceding *Exam.* 3. ſhews. And if it is a perfect Power, the Root found will never be in its leaſt Terms, as is manifeſt; becauſe the leaſt Terms are the Root of the Numerator and Denominator of the leaſt Terms of the given Fraction.

DEMON. 1. The given Fraction being $\frac{A}{B}$, multiply both Terms by B, or by A; and then $\frac{A}{B}=\frac{AB}{BB}=\frac{AA}{AB}$. Suppoſe AB is a compleat Square, whoſe Root is *a*, ſo that $AB=aa$; then are all theſe Expreſſions equal, *viz.* $\frac{aa}{BB}=\frac{AB}{BB}=\frac{A}{B}=\frac{AA}{AB}=\frac{AA}{aa}$; conſequently $\frac{a}{B}$ the Square Root of $\frac{aa}{BB}$, and $\frac{A}{a}$ the Square Root of $\frac{AA}{aa}$ are each a true Root to $\frac{A}{B}$.

2. If AB is not a perfect Square, then neither is $\frac{A}{B}$ in any Senſe. For ſuppoſe $\frac{M^2}{N^2}=\frac{A}{B}=\frac{AB}{BB}$, then, by equal Multiplication, according to the Nature of Fractions, it is $\frac{M^2\times B^2}{N^2}=AB$. But $M^2\times B^2=\overline{MB}^2$, therefore $\frac{M^2\times B^2}{N^2}=\frac{\overline{MB}^2}{N^2}$; which is plainly an immediate Square, whoſe Root is $\frac{MB}{N}$, and is therefore a true Root to AB, which is abſurd; for AB is ſuppoſed not to have a true Square Root in Integers, and conſequently has no ſuch Root true, (*Theor.* 19. *Chap.* 1.) therefore $\frac{MB}{N}$ is not the Square Root of AB; Nor is $\frac{M^2\times B^2}{N^2}$ equal to AB; nor $\frac{M^2}{N^2}$ equal to $\frac{AB}{BB}\left(=\frac{A}{B}\right)$ as was ſuppoſed, *i.e.* no immediate Square $\frac{M^2}{N^2}$ can be equal to $\frac{A}{B}$, or $\frac{A}{B}$ is not a Square in any Senſe.

3. Suppoſe *a* the greateſt Integral Root of AB, ſo that *aa* is leſs than AB; then is $\frac{aa}{BB}$ leſs than $\frac{AB}{BB}\left(\frac{A}{B}\right)$ *i.e.* $\frac{a}{B}$ is a deficient Root to $\frac{A}{B}$. Alſo ſince *aa* is leſs than AB, therefore $\frac{AA}{aa}$ is greater than $\frac{AA}{AB}\left(=\frac{AB}{BB}=\frac{A}{B}\right)$ *i. e.* $\frac{A}{a}$ is an exceſſive Root to $\frac{A}{B}$.

2. *For the Cube Root.*

Multiply the Numerator by the Square of the Denominator; the Product will be a compleat Cube, if the given Fraction is ſo in any Senſe: the Cube Root of this Product ſet fractionally over the Denominator of the given Fraction, is the fractional Root ſought.

ſought. Or *thus:* Multiply the Denominator by the Square of the Numerator, and over the Cube Root of the Product ſet the given Numerator fractionally; and this alſo is the Root ſought. And *obſerve,* that if the given Fraction is a proper one, the laſt is the beſt Method; but if it is improper, take the firſt Method. But if the Product mention'd is not a compleat Cube, neither is the given Fraction. And if we take the greateſt Integral Root of that Product, and uſe it as directed, we have a deficient or exceſſive Root, according as we chuſe the firſt or ſecond Method.

Example 1. To find the Cube Root of $\frac{8}{27}$, take $27 \times 27 = 729$, then $729 \times 8 = 5832$, whoſe Cube Root is 18; then is $\frac{18}{27} = \frac{2}{3}$ the Root ſought. Or thus; $8 \times 8 = 64$, then $64 \times 27 = 1728$, whoſe Cube Root is 12; therefore $\frac{8}{12} = \frac{2}{3}$ is alſo the Root ſought.

Example 2. For $\frac{15}{54}$, take $54 \times 54 = 2916$, then $2916 \times 15 = 43740$, which has not a Cube Root, and ſo $\frac{15}{54}$ is not a Cube; but the greateſt integral Cube Root of 43740 being 35, therefore $\frac{35}{54}$ is a deficient Root to $\frac{15}{54}$; or if we take $15 \times 15 = 225$, then $225 \times 54 = 12150$, whoſe neareſt Cube Root is 22, and $\frac{15}{22}$ is an Exceſſive Root.

DEMON. 1. For the firſt Method, multiply each Term of the Fraction $\frac{A}{B}$ by B^2, and it is $\frac{AB^2}{B^3} = \frac{A}{B}$; and if AB^2 is a Cube, let its Root be m, then $\frac{m}{B}$ is the Cube Root of $\frac{AB^2}{B} = \frac{A}{B}$. Again; if AB^2 is not a Cube, neither is $\frac{A}{B}$; for if we ſuppoſe $\frac{m^3}{n^3} = \frac{AB^2}{B^3}$, then is $AB^2 = \frac{m^3 \times B^3}{n^3} = \frac{\overline{mB}^3}{n^3}$, whoſe Cube Root is $\frac{mB}{n}$; *i. e.* AB^2 is a Cube, which is contrary to Suppoſition, if $\frac{mB}{n}$ is Integer; and if it's not Integer, it cannot be the Cube Root of AB^2, which has no Cube Root in Integers (*Theor.* XIX. *Ch.* 1.). But if we ſuppoſe m the greateſt integral Cube Root contained in AB^2, ſo that m^3 is leſs than AB^2, it's plain that $\frac{m^3}{B^3}$ is leſs than $\frac{AB^2}{B^3} = \frac{A}{B}$, or $\frac{m}{B}$ is a deficient Root.

2. For the ſecond Method, multiply each Term of $\frac{A}{B}$ by A^2, and it is $\frac{A^3}{A^2B} = \frac{A}{B}$, ſo that A^2B being a true Cube, whoſe Root is n, then $\frac{A}{n}$ is the Root of $\frac{A^3}{A^2B} = \frac{A}{B}$; but if A^2B is not a Cube, neither is $\frac{A}{B}$; for ſuppoſe $\frac{m^3}{n^3} = \frac{A^3}{A^2B}$, then is $\frac{m^3}{n^3} \times A^2B = A^3$, and $A^2B = A^3 \div \frac{m^3}{n^3} = \frac{A^3 \times n^3}{m^3} = \frac{\overline{An}^3}{m^3}$, whoſe Cube Root is $\frac{An}{m}$; *i. e.* A^2B is a Cube, contrary to Suppoſition: But if we take m the greateſt integral Cube Root contained in A^2B, ſo that m^3 is leſs than A^2B, then is $\frac{A^3}{m^3}$ greater than $\frac{A^3}{A^2B}$, and conſequently $\frac{A}{m}$ is an exceſſive Root to $\frac{A}{B}$.

SCHOLIUM. Both theſe, and the Extraction of all higher Roots, may be comprehended in one general *Rule,* thus:

General RULE *for all the Roots of* Fractions, *after the Manner of the preceding Particular* Rules.

Raiſe the Denominator of the given Fraction to a Power, whoſe Index is 1 leſs than that of the Root to be extracted, and multiply this Power by the given Numerator: Extract the propoſed Root of this Product (which is a compleat Power, if the given Fraction is ſo), and ſet it fractionally over the given Denominator, and this makes the Root ſought. But if that Product has not ſuch a Root, neither has the given Fraction; and taking the greateſt integral Root contained in that Product, it makes, with the given Denominator, a deficient Root to the given Fraction.

Or alſo thus: Raiſe the Numerator to the Power directed, and by that multiply the Denominator; extract the propoſed Root of the Product if it has one, or take the greateſt integral Root contained in it; over this Root ſet the given Numerator, and it makes an exceſſive Root to the given Fraction.

Example. To find the 4th Root of $\frac{A}{B}$, I multiply B^3 by A, and ſet the 4th Root of the Product AB^3, or the greateſt integral 4th Power contained in it, over B, and it makes the Root ſought, or a deficient one; or alſo over the 4th Root of A^3B ſet A, it makes the Root ſought, or an exceſſive one.

DEMON. 1. For the n Root of $\frac{A}{B}$: If AB^{n-1} is a true Power of the Order n, let its Root be m, then it's plain that $\frac{m}{B}$ is the n Root of $\frac{AB^{n-1}}{B^n} = \frac{AB^{n-1}}{B \times B^{n-1}} = \frac{A}{B}$. And if m is only the Root of the greateſt integral Power contained in AB^{n-1}, it ſelf not being one, then is $\frac{m}{B}$ plainly a deficient Root to $\frac{A}{B}$, which in this Caſe has no true Root; for if we ſuppoſe $\frac{m^n}{n^n} = \frac{AB^{n-1}}{B^n}$ $\left(= \frac{A}{B}\right)$, then is $AB^{n-1} = \frac{m \times B^n}{n^n} = \frac{\overline{mB}^n}{n^n}$, whoſe n Root is $\frac{mB}{n}$; *i.e.* AB^{n-1} is a Power of the Order n, contrary to Suppoſition.

2. For the 2d Method; $\frac{A^n}{BA^{n-1}} = \frac{A}{B}$, and if the n Root of BA^{n-1} is n, then is $\frac{A}{n}$ the Root ſought: But if n is only the Root of the greateſt integral Power contain'd in BA^{n-1}, it ſelf not being one, then is $\frac{A}{n}$ an exceſſive Root to $\frac{A}{B}$, which in this Caſe has no true Root; for if $\frac{m^n}{n^n} = \frac{A^n}{BA^{n-1}}$, then is $BA^{n-1} = \frac{A^n \times n^n}{m^n}$, whoſe n Root is $\frac{A^n}{m}$; *i.e.* BA^{n-1} is a Power of the Order n, contrary to Suppoſition.

SCHOLIUM. If the Denominator of the Root is a Compound Number; *i.e.* the Product of two or more Integers, the Extraction may be made by ſeveral more ſimple Extractions, in the manner already explained, which needs not to be further inſiſted on.

§. II.

§. II. Part III.

Problem 5. *Of the* Approximation *of* Roots.

Definition.

WE have already obſerved, that tho' a Number has no determinate Root, yet it has what we may call an indeterminate one (ordinarily called a *Surd* Root); *i. e.* there is a certain Series of Numbers decreaſing, which can be carried on by a certain Law or Order *in infinitum*, whoſe Sum taken from the beginning is a Root whoſe Power approaches nearer and nearer to the given Number, as the Series goes on; and tho' it is never equal to it preciſely, it may be brought within any aſſignable Difference: The *Invention*, or carrying on of this Series is what we here call the *Approximation* of the Root; and if we take the Series of the Sums invented, it may be called the Series of *Approaching* Roots. Obſerve alſo, that they may be found approaching yet either ſtill leſs or ſtill greater than true Roots.

I. *For Roots of* Integers.

Rule. Whatever Root is propoſed, after the Root of the greateſt integral Power contained in the given Number is found, by the preceding Rules; To the Remainder prefix a Period of 0's according to the Index; thus 00 for a Square Root, 000 for a Cube, 0000 for a 4th Power, and ſo on: Then form a Diviſor, and find a new Figure in the Root the ſame way as in the preceding Steps of the Work: To every ſucceeding Remainder prefix a Period of 0's, and find a new Figure of the Root, and this Work will go on for ever, becauſe there will always be a Remainder. The Figures thus found are all Decimal Places in the Root, the decimal Point being placed immediately after the integral Part, and before theſe new Figures. And thus we have a Mixt Number for the Root; which is ſtill nearer and nearer to the true Root of the given Number, the further the Operation is carried on, but is ſtill deficient, becauſe there is ſtill a Remainder. Again; *Obſerve*, that if to the laſt Figure found in the Root you add 1, the Sum will make an exceſſive Root; and thus you may have a Series of Roots nearer and nearer, but ſtill exceſſive.
The following Example of a Square Root will ſufficiently illuſtrate this Practice.

Operation.

```
387 (19.672, &c.
1
————
287
261
————
 2600
 2316
 ————
  28400
  27489
  —————
   91100
   78684
   —————
   12416
    &c.
```

To find the Square Root of 387. The Root of the greateſt integral Square contained in it is 19. Then by one Period of 0's the Root becomes 19.6; by a 2d it is 19.67; by a 3d it is 19.672; and may be carried further at pleaſure; and each of theſe Roots are deficient; *i. e.* their Figures are leſs than 387; but the Difference is ſtill leſs and leſs: and what I called the Series of Numbers decreaſing, whoſe Sums make the Series of approaching Roots, tho' ſtill defective, are theſe 19, .6, .07, .002, *&c.* and the Series of their Sums, which make the approaching Root, is 19, 19.6, 19.67, 19.672, *&c.* And laſtly, by adding 1 to each of theſe, we have a Series of approaching Roots, but ſtill exceſſive, tho' the Differences grow ſtill leſs. Thus, 20, 19.7, 19.68, 19.673, *&c.*

Demon. 1. If any compleat integral Power of any Order is multiplied into a Number which is not a Power of that Order, the Product is not a Power of that Order; or has not a perfect Root of that Order. Thus; If A is not a Power of the Order n, neither is $A \times B^n$, as has been demonſtrated in *Theor.* II. *Coroll.* 4. *Chap.* I.

2. If

2. If the Root of the greatest integral Power contained in AB^n is divided by B, which is the n Root of the Multiplier B^n, the Quote is less than a true Root to the given Number. For suppose r to be the n Root of the greatest integral Power of the Order n contained in AB^n, and it's plain that r^n is less than AB^n, therefore take their like aliquot Parts, and $\frac{r^n}{B^n}$ is less than $\frac{AB^n}{B^n}$, or A; *i.e.* $\frac{r}{B}$, the n Root of $\frac{r^n}{B^n}$, is less than a true Root to A. Again; If to r, the greatest integral Root of AB^n, we add 1, and call the Sum s, then s^n is greater than AB^n; and consequently $\frac{s^n}{B^n}$ greater than $\frac{AB^n}{B^n}$, or A; *i.e.* $\frac{s}{B}$ is an excessive Root to A.

From these two Articles we shall easily demonstrate the Rule of *Approximation*, thus:

3. The greatest integral Root, or Root of the greatest integral Power contained in the given Number being found, what remains to be proved is this only, That the Extraction will go on in this manner without end; *i.e.* that there will always be a Remainder, and so a new decimal Fraction will at every Step be added to the preceding Root, making the whole greater and greater; yet so that the Mixt Root will still be deficient, or its Power still less than the given Number, tho' still nearer *in infinitum.* To shew this Truth, *consider*, that by prefixing Periods of 0's to any Number, we do really multiply it by a Number consisting of 1 with as many 0's as are thus prefix'd; but it's the same thing to multiply the given Number (whose Root we seek) by prefixing 0's, and then bringing them down to the Remainders, or prefixing them only to the Remainders; for either way we find the Root of the Product (or the greatest integral Root contained in it). Thus, for a Square Root one Period 00 multiplies the given Number by 100, two Periods multiplies by 10000, *&c.* For a Cube Root one Period 000 multiplies by 1000, and two Periods multiplies by 1000000, *&c.* and so of other Powers. But these Multipliers are evidently true and compleat Powers of their several Orders, whose Roots are 1, with as many 0's as we have used Periods of 0's; therefore, by the first Article, however far the Extraction is carried by Periods of 0's thus prefix'd to the Remainders, *i.e.* however great the Power is by which we have thus multiplied the given Number, there will always be a Remainder, because the given Number not being a true Power, tho' the Multiplier is, yet the Product is not. Again; By putting all the Figures found by means of these Periods of 0's, in decimal Places, we do evidently divide the Root of the Product, (*i.e.* the Root of the greatest integral Power contain'd in it) by the Root of the Number multiplied into the given Number: Because for every Period annex'd we have one Place in Decimals; which is plainly dividing the Root found, considered all as a whole Number, by 10, or 100, *&c.* according to the Number of Periods of 0's employ'd. Therefore, by the 2d Article, this Mixt Root will always be less than a just Root to the given Number, tho' still approaching nearer, which demonstrates the Rule as to the Series of deficient Roots: and as to the excessive Roots, it's evident that adding an Unit to the last Place of the Root already found, is adding 1 to the Root of the greatest integral Power contained in the given Number, or to its Product by the Power which multiplies it: Therefore, by the 2d Article, the Root becomes excessive. Or it's found by this Consideration, That 1 in any Place of a Number either integral or decimal, is of more value than all the rest of the Number standing on the Right of that Place, however many Figures there be.

SCHOLIUMS.

1. The Proof of this Operation is made the same way as has been already explain'd, *viz.* either by raising the Root found to its Power, and adding the Remainder; or by casting out of the 9's.

As to the former Method, *Observe*, That we need to take no notice of the Root's being a Mixt Number, but take it all as a whole Number, and the Remainder so also; and then

then the Sum of the Power and Remainder muſt have as many Periods of o's on the Right as were uſed in the Operation; becauſe when the Root and Remainder are taken for Integers, ſo many Periods of o's belong to the ſuppoſed Power or Number, whereof that Root is the greateſt integral Root: But if we take the Root as it really is, a Mixt Number, then the Remainder is a decimal Fraction, whoſe Denominator is 1, with as many o's as were added to all the Remainders in the Operation, and in this value it is to be added to the Power of the Mixt Root: Thus in the preceding Example, the Root found is 19.672, whoſe Square is 386.987584, and the Remainder is, in its true value, .012416; for two Periods, or 6 o's, were employed in the Operation; and the Sum of 386 987584, + .012416 is = 387.000000 (= 387), which is the ſame as if the Quote and Remainder had been taken for Integers, and the given Number had been 387000000.

As to the Method by caſting out 9's; When we ſubtract the Remainder from the given Number, we may take it either in its real Value, or as a whole Number, and then we muſt take the given Number, with as many o's after it as were uſed in the Operation. For it is the ſame to the preſent Purpoſe, to take .012416 from 387, or 12416 from 387000000, the Remainder in both Caſes being the ſame Figures, *viz.* 386.987584, or 386987584.

2. If we point the true Value of the Remainder at every Step of the Approximation, this will ſhew gradually how much the propoſed Power of the Root found wants of the given Number; and as the Root, ſo conſequently its Power continually increaſes; therefore theſe Remainders will continually diminiſh; ſo that by obſerving this, we can carry on the Work till that Difference or Remainder be as little as we pleaſe, or leſs than any aſſigned Difference.

But if inſtead of this, it ſhould be required to extract the Root ſo near to a true and perfect Root to the given Number that it ſhall want leſs than an aſſigned Difference, *i. e.* ſo that this Difference added to the Root found, the Power of the Sum ſhall exceed the given Number, it's done thus; Suppoſe any Fraction $\frac{a}{r}$ to be the given Difference, within which the Root is to be brought; then extract the Root to a Number of decimal Places equal to the Number of Figures in r, and you have done; for $\frac{1}{r}$ is leſs than $\frac{a}{r}$, if a is greater than 1; and a decimal Denominator having as many o's as r has Figures, is a greater Number than r; and ſo a Fraction whoſe Numerator is 1, and its Denominator that decimal one, is leſs than $\frac{a}{r}$, becauſe the Denominator is greater, and the Numerator not. Laſtly, ſince, as has been ſhewn, 1 added to the laſt Figure of the Root would make it exceed a true Root; therefore, in whatever Place of Decimals the laſt Figure of the Root ſtands, the whole does not want of a true Root to the given Number, an Unit of the Value of that Place, and conſequently, if the Denominator of that laſt Place is a Number greater than r, the Root is within $\frac{1}{r}$ of a true Root, becauſe it's within a leſſer Fraction, and much more is it within $\frac{a}{r}$, which is greater than either of the former. Example: Let r be a Number of three Figures; if the Root have three decimal Places whereby the Denominator is 1000, the Root is within $\frac{1}{1000}$, which is leſs than any proper Fraction whoſe Denominator is a Number of three Figures; ſo in the preceding Example 19.672 is within $\frac{1}{1000}$ of a perfect Root to 387.

II. *For*

II. *For the Roots of* Fractions.

The Approximation of the Roots of Fractions is performed thus: Let that Root which the particular Rules for Square and Cube, or the general Rule following these, prescribes to be extracted, be carried on to what Length of decimal Places you please, and then divide it by the Denominator of the given Fraction, if you chuse the first Method of these Rules; and thus you have a Root still less, but approaching to a perfect one: But if you chuse the 2d Method of the Rule, divide the given Numerator by that Root, and you have a Root approaching, but still excessive; and the further the Approximation of that Root is carried, the truer will each of these fractional Roots be.

Example. For the Square Root of $\frac{13}{24}$, I take $13 \times 24 = 312$, whose Root to 2 Places of decimals is 17.66, which divided by 24 Quotes $\frac{1766}{2400} = \frac{883}{1200}$, less than a true Root; or it is $13 \div 17.66 = \frac{1300}{1766} = \frac{650}{883}$, greater than a true Root.

Demon. The Reason is manifest from the preceding Rules; for the Square Root of $\frac{A}{B}$ is $\overline{AB}|^{\frac{1}{2}} \div B$, or $A \div \overline{AB}|^{\frac{1}{2}}$. Universally, the n Root of $\frac{A}{B}$ is $\overline{AB^{n-1}}|^{\frac{1}{n}} \div B$, or $A \div \overline{BA^{n-1}}|^{\frac{1}{n}}$ if these Roots are perfect; and if they are not, yet by approximating them we make the fractional Root also truer, tho' never perfect.

Coroll. Tho' a given Integer is not a perfect Power of any Order, yet there is a greatest Power of that Order, which is a lesser Number than the given one; and also there is a least Power of the same Order, which is a greater Number than the given: But in Fractions there is no such greatest and least Power; because we can find new Roots increasing for ever, or decreasing, yet so as the Powers are still less or greater than the given Number.

Scholium. There remains one curious Problem relating to the Extraction of Roots, which goes a little deeper into the Algebraick Art than at first I designed: but without it, I found I must omit several other curious things: and since among several ways of solving this Problem, there is one that arises very easily and naturally from the Consideration of Square Numbers, especially the Square of a Binomial Root (already sufficiently explained) therefore I was determined to give it a place here.

Problem VI.

Having the Sum or Difference of any Square Number, and a certain Multiple of the Root; also having the Multiplier of the Root; to find the Root. Thus: Suppose $R = a^2 + ac$, or $R = a^2 - ac$, or $R = ac - a^2$. Then if the Numbers expressed by R and c are given, we can find the Number expressed by a by the following Rules.

Case I.

When the Sum and Multiplier are given to find the Root, *i. e.* if $R = a^2 + ac$; and R, c are given to find a.

Rule. To the Sum add the Square of half the Multiplier, (or a 4th of the Square of the Multiplier.) Extract the Square Root of this Sum; and from it subtract half of the Multiplier, the Remainder is the Root sought. Which Rule is expressed in Characters thus: $a = \overline{R + \frac{cc}{4}}|^{\frac{1}{2}} - \frac{c}{2}$.

Exam-

Example. $R=21$, $c=4$; then is $a=3$; for $4\times4=16$, whose 4th is 4; then $21+4=25$, whose Square Root is 5, from which take 2 (= the half of the Multiplier 4) the Remainder is 3 the Root. *Proof:* $3\times3=9$, $3\times4=12$, and $12+9=21$.

DEMON. Since by Supposition, $R=a^2+ac$; add $\frac{cc}{4}$ to each Side, then $R+\frac{cc}{4}=a^2+ac+\frac{cc}{4}$; which last Expression is a compleat Square, whose Root is $a+\frac{c}{2}$: therefore $a+\frac{c}{2}=\overline{R+\frac{cc}{4}}\big|^{\frac{1}{2}}$ (*Ax.* 1.) and subtracting $\frac{c}{2}$ from both, it is, $a=\overline{R+\frac{cc}{4}}\big|^{\frac{1}{2}}-\frac{c}{2}$. Which is the Rule.

CASE II.

If the Difference and Multiplier are given, to find the Root;
Here there are two Rules, according as the Square or Multiple is supposed to be greatest.

1. Suppose the Square greater than the Multiple, *i.e.* $R=a^2-ac$.

RULE. To the Difference, add the 4th of the Square of the Multiplier; and to the Square Root of the Sum, add half the Multiplier: this Sum is the Root sought. Thus: $a=\overline{R+\frac{cc}{4}}\big|^{\frac{1}{2}}+\frac{c}{2}$.

Example. $R=28$, $c=3$; then is $a=7$: for $\frac{cc}{4}=\frac{9}{4}=2\frac{1}{4}$, and $R+\frac{cc}{4}=28+2\frac{1}{4}=30\frac{1}{4}=\frac{121}{4}$, whose Square Root is $\frac{11}{2}$ or $5\frac{1}{2}$; to which add $\frac{3}{2}$ or $1\frac{1}{2}$, the Sum is 7. *Proof.* $7\times7=49$, and $3\times7=21$; then $49-21=28$.

DEMON. Since $R=a^2-ac$, add $\frac{cc}{4}$ to both Sides; then is $R+\frac{cc}{4}=a^2-ac+\frac{cc}{4}$. Which last Expression is the Square of $a-\frac{c}{2}$. Wherefore $a-\frac{c}{2}=\overline{R+\frac{cc}{4}}\big|^{\frac{1}{2}}$; and adding $\frac{c}{2}$ to both Sides, it is $a=\overline{R+\frac{cc}{4}}\big|^{\frac{1}{2}}+\frac{c}{2}$. *Observe,* Tho' $a^2-ac+\frac{cc}{4}$ is the Square either of $a-\frac{c}{2}$, or $\frac{c}{2}-a$, yet we cannot here use $\frac{c}{2}-a$; for if a is less than $\frac{c}{2}$, a^2 is less than ac, contrary to Supposition.

2. Suppose the Multiplier greater than the Square, *i. e.* $R=ac-a^2$.

RULE. From the 4th of the Square of the Multiplier subtract the given Difference, (which cannot exceed the Multiplier, if the Problem is possible); then extract the Square Root of the Remainder; and either add it to, or subtract it from half the Multiplier, (which is greater than the other, if the Problem is possible); the Sum or Difference will either of them solve the Problem. Thus: $a=\frac{c}{2}+\overline{\frac{cc}{4}-R}\big|^{\frac{1}{2}}$, or also $a=\frac{c}{2}-\overline{\frac{cc}{4}-R}\big|^{\frac{1}{2}}$.

Example. $R=6$, $c=5$; then is $a=3=\frac{5}{2}+\overline{\frac{25}{4}-6}\big|^{\frac{1}{2}}$, or $\frac{5}{2}+\overline{\frac{1}{4}}\big|^{\frac{1}{2}}=\frac{5}{2}+\frac{1}{2}=\frac{6}{2}$.

Proof. $ac=15$, and $ac-a^2=15-9=6=R$. Also, $a=2=\frac{5}{2}-\frac{1}{2}=\frac{4}{2}$. *Proof.* $ac-a^2=10-4=6=R$.

Demon. Since $R = ac - a^2$. Subtract each of these from $\frac{cc}{4}$, then is $\frac{cc}{4} - R = \frac{cc}{4} - ac + a^2$; which last is the Square either of $a - \frac{c}{2}$, or $\frac{c}{2} - a$. Wherefore $a - \frac{c}{2}$, or $\frac{c}{2} - a$ (according as a is greater or lesser than $\frac{c}{2}$) is $= \overline{\frac{cc}{4} - R}|^{\frac{1}{2}}$. Hence, in the 1st *Case*, or taking $a - \frac{c}{2}$; by adding $\frac{c}{2}$ to both Sides, it is $a = \frac{c}{2} + \overline{\frac{cc}{4} - R}|^{\frac{1}{2}}$; and taking $\frac{c}{2} - a$, add $a - \overline{\frac{cc}{4} - R}|^{\frac{1}{2}}$ to both Sides, it is $a = \frac{c}{2} - \overline{\frac{cc}{4} - R}|^{\frac{1}{2}}$.

There remains yet to be demonstrated, That $\frac{cc}{4}$ can never be less than R, if the Problem is possible; and that $\frac{c}{2}$ is greater than $\overline{\frac{cc}{4} - R}|^{\frac{1}{2}}$. Now it is plain, that the Solution is impossible, according to this Rule, if R is greater than $\frac{cc}{4}$; and that if R does not exceed $\frac{cc}{4}$, one of the Solutions is good. But to shew that the Problem will always necessarily have the two Solutions explained, it must be shewn that R cannot exceed $\frac{cc}{4}$, when it is $= ac - a^2$; and then the other Part will easily follow.

To demonstrate this, we must first *observe*, That a may be either greater or lesser than $\frac{c}{2}$ consistently enough with $R = ac - aa$. For this requires no more than that ac be greater than aa, which requires again that a be less than c; consequently, whether a be greater or less than $\frac{c}{2}$, providing it be less than c, (as it may be) ac will be greater than aa. *Again*; Whether we take $a - \frac{c}{2}$ or $\frac{c}{2} - a$, the Square of it is $a^2 - ac + \frac{cc}{4} = \frac{cc}{4} - ac + a^2$, which is also $= \frac{cc}{4} - \overline{ac - aa} = \frac{cc}{4} - R$, (because $ac - aa = R$) But the Root being real or positive, so must the Square be; *i. e.* $\frac{cc}{4}$ is greater than R. Or, if $a = \frac{c}{2}$, then $a^2 = \frac{cc}{4}$, and $2a = c$; also $2aa = ac$; consequently $ac - aa = 2aa - aa = aa$; and $\frac{cc}{4} = ac - aa = R$. So that R can never be greater than $\frac{cc}{4}$, tho' it may be either equal or less. And *observe*, if they are equal, then there is but one Solution, *viz.* $a = \frac{c}{2}$; for here both the Solutions coincide.

For the second thing, *viz.* that $\frac{c}{2}$ is greater than $\overline{\frac{cc}{4} - R}|^{\frac{1}{2}}$; consider that $\frac{cc}{4}$ is greater than $\frac{cc}{4} - R$, and consequently $\frac{c}{2}$ (the Square Root of $\frac{cc}{4}$) is greater than $\overline{\frac{cc}{4} - R}|^{\frac{1}{2}}$. Or, we have this in the very Supposition: for, by the first Part of the Demonstration of this Rule, the

Value

Value of a is $= \frac{c}{2} - \overline{\frac{cc}{4} - R}|^{\frac{1}{2}}$, upon that very Suppoſition that $\frac{c}{2}$ is greater than a; whence it was ſhewn that $\overline{\frac{cc}{4} - R}|^{\frac{1}{2}} = \frac{c}{2} - a$, and conſequently $\frac{c}{2}$ greater than $\overline{\frac{cc}{4} - R}|^{\frac{1}{2}}$.

SCHOLIUMS.

1. If the Difference of the Square and Multiple of the Root is given, without determining which of them is greateſt, then we muſt try both Rules.

2. This Problem is what the *Algebraiſts* call, *Extracting the Root of an affected Square*, (*i. e.* wherein the Number given is the Sum or Difference of a Square, and a certain Multiple of the Root; whoſe Multiplier is alſo given.) The Solutions explained are all that are real and poſitive; yet the Algebraick Art conſiders two Roots or Solutions in every Caſe: But the Roots that I have not explained are only negative and imaginary; and to ſay any thing farther about them, were to exceed the Limits preſcribed to this Work; and for the ſame Reaſon I am obliged to ſpeak nothing of extracting the Roots of higher Powers that are affected.

CHAP. III.

The Arithmetick *of* SURDS.

WHAT a *Surd* is has been already explained: It has been demonſtrated that every Number has not a perfect and determinate Root; but yet that we can find an Approximate Root within any aſſignable Difference of a true and compleat one; ſo that it may be truly ſaid, that the Quantity which hinders any Number from being a compleat Power of any kind, is infinitely little; or that a Quantity infinitely little (or leſs than any aſſigned one) being taken from the Quantity expreſſed by any given Number, the Remainder is a Quantity expreſſible by a Number (of the ſame Parts) which is a true Power of the Order propoſed; with this Difference, that it will be a fractional Power and not an integral. Now ſince *Surds*, or indeterminate Roots, can be determined infinitely near; and ſince the indeterminate Series goes on by a certain Law or Condition, it may be conceived as ſome whole and compleat thing of its own kind; and therefore, taking Surds under the general Expreſſion of Roots, as $N^{\frac{1}{n}}$, we may apply all the *Theory* of Chap. I. and all the *Operations* of Arithmetick to them, as if they were determinate: For thus we can form general Ideas of Sums, Differences, Products and Quotes of Surds, imagined under the Notion of compleat Quantities of their own kind, the ſame way as we do of rational or determinate Roots expreſſed after the ſame general manner. And hereby we can diſcover certain Connections and Relations of Quantities thus repreſented, which may lead us to ſome other particular Truths we would diſcover.

It's true indeed, that as to any actual Operation with ſuch Roots it can only be made in an imperfect manner, by way of Approximation; yet ſince we can approximate or determine the Root ſo far, that taking it for true and compleat, the Error it can make in any Operation or Concluſion ſhall be within any aſſignable Difference of what it would be if the compleat Value of the Surd could poſſibly be determined and uſed in the Operation: Therefore our arguing with them as we do with rational and determinate Roots, is ſo far at leaſt juſt and concluſive; and is indeed abſolutely ſo, taking them in general and abſtractly.

 To

To illuſtrate this by a few *Examples:* The Sum of $\sqrt{8}$ and $\sqrt{12}$ may be expreſſed in general $\sqrt{8}+\sqrt{12}$, whatever theſe are in themſelves; and if we would apply this by an actual Operation, then we can approximate each of theſe Roots ſo near, that their Sum ſhall want leſs than any aſſigned Difference of what it would be if the Roots could be determined.

Example 2. To multiply $\sqrt{8}$ by 3, it may be expreſſed thus, $3\times\sqrt{8}$; and by Approximation we can find a determinate Number for $\sqrt{8}$, which multiplied by 3, the Product ſhall want leſs than any aſſigned Difference of what it would be if the Root could be compleatly determined.

Example 3. The Product of $\sqrt{8}$ and $\sqrt{6}$ may be expreſſed $\sqrt{8}\times\sqrt{6}$; and by Approximation we can take $\sqrt{8}$ and $\sqrt{6}$ ſo near, that multiplying them together at every Step, the Products ſhall ſtill increaſe and come within any aſſigned Difference of what it would be were the Roots determined.

Again: Tho' Surds can never be reduced to determinate Numbers (for then they were not Surds), yet in many Caſes their Sums, Differences, Products and Quotes can be expreſſed after different ways (by means of the *Theory* explained in *Chap.* I.), which are more or leſs ſimple and convenient; ſo that what by the more general Rules can be expreſſed only by Signs of Addition, *&c.* may be expreſſed more ſimply, either by one Surd, or by an Expreſſion partly ſurd, partly rational, and in ſome Caſes altogether rational. Now to this tends the more particular Practice or Arithmetick of Surds; which depending upon certain different Forms in which the ſame Surd may be expreſſed, therefore the firſt thing to be explained is *The Reduction of Surds*; the Demonſtration of which depends upon the *Theorems* in *Chap.* I. applied to *Surds*.

Obſerve alſo, That all the following Practice is equally applicable to rational Roots expreſſed in the general radical Form; for when we take general Expreſſions they comprehend all poſſible Caſes; and the Practice propoſed is often as convenient with reſpect to Rationals as Surds, becauſe it's convenient ſometimes to expreſs even rational Numbers in this radical Form; and therefore, tho' it's commonly called the *Arithmetick of Surds*, it were as proper to call it the *Arithmetick of Radicals*.

Reduction of Surds (*or* Radicals).

CASE I. To expreſs any Number in Form of a Surd (*i.e.* in a radical Form). RULE: Raiſe the given Number to the Power of the Surd, and then apply the Surd Index, thus; $8=64^{\frac{1}{2}}$, for $8\times 8=64$. Univerſally, $A=\overline{A^n}|^{\frac{1}{n}}$.

The *Reaſon* is manifeſt from the *Definitions*, and *Ax.* I.

CASE II. To reduce a Surd with a Mixt Index (*i.e.* whoſe Numerator is greater than 1) to another, having a ſimple radical Index (*i.e.* whoſe Numerator is 1.) RULE: Involve the Number given to a Power whoſe Index is the Numerator of the Mixt Index, and to the Number found apply the Denominator radically. *Example*: $8^{\frac{2}{3}}=64^{\frac{1}{3}}$; for $8^2=64$. Univerſally, $A^{\frac{n}{r}}=\overline{A^n}|^{\frac{1}{r}}$.

The *Demonſtration* of this is plainly in the Definition: For $A^{\frac{n}{r}}$ expreſſes the *r* Root of the *n* Power (which is alſo the *n* Power of the *r* Roots by *Theor.* X.)

CASE III. To reduce two Unlike Surds to Like: *i.e.* having two unlike Surds of the ſame or different Numbers, to find other two Surds equal reſpectively to the given ones, but having the ſame Index, and that alſo the leaſt poſſible; and ſuch too, that the Numbers under the common Index be the leaſt poſſible. RULE.

Rule. Reduce their Indexes to one common Denominator by the Rules of Fractions. Again, find the greateſt common Meaſure to both the new Numerators, (*i. e.* the greateſt Number which will divide them both without a Remainder,) by the Method taught in the Reduction of the Numerator and Denominator of a Fraction to their leaſt Terms; make that common Meaſure the common Numerator to the common Denominator before found: The loweſt Terms of this Fraction is the common Index ſought.

Again; Divide the new Numerators mentioned by their greateſt common Meaſure, and mark the Quotes; then involve each given Number to a Power whoſe Index is the reſpective Quote; and that is the Number to which if the common Index is applied the Caſe is compleatly ſolved.

Example: To reduce $8^{\frac{1}{2}}$ and $15^{\frac{1}{3}}$ to Like Surds, with the other Conditions propoſed. 1. They are $8^{\frac{3}{6}}$ and $15^{\frac{2}{6}}$, by reducing the Indexes $\frac{1}{2}$ and $\frac{1}{3}$ to one Denominator; Then the greateſt common Meaſure of the Numerators 3, 2, is 1, and the common Index is $\frac{1}{6}$; and to have Numbers to which it muſt be applied, I raiſe 8 to the 3d Power, and 15 to the 2d, (for here the common Meaſure of 3 and 2 is 1, which makes the Quotes the ſame,) theſe Powers are 512, 225; wherefore the Surds ſought are $512^{\frac{1}{6}} = 8^{\frac{1}{2}}$, and $225^{\frac{1}{6}} = 15^{\frac{1}{3}}$.

Example 2. To reduce $4^{\frac{2}{3}}$ and $5^{\frac{4}{7}}$: They are firſt $4^{\frac{14}{21}}$, $5^{\frac{12}{21}}$, then the greateſt common Meaſure of 14, 12, is 2; and ſo $\frac{2}{21}$ is the common Index, which is in its leaſt Terms. Again; The Numerators 14, 12, divided by their greateſt Meaſure 2, the Quotes are 7, 6, and $4^7 = 16384$. $5^6 = 15625$. Then laſtly, $16384^{\frac{2}{21}} = 4^{\frac{2}{3}}$. $15625^{\frac{2}{21}} = 5^{\frac{4}{7}}$.

Example 3. To reduce $3^{\frac{1}{6}}$ and $4^{\frac{1}{3}}$: They are firſt $3^{\frac{3}{18}}$, $4^{\frac{6}{18}}$, and the greateſt Meaſure of the Numerators 3, 6, being 3, the common Index is $\frac{3}{18} = \frac{1}{6}$ in its leaſt Terms; then 3, 6, divided by 3, the Quotes are 1, 2; and $3^1 = 3$: $4^2 = 16$: Wherefore, laſtly, $3^{\frac{1}{6}}$, $16^{\frac{1}{6}}$ are the Surds ſought.

Demon. Let $A^{\frac{r}{s}}$, $B^{\frac{n}{u}}$ be any two Surds, (where, if r or n are 1, the Surds are ſimple.) Theſe are firſt equal to $A^{\frac{ru}{su}}$, $B^{\frac{ns}{su}}$ by Reduction of the Indexes to one Denominator, (*Theor* XI. *Ch.* I.) Suppoſe m to be the common Meaſure to ru, ns, and let the Quotes be $ru \div m = x$; $ns \div m = y$; ſo that $ru = mx$, and $ns = my$: Then the Surds are $A^{\frac{mx}{su}}$, $B^{\frac{my}{us}}$, that is, (by *Theor.* XII.) $\overline{A^x}|^{\frac{m}{su}}$, $\overline{B^y}|^{\frac{m}{su}}$; which is exactly the Expreſſion of the Rule, ſuppoſing m the greateſt common Meaſure of ru, ns, and $\frac{m}{su}$ to be in its leaſt Terms; or if it's not in loweſt Terms, yet its loweſt Terms being put in its Place makes an equivalent Expreſſion (*Theor.* XI.) *Obſerve* alſo, that tho' m is not the greateſt common Meaſure of the new Numerators ru, ns, yet we have the Surds reduc'd to Like Surds, tho' not in the loweſt Expreſſions; which it's plain will then, and then only happen when m is the greateſt common Meaſure.

SCHOLIUMS.

1. It will be the ſame thing if we firſt reduce the given Surds to ſimple Indexes, if they are Mixt, and then reduce theſe new Indexes to one Denominator, and go on with the reſt as in the Rule.

2. When

2. When there are more Surds propofed, the Operation and the Reafon of it is the fame; except that we have not yet learn'd how to find the greateft common Meafure to 3 or more Numbers (which you'll find in *Book* IV. *Ch.* I.) and therefore, till that be learn'd, we muft be content to reduce the Surds to Likes, tho' not in their loweft Terms, by ufing 1 as a common Meafure, which makes the Dividend and Quote the fame.

CASE IV. To reduce a Surd having a fimple Index to lower Terms; *i. e.* to an equivalent Expreffion in which there is a fimilar Surd of a leffer Number multiplied into fome rational Number.

RULE. Among the Numbers greater than 1, which meafure the given Number, (or Surd Power) feek one which is a fimilar and rational Power, by which divide the given Number: Take the Quote, and to it apply the given Index; and multiply that Root by the Root of the Divifor: This Product is the Expreffion fought.

Example 1. $8^{\frac{1}{2}} = 2 \times 2^{\frac{1}{2}}$; for $8 \div 4 = 2$, and $4^{\frac{1}{2}} = 2$.

Example 2. $648^{\frac{1}{3}} = 2 \times 81^{\frac{1}{3}} = 3 \times 24^{\frac{1}{3}}$; for $648 \div 8 = 81$, and $8^{\frac{1}{3}} = 2$; whence, by the Rule, the 1ft Solution is $2 \times 81^{\frac{1}{3}}$. Again; $648 \div 27 = 24$, and $27^{\frac{1}{3}} = 3$; whence the 2d Solution is $3 \times 24^{\frac{1}{3}}$.

DEMON. Suppofe $A \div D^n = B$, fo that $A = B \times D^n$, then is $A^{\frac{1}{n}} = \overline{B \times D^n}|^{\frac{1}{n}}$, (*Ax.* I.) and $\overline{B \times D^n}|^{\frac{1}{n}} = D \times B^{\frac{1}{n}}$, (*Theor.* III) which is precifely conform to the Rule; A reprefenting the given Number.

SCHOLIUMS.

1. If the Power by which we meafure the given Number is the greateft Like Power which meafures it, then we find the loweft Terms of the given Surd.

2. As to the finding the Numbers that meafure any given Number, you'll have it more particularly explained in *Book* IV. *Chap.* I.: Here we fuppofe thefe to be given; becaufe from the Nature of the Thing this Rule for finding them is obvious, *viz.* To try all the Numbers not exceeding the half of the given Number; for all of thefe which meafure it, together alfo with the Quotes, make all the Numbers that meafure it. But unlefs thefe Meafures that ferve the prefent Problem are obvious, the finding them out is more Trouble than is always neceffary.

3. If the given Surd has a mixt Index, the fame kind of Reduction may be performed by reducing it firft to a Surd with a fimple Index; and then applying the prefent Rule. And again; If the Number under the radical Sign in the Anfwer thus found, is a rational Power of the Order expreffed by the Numerator of the given Index, then by taking the Root of it we may alfo reduce the whole to a Surd with the given mixt Index. Thus, $192^{\frac{2}{3}}$ is firft $= 36864^{\frac{1}{3}}$ $(= \overline{192^{2}}|^{\frac{1}{3}})$; which again reduced is $= 16 \times 9^{\frac{1}{3}}$; for $16^3 = 4096$, and $36864 \div 4096 = 9$: And becaufe 9 is a Square, therefore $9^{\frac{1}{3}} = 3^{\frac{2}{3}}$; and hence $16 \times 9^{\frac{1}{3}} = 16 \times 3^{\frac{2}{3}} = 192^{\frac{2}{3}}$.

COROLL. Hence we fee plainly, That one fimilar Surd may be a Multiple or aliquot Part of another. But obferve that in applying this to Practice, all we can make of it is, That the greater Surd approximate to a certain degree, and divided by the other approximate to the fame degree, the Quote will be within a certain Difference of that Number, which, by this Reduction, appears to be the Quote: But being approximate nearer and nearer *in infinitum*, the Quote will be nearer *in infinitum* to that other, which we here call the True and Compleat Quote. But if the fame Dividend be divided by any

any other Surd or Number whatever, the Quote can be brought to exceed that true Quote, or will never be brought within an assignable Difference of it; and therefore it's justly called the true Quote of these two Surds.

CASE V. To reduce any two Surds to Expressions, having a common Surd; *i. e.* to Expressions that are Products of rational Numbers into a common Surd.

RULE. Reduce the given Surds to the same simple Index, if they are not so already (by *Case* 3.): Then find the greatest common Measure of the Powers (or Numbers under the radical Signs); and taking the Quotes, examine by Extraction if they are rational and similar Powers of the Order expressed by the Denominator of the common simple Index; if they are, their Roots are the rational Numbers sought; and the surd Root of the common Measure is the surd Part sought: But if these Quotes are not such similar Powers, the Question is impossible.

Example: To reduce $12^{\frac{1}{2}}$ and $27^{\frac{1}{2}}$: The greatest common Measure of 12, 27, is 3, and the Quotes are 4, 9 which being rational Squares, I take their Roots 2, 3, and multiply them into the common Surd $3^{\frac{1}{2}}$, and the Expressions sought are $2 \times 3^{\frac{1}{2}} = 12^{\frac{1}{2}}$, and $3 \times 3^{\frac{1}{2}} = 27^{\frac{1}{2}}$.

DEMONSTR. Let $A^{\frac{1}{n}}$, $B^{\frac{1}{n}}$ be the given Surds (or the Expressions to which they are reduced): Suppose $A \div m = a^n$, and $B \div m = b^n$, so that $A = m \times a^n$, and $B = m \times b^n$; then $A^{\frac{1}{n}} = \overline{m \times a^n}|^{\frac{1}{n}}$ (*Ax.* I.) $= a \times m^{\frac{1}{n}}$, (*Theor.* I. *Cor.*) Also $B^{\frac{1}{n}} = \overline{m \times b^n}|^{\frac{1}{n}}$, $= b \times m^{\frac{1}{n}}$, which is exactly according to the Rule, supposing m to be any common Measure: And the Reason why it's in the Rule called the greatest common Measure, is, because if the greatest will not quote similar rational Powers, none of the other common Measures will; which remains to be demonstrated. Thus; take the given Surd Powers fractionwise, $\frac{A}{B}$; This is not an immediate fractional Power of the Order n, because by Supposition, neither A or B are rational Powers of that Order; but if any other Fraction equivalent to $\frac{A}{B}$ is an immediate Power of that Order, the least Terms of $\frac{A}{B}$ will be so; and if the least Terms are not so, no other Terms can be so, (as has been demonstrated in the Rule for Extracting the Roots of Fractions) *i. e.* if A, B, being divided by their greatest common Measure do not give for Quotes similar rational Powers of the Order n, neither can their Quotes by any other common Measure do so.

SCHOLIUMS.

1. The greatest common Measures quoting similar rational Powers, is a certain Character of the Problems being possible, tho' none of the other common Measures should make such Quotes; but if any of these others do so, these would make so many different Solutions to the Problem; in which this Difference is to be *observed*, that the lesser the common Measure is which we use, the lesser Terms will the Solution be in, as to the Surd Part: And the Reason why we chuse the greatest Measure in the Rule is, because that tho' from any other Measure's giving Quotes which are rational Powers we are sure that the Problem is possible, yet we can conclude it impossible from no other but the greatest common Measure giving Quotes which are not Like Powers.

2. If the two Surd Powers are Fractions, then reduce them to any common Denominator; and if the new Numerators are reducible according to this Rule, so are the given Surds.

Surds. One Example will ſhew this: Suppoſe $\overline{\frac{12}{15}}\Big|^{\frac{1}{2}}$, $\overline{\frac{3}{15}}\Big|^{\frac{1}{2}}$, the Numerators 12, 3, taken radically, *viz.* $12^{\frac{1}{2}}$, $3^{\frac{1}{2}}$, are reducible to theſe, $2 \times 3^{\frac{1}{2}}$, $1 \times 3^{\frac{1}{2}}$, or $3^{\frac{1}{2}}$; wherefore the given Surds are reduced to theſe, $2 \times \overline{\frac{3}{15}}\Big|^{\frac{1}{2}}$, and $\overline{\frac{3}{15}}\Big|^{\frac{1}{2}}$.

Again; If either the Numerators or Denominators of two Fractions, affected with a ſimple radical Sign, or reduced to that State, are rational Powers of the Order expreſſed by the Denominator of the Index. The Fractions need not be reduced to a common Denominator; for we need only examine if the other Terms are reducible to a common Surd Power: Thus; Suppoſe $\overline{\frac{50}{16}}\Big|^{\frac{1}{2}}$, $\overline{\frac{72}{25}}\Big|^{\frac{1}{2}}$: Here the Denominators, 16, 25, are Squares, whoſe Roots are 4, 5. Again; $50^{\frac{1}{2}} = \overline{2 \times 25}|^{\frac{1}{2}} = 5 \times 2^{\frac{1}{2}}$. Alſo $72^{\frac{1}{2}} = \overline{2 \times 36}|^{\frac{1}{2}} = 6 \times 2^{\frac{1}{2}}$; whence it's plain that the given Surds are $2^{\frac{1}{2}} \times \frac{5}{4}$, $2^{\frac{1}{2}} \times \frac{6}{5}$. And had the given Surds been $\overline{\frac{16}{50}}\Big|^{\frac{1}{2}}$, $\overline{\frac{25}{72}}\Big|^{\frac{1}{2}}$, the Solution is $\frac{4}{5} \times \overline{\frac{1}{2}}\Big|^{\frac{1}{2}}$, $\frac{5}{6} \times \overline{\frac{1}{2}}\Big|^{\frac{1}{2}}$; for $16^{\frac{1}{2}} = 4$, and $50^{\frac{1}{2}} = 5 \times 2^{\frac{1}{2}}$, then is $\overline{\frac{16}{50}}\Big|^{\frac{1}{2}} = \frac{4}{5 \times 2^{\frac{1}{2}}} = \frac{4}{5} \times \frac{1}{2^{\frac{1}{2}}}$, or $\frac{4}{5} \times \overline{\frac{1}{2}}\Big|^{\frac{1}{2}}$, and ſo of the other.

3. This Caſe is commonly called *Finding, if two Surds are commenſurable*; *i. e.* if they have a common Meaſure, or if there is any Surd which meaſures or is an aliquot Part of each of them; whereby they are reducible to Expreſſions which are the Products of that common Surd into the reſpective Quotes. *Obſerve* alſo, That the Meaſure of a Surd muſt be a Surd, which is manifeſt; for if any rational Number ſhould meaſure a Surd, or be an aliquot Part of it, then that aliquot Part and its Denominator (or the Meaſure and Quote) would produce the Dividend, *i. e.* two rational Numbers would produce a Surd, which is impoſſible.

The Uſe of theſe Reductions in the common Operations of Addition, *&c. I ſhall briefly ſhew thus:*

In Addition and Subtraction of Surds.

If one Surd is to be added to or ſubtracted from another, and if they are commenſurable, *i. e.* reducible to a common Surd, by *Caſe* 5. this Reduction being made, or if the given Expreſſions are of this kind, the Sum or Difference of the rational Parts multiplied into the common Surd Part is the Sum or Difference ſought, in a more ſimple and convenient Form than connecting the given Numbers by the general Signs of Addition and Subtraction, which is the general Rule for all other Caſes.

Example 1. $8^{\frac{1}{2}} + 50^{\frac{1}{2}} = 2 \times 2^{\frac{1}{2}} + 5 \times 2^{\frac{1}{2}} = 7 \times 2^{\frac{1}{2}}$.

Example 2. $54^{\frac{1}{3}} - 16^{\frac{1}{3}} = 3 \times 2^{\frac{1}{3}} - 2 \times 2^{\frac{1}{3}} = \overline{3 - 2} \times 2^{\frac{1}{3}} = 2^{\frac{1}{3}}$.

The Sum or Difference of two Square Roots may be alſo expreſs'd. *Thus*: Take the Surd Powers, or Numbers under the radical Sign; to the Square Root of double their Product, add their Sum, or ſubtract that Root from this Sum; the Square Root of this Sum or Difference expreſſes the Sum or Difference ſought. *Example*: $5^{\frac{1}{2}} + 3^{\frac{1}{2}} = \overline{8 + 30^{\frac{1}{2}}}|^{\frac{1}{2}}$, and $5^{\frac{1}{2}} - 3^{\frac{1}{2}} = \overline{8 - 30^{\frac{1}{2}}}|^{\frac{1}{2}}$.

DEMON. Suppose $A^{\frac{1}{2}}=a$, $B^{\frac{1}{2}}=b$; then is $ab=A^{\frac{1}{2}}\times B^{\frac{1}{2}}=\overline{AB}^{\frac{1}{2}}$ (*Theor.* 3.) and $A^{\frac{1}{2}}+B^{\frac{1}{2}}=a+b=\overline{a^2+b^2+2ab}|^{\frac{1}{2}}$. Also $A^{\frac{1}{2}}-B^{\frac{1}{2}}=a-b=\overline{a^2+b^2-2ab}|^{\frac{1}{2}}$, which is exactly according to the Rule.

For Multiplication and Division of SURDS.

If one Surd is to be multiplied or divided by another; then if they are unlike, reduce them to Likes, and examine if they are commensurable, *i. e.* reducible to Expressions, wherein the same common Surd is multiplied into rational Numbers; and if it is so, multiply or divide the rational Parts, the Product multiplied again into the Square of the common Surd is the Product sought: so that if the given Surds are Square Roots, the Product is rational. But in Division the Quote of the rational Parts alone is the Quote sought; which is therefore rational.

Example. $72^{\frac{1}{2}}\times 32^{\frac{1}{2}}=48$. For $72=8\times 9$, and $9^{\frac{1}{2}}=3$; therefore $72^{\frac{1}{2}}=\overline{8\times 9}^{\frac{1}{2}}=9^{\frac{1}{2}}\times 8^{\frac{1}{2}}=3\times 8^{\frac{1}{2}}$. Again, $32^{\frac{1}{2}}=\overline{4\times 8}^{\frac{1}{2}}=4^{\frac{1}{2}}\times 8^{\frac{1}{2}}=2\times 8^{\frac{1}{2}}$: so that $72^{\frac{1}{2}}\times 32^{\frac{1}{2}}=3\times 8^{\frac{1}{2}}\times 2\times 8^{\frac{1}{2}}=3\times 2\times 8^{\frac{1}{2}}\times 8^{\frac{1}{2}}=6\times 8=48$. And $72^{\frac{1}{2}}\div 32^{\frac{1}{2}}=3\times 8^{\frac{1}{2}}\div 2\times 8^{\frac{1}{2}}=3\div 2$.

DEMON. Suppose $A^{\frac{1}{n}}=a\times R^{\frac{1}{n}}$, and $B^{\frac{1}{n}}=b\times R^{\frac{1}{n}}$; then $A^{\frac{1}{n}}\times B^{\frac{1}{n}}=ab\times R^{\frac{1}{n}}\times R^{\frac{1}{n}}=ab\times R^{\frac{2}{n}}$. Wherefore if $n=2$, the Product is abR. Also $A^{\frac{1}{n}}\div B^{\frac{1}{n}}=a\times R^{\frac{1}{n}}\div b\times R^{\frac{1}{n}}=a\div b$.

SCHOLIUMS.

1. To multiply similar Surds: If we multiply the Surd Powers, and apply the same Index to that Product; this expresses the Product sought more simply, than by the general Sign of Multiplication. Thus: $A^{\frac{1}{n}}\times B^{\frac{1}{n}}=\overline{AB}^{\frac{1}{n}}$ (*Theor.* 3.) Again; if this is reducible, bring it to lowest Terms, and you'll have in many Cases the same Product that the preceding Rule brings out; and it's always the best we can make of it, when the given Surds are not commensurable. In the preceding Example, $72^{\frac{1}{2}}\times 32^{\frac{1}{2}}=\overline{72\times 32}^{\frac{1}{2}}=\overline{2304}^{\frac{1}{2}}=48$.

2. If a rational Number is multiplied into a Surd, it may be sometimes convenient to express it altogether radically; for which you have a Rule in *Theo.* 3. *Cor.* Thus: Raise the rational Part to the Power whose Index is the Denominator of the surd Part, and multiply this Power into the surd Power; then apply the radical Index. *Exam.* $A\times B^{\frac{1}{n}}=\overline{A^nB}^{\frac{1}{n}}$, and $A\times B^{\frac{r}{n}}=\overline{A^nB^r}|^{\frac{1}{n}}$, for $B^{\frac{r}{n}}=\overline{B^r}^{\frac{1}{n}}$.

3. If a Surd and rational Number are multiplied; and if the Surd is reducible to lower Terms, the whole Product is so. Thus: $6\times 45^{\frac{1}{2}}=6\times 3\times 5^{\frac{1}{2}}=18\times 5^{\frac{1}{2}}$; for $45=9\times 5$, and $45^{\frac{1}{2}}=9^{\frac{1}{2}}\times 5^{\frac{1}{2}}=3\times 5^{\frac{1}{2}}$.

You may apply the same Observations to Division. So for the 1st, $72^{\frac{1}{2}}\div 32^{\frac{1}{2}}=\overline{72\div 32}^{\frac{1}{2}}=\overline{9\div 4}^{\frac{1}{2}}$, or $\frac{9}{4}|^{\frac{1}{2}}=\frac{3}{2}$.

And from this Example, wherein the Product or Quote becomes rational, we have a farther remarkable Proof of the Reasonableness and Usefulness of our treating Surds, and working with them in all respects as with Rationals or compleat Roots; for if any other Number than 48 is supposed to be the Product of $72^{\frac{1}{2}}\times 32^{\frac{1}{2}}$, we can prove it to be false.

Thus:

Thus: Since $72^{\frac{1}{2}}$ and $32^{\frac{1}{2}}$ can be Approximate *in infinitum*, the Products of these approximate Roots will grow *in infinitum* towards a certain Limit; which must necessarily be $\overline{72 \times 32}^{\frac{1}{2}} = 48$; because if $72^{\frac{1}{2}}$ and $32^{\frac{1}{2}}$ are both rational, then is $\overline{72 \times 32}^{\frac{1}{2}}$ rational, being equal to $72^{\frac{1}{2}} \times 32^{\frac{1}{2}}$ (*Theor.* 3.) And tho' $72^{\frac{1}{2}}$, $32^{\frac{1}{2}}$ are surd, yet being infinitely approximable, their Product will grow infinitely near to $\overline{72 \times 32}^{\frac{1}{2}} = 48$; which is therefore the true Limit or compleat Value of $72^{\frac{1}{2}} \times 32^{\frac{1}{2}}$, beyond which it cannot possibly grow; nor can it be supposed less, because we can approximate $72^{\frac{1}{2}}$ and $32^{\frac{1}{2}}$ so far, that the Product shall exceed any assignable Number less than $48 = \overline{72 \times 32}^{\frac{1}{2}}$; for else they were not infinitely approximable, as is supposed and demonstrated.

CHAP. IV.

Containing several THEOREMS *relating to the* Powers *of Numbers.*

IN the following *Theorems* and *Corollaries* are comprehended all the Propositions of the *Second Book* of EUCLID that are applicable to Numbers, excepting four, which are in effect already explained in this Work; but that you may see them all in this Place, I briefly repeat these four.

1. If one Number A, (or Line, as it is in *Euclid, Book* II. *Theor.* 1.) is multiplied severally into all the Parts of another $B = a + b + c$, &c. the Sum of the Products is the Product of the two Wholes; thus $Aa + Ab + Ac$, &c. $= AB$. This you have already in *Lemma* 3. *Ch.* 5. *Book* I. which, observe, is equally applicable to Fractions and Integers.

2. If any Number is multiplied into all its own Parts severally, the Sum of the Products is equal to the Square of the Whole; which is the Consequence of the last; thus, if $N = a + b$, then is $N^2 = Na + Nb$ (*Euclid,* Theor. 2.)

3. If a Number is divided into two Parts, the Product of the Whole and one Part, is equal to the Sum of the Square of this Part, and the Product of the Parts: This is also a Consequence, or particular Case of the 1st: Thus, if $N = a + b$, then $Na = a^2 + ab$, (*Euclid,* Theor. 3.)

4. If a Number is divided into two Parts, the Square of the Whole is equal to the Sum of the Squares of the Parts, and twice the Product of the Parts: This proceeds also from the 1st: Thus, if $N = a + b$, then $N^2 = a^2 + b^2 + 2ab$, (*Euclid,* Theor. 4.)

The rest of *Euclid* you have in the following *Theorems.*

THEOREM I.

THE Square of any Number is equal to the Difference of the Products of that Number, multiplied into any greater Number, and into the Difference of these Numbers. Or it is equal to the Sum of the Products of that Number multiplied into any lesser, and into the Difference of these Numbers.

DEMONSTR. (1.) Let two Numbers be a, $a + d$, wherein d is the Difference; then $a \times \overline{a + d} = a^2 + ad$; hence $a^2 = a \times \overline{a + d} - ad$. (2.) Let the Numbers be $a - a$, a, wherein d is also the Difference; then $a \times \overline{a - d} = a^2 - ad$, and $a^2 = a \times \overline{a - d} + ad$.

THEO-

THEOREM II.

THE Sum of the Squares of two Numbers is equal to the Sum of twice their Product and the Square of their Difference.

DEMONSTR. Let the Roots be a, b, then $a^2 + b^2 = 2ab + \overline{a-b}^2$; for $\overline{a-b}^2 = a^2 - 2ab + b^2$; whence the *Theorem* is manifest.

Or let the Roots be a, $a+b$, wherein b is the Difference; then is $a^2 + \overline{a+b}^2 = a^2 + a^2 + 2ab + b^2 = 2a^2 + 2ab + b^2 = 2a \times \overline{a+b} + b^2$.

SCHOLIUM. In this last Form we see plainly comprehended *Euclid's Theor.* 7. which is this; If a Number consist of two Parts, (a, b,) the Sum of the Squares of the Whole and of one Part, (*viz.* $\overline{a+b}^2 + a^2$) is equal to double the Product of the whole into this Part, and the Square of the other Part (*viz.* $2a \times \overline{a+b} + b^2$).

THEOREM III.

THE Sum of the Squares of two Numbers is equal to the $\frac{1}{2}$ Sum of the Squares of their Sum and Difference.

DEMONSTR. The two Numbers being a, b, then $a^2 + b^2 = \frac{\overline{a+b}^2 + \overline{a-b}^2}{2}$, for $\overline{a+b}^2 = a^2 + 2ab + b^2$, and $\overline{a-b}^2 = a^2 - 2ab + b^2$, and the Sum of these two Squares is $2a^2 + 2b^2$, whose $\frac{1}{2}$ is $a^2 + b^2$.

COROLL. The Sum of two Squares is double the Sum of the Squares of their half Sum and half Difference; for $a+b$, $a-b$, may represent any two Numbers, whose half Sum is a, and their half Difference is b; but we see above that $2a^2 + 2b^2 = \overline{a+b}^2 + \overline{a-b}^2$.

SCHOLIUMS.

1. This *Corrollary* is in effect the same as *Euclid*, Theor. 9. *viz.* If a Number is divided into two equal Parts, a, a, and into two unequal Parts $a+b$. $a-b$, (whose Sum is $2a$) the Sum of the Squares of the unequal Parts, (*viz.* $\overline{a+b}^2 + \overline{a-b}^2$) is equal to twice the Square of the half, (*viz.* $2 \times a^2$) and twice the Square of the middle Part, or half Difference of the unequal Parts (*viz.* $2 \times b^2$).

2. If we express the supposed Numbers thus, a, $a+b$, then the *Theorem* is $a^2 + \overline{a+b}^2 = \frac{\overline{2a+b}^2 + b^2}{2}$. Also by taking b and $\overline{2a+b}$ for two Numbers, whose Sum is $2a + 2b$, and their half Sum $a+b$, and half Difference a, the preceding *Corollary* is thus expressed, $\overline{2a+b}^2 + b^2 = 2 \times \overline{a+b}^2 + 2 \times a^2$: Which is in effect the same as *Euclid*, Theor. 10. *viz.* If a Number is equally divided into two Parts, a, a, and to the whole $2a$ another Number be added as b, the Square of the Sum, *viz.* $\overline{2a+b}^2$, and the Square of the

 the

the Number added, *viz.* b^2, are together equal to double the Squares of $\frac{1}{2}$ the 1st Number, and of the Sum of that half and the Number added, *viz.* $2a^2 + 2 \times \overline{a+b}^2$.

THEOREM IV.

THE Sum of the Squares of two Numbers is equal to the Difference betwixt the Square of their Sum and double their Product.

DEMONSTR. The two Numbers being a, b, the *Theorem* is $a^2+b^2=\overline{a+b}^2-2ab$, for $\overline{a+b}^2=a^2+2ab+b^2$, from which take $2ab$, remains a^2+b^2.

Or thus; Let the two Numbers be $2a$, b, then $4a^2+b^2=\overline{2a+b}^2-4ab$.

COROLL. The Square of the Sum of two Numbers is 4 times their Product, more the Square of their Difference. This appears by making a, $a+b$, the two Numbers, whereby $2a+b$ is their Sum, and then adding $4ab$ to both Sides, whereby $\overline{2a+b}^2 = 4a^2+4ab+b^2=4\times a\times\overline{a+b}+b^2$.

SCHOLIUM. This *Coroll.* is the same in effect as *Euclid*, Theor. 8. *viz.* If a Number is divided into two Parts, as a, $a+b$; then 4 times the Product of the whole, and one Part, more the Square of the other Part, is equal to the Squares of the Sum of the whole and that Part.

THEOREM V.

THE Square of the Sum of two Numbers is equal to the Sum of the Square of one of them; and the Product of the other into the Sum of this other and double the former. Also the Square of the Difference of two Numbers is equal to the Difference of the Square of one of them, and the Product of the other into the Difference of this other, and double the former.

DEMONSTR. 1. $\overline{a+b}^2=a^2+2ab+b^2=a^2+\overline{2a+b}\times b$.

2. $\overline{a-b}^2=a^2-2ab+b^2=a^2-\overline{2a-b}\times b$.

SCHOLIUM. The first Part of this comprehends *Euclid, Theor. 6. viz.* If a Number is divided into two equal Parts a, a, and to the whole $2a$, another Number b is added; the Product of the Sum $2a+b$ by the Number added b, *viz.* $\overline{2a+b}\times b$, *&c.* together with the Square of $\frac{1}{2}$ the first Number, *viz.* a^2, is equal to the Square of the Sum of this half Number, and the Number added, *viz.* $\overline{a+b}^2$

THEOREM VI.

THE Difference of the Squares of two Numbers is equal to the Product of their Sum and Difference.

DEMON. $\overline{a+b}\times\overline{a-b}=a^2+ab-ab-b^2=a^2-b^2$.

COROLLARIES.

1. Of two unequal Numbers, $a+b$, $a-b$, the Square of half their Sum, *viz.* a^2 (for $2a$ is the Sum) is the Sum of their Product, *viz.* a^2-b^2, and the Square of their Difference, *viz.* b^2.

SCHOL.

SCHOLIUM. This is *Euclid's Theor.* 5. *viz.* If a Number is divided into two equal Parts, a, a; and into two unequal Parts, $a+b, a-b$; the Product of the unequal Parts (*viz.* $\overline{a+b} \times \overline{a-b} = a^2 - b^2$) together with the Square of the middle Part, (*i. e.* of half their Difference, b^2) is equal to the Square of half the given Number, *viz.* a^2.

2. The Sum and Difference of two Numbers are the reciprocal aliquot Parts of the Difference of their Squares.

3. The Square of any Number may be produced after a new Method. Thus: Let the given Root be N, assume any other Number A; the Product of their Sum and Difference, which call p, is the Difference of their Squares, *i. e.* $p = N^2 - A^2$, or $A^2 - N^2$. Hence $N^2 = A^2 + p$, or $A^2 - p$.

4. Take any Number A, and make a Series from it continually decreasing by 1, till the least Term be 1; also a Series increasing by 1, to the same Number of Terms; multiply the corresponding Terms of the two Series together; the Products make a Series decreasing in such a manner, that comparing each Product to the Square of A, the Differences are the Series of Squares of the natural Progression, 1, 2, 3, *&c.* the Deduction of which from the *Theorem* is easy. Thus: $\overline{A-n} \times \overline{A+n} = A^2 - N^2$. But the Differences betwixt A, and the several Terms of the Series, are, by Supposition, 1, 2, 3, 4, *&c. that is, n* is gradually 1, 2, 3, *&c* Consequently the Differences of the several Products of the corresponding Terms of the two Series from A^2 the first Product, are gradually the Squares of these Roots, 1, 2, 3, *&c.* Hence again,

A =	8. 7. 6. 5. 4. 3. 2. 1.
A =	8. 9. 10 11. 12 13. 14. 15.
Products - - -	64 63. 60. 55. 48. 39. 28 15.
Differs from 64. - -	1. 4. 9. 16. 25 36. 49.

5. We have this Rule for summing the Series of the Squares of the natural Progression 1, 2, 3, *&c. viz.* Take any Number A greater than n, the greatest of the Roots whose Squares are to be summed; then beginning at $A-1$, and $A+1$, continue a Series downwards from $A-1$, and upwards from $A+1$, with the common Difference 1, till the Number of Terms be n; then taking the Products of the two Series as before, subtract their Sum from $n \times A^2$; the Remainder is the Sum sought. The Reason is plain; for the Sum of the Products is n times A^2, wanting the Sum of the Series of Squares 1, 4, 9, *&c.* taken to a Number of Terms equal to n; therefore also the Sum of the Squares is $n \times A^2$ wanting the Sum of the Products.

$$\begin{array}{l} A-1 : A-2 : A-3 : A-4, \&c. \\ \underline{A+1 : A+2 : A+3 : A+4, \&c.} \\ A^2-1 : A^2-4 : A^2-9 : A^2-16, \&c. \end{array}$$

THEOREM VII.

The Sum of any Number of different Powers of the same Root, which stand all next together in the Series or Order of Powers, [*i. e.* whose Indexes follow one another in the natural Series of Numbers 1, 2, 3, *&c.* but beginning at any Power, or Place of the Series] is equal to the Quote of the Difference of the least of these Powers, and that next above the greatest of them, divided by the Difference of the Root and 1. Thus:

Example 1. $a^4 + a^5 + a^6 = \dfrac{a^7 - a^4}{a-1}$

2. $a + a^2 + a^3 + a^4 = \dfrac{a^5 - a}{a-1}.$

Ex. 3. $a + a^2 + a^3, \&c. + a^n = \dfrac{a^{n+1} - a}{a-1}.$

4. $a^r + a^{r+1} + a^{r+2} +, \&c. a^{r+n} = \dfrac{a^{r+n+1} - a^r}{a-1},$

DEMON. Take the Series $a + a^2 + a^3 + a^4$ to a^n, multiply it by $a-1$; the Product is $a^{n+1} - a$, as the annex'd Scheme of the Operation manifestly shews: For the given Series being multiplied by a, the Series of Products is the same as the given Series from the second Term, taking

$$\begin{array}{l} a + a^2 + a^3 + a^4, \&c.\ a^n. \\ \underline{\qquad a - 1 \qquad} \\ \quad a^2 + a^3 + a^4 + a^5, \&c. + a^{n+1}. \\ \underline{a + a^2 + a^3 + a^4, \&c. + a^n.} \\ a^{n+1} - a. \end{array}$$

taking in the Power next above the greatest given Power; then the other Part of the Multiplier being 1, its Product is the given Series. But this is to be subtracted from the former, (because the Multiplier is $a-1$); and all the Terms of the two Series of Products being the same, except the greatest of the first Series, and the least of the other, it's manifest that the Difference, *i. e.* the Product sought, is $a^{n+1}-a$, that is, $\overline{a+a^2+a^3, \&c\ a^n} \times \overline{a-1}=a^{n+1}-a$. Hence dividing both Sides by $a-1$, it is $a+a^2+a^3, \&c.\ a^n=\frac{a^{n+1}-a}{a-1}$.

Whatever Power the Series begins at, the Reason of the Rule is the same; for the Products by a and by 1 will be the same Series, except the greatest of the former, (which will be the next Power above the greatest given Power,) and the least of the other, (which is the least given Power); so that the Difference of the two Series must be the Difference of these two.

THEOREM VIII.

THE Difference betwixt any two Powers of the same Root, is the Product of the Difference of the Root and 1, multiplied by the Sum of all the Powers of that Root from the lesser given Power to that next below the greater.

Example 1. $a^n-a^r=\overline{a-1}\times\overline{a^r+a^{r+1}+, \&c.+a^{n-1}}$.

2. $a^n-a=\overline{a-1}\times\overline{a+a^2+a^3, \&c.\ a^{n-1}}$.

DEMON. This is a manifest Consequence of the preceding.

THEOREM IX.

TAKE the Series of Powers of any two Numbers (or Roots) to any the same Length or Index. To each of these Series prefix 1; then set the one of these Series under the other in a reverse Order, and multiply the corresponding Terms of the one Series into those of the other; then take the Sum of the Products: I say, if this Sum is multiplied by the Difference of the given Roots, the Product is equal to the Difference of their similar Powers of the degree next above the highest in the Series.

$$\begin{array}{ccccc} a^4, & a^3, & a^2, & a, & 1. \\ 1, & b, & b^2, & b^3, & b^4. \\ \hline a^4+ & ba^3+ & b^2a^2+ & ab^3+ & b^4. \\ & & a-b. & & \\ \hline & & a^5-b^5 & \text{Product.} & \end{array}$$

Example. Take any two Roots a, b; take their Powers to the 4th; the two Series formed and multiplied as in the Margin make the Series of Products, $a^4+ba^3+a^2b^2+ab^3+b^4$; which multiplied by $a-b$ produces a^5-b^5.

Universally:

$$\begin{array}{ccccccc} a^n, & a^{n-1}, & a^{n-2}, & \&c. & a^2, & a, & 1. \\ 1, & b, & b^2, & \&c. & b^{n-2}, & b^{n-1}, & b^n. \\ \hline a^n+ & ba^{n-1}+ & b^2a^{n-2}, & \&c & a^2b^{n-2}+ & ab^{n-1}+ & b^n. \\ & & & a-b. & & & \\ \hline & & & a^{n+1}-b^{n+1}. & & & \end{array}$$

DEMON. The Reason of this appears the same way as that of *Theor.* 7. for $a^n\times a=a^{n+1}$, and $b^n\times b=b^{n+1}$; then the Product of a into every Term after a^n, is destroyed by that of b (which is to be subtracted) into the preceding.

COROL. Hence the Difference of any two Roots is an aliquot Part of the Difference of any their similar Powers.

SCHOL. From the Doctrine of the next Book, *Chap.* 3. you'll find the Investigation, and another Demonstration of this and *Theor.* 7, and 8. *viz.* from the Consideration of Geometrical Progressions. But I have placed them here, because they have a Demonstration independent of these Progressions: And *Theor.* 7. furnishes us another Demonstration for the summing of these Progressions.

ARITH-

ARITHMETICK.

BOOK IV.

The Doctrine of PROPORTION.

CHAP. I.

Explaining the general Nature of Proportion.

DEFINITIONS.

I. NUMBERS are compared in order to discover certain Relations they have to one another; and as to every Comparison there must be two Terms or Things, *viz.* one which is compared, and another to which it is compared, so must it be also in Numbers; where more particularly the Number compared is called the *Antecedent*; and the Number to which it is compared is called the *Consequent*. For Example; if we compare 3 to 4, 3 is called the *Antecedent* and 4 the *Consequent*.

SCHOL. Every Comparison is reciprocal, and includes a Comparison of each Term to the other; but because the Relation may be different according as the one or the other is made the Antecedent (as we shall see below); and because all Comparisons must be of things of like Species, therefore, in order to compare again the Relations of different Couplets of Numbers (as you'll find afterwards), we must carefully distinguish the two particular Comparisons that may be made in every Couplet; which is done by this Distinction of Antecedent and Consequent, according as they are applied to the two Numbers.

II. The Comparisons and consequent Relations of Numbers are of two kinds, distinguished by the Names of Arithmetical and Geometrical.

(1.) *Of* Arithmetical *Relation.*

If we compare Numbers so as to consider their simple Differences, or how much the Antecedent is greater or lesser than the Consequent; their Relation in that View is called *Arithmetical*; and the Difference of these two Numbers is called the *Arithmetical Ratio,* or *Exponent* of the Arithmetical Relation of the Antecedent to the Consequent. Example;

If

If we compare 3 to 5, the Arithmetical Ratio is 2, ſignifying this Relation, *viz.* that 3 is leſs than 5 by 2; and reciprocally, if we compare 5 to 3 the Ratio is alſo 2, ſignifying, in this Caſe, that the Antecedent 5 is greater than the Conſequent 3, by 2.

If the Numbers compared are equal, their Arithmetical Ratio is 0, ſignifying their Equality.

Here then we ſee Arithmetical Relation diſtinguiſhable into two Kinds, *viz.* a Relation of Equality and Inequality; and the laſt again diſtinguiſhable into two Species, *viz.* a Relation of *Exceſs*, when the Antecedent is greateſt, and of *Defect*, when the Antecedent is leaſt. Now it would ſeem reaſonable that the Exponents of different Relations, as the mutual Relations of unequal Numbers are, ſhould be different; yet here it is the ſame Number, whether the Antecedent is greateſt or leaſt; and therefore that it may determine the Species of the Relation, we muſt apply the Words *Exceſs* and *Defect*; or ſome Mark to ſignify it in Writing. So if you ſay that 3 is the Arithmetical Ratio betwixt two Numbers, as 4 and 7, it may be applied two ways, according as we ſuppoſe the Antecedent greateſt or leaſt, for there are neceſſarily two mutual Relations betwixt two Things; yet by ſaying 3 in Exceſs or Defect, or uſing ſome Mark of Diſtinction in Writing, the Compariſon is determined. But the Terms compared being known, the moſt ſimple and natural Method of determining the Compariſon is to put the Antecedent always before the Conſequent (as theſe Names do import), with the Word *To* betwixt them. For Example; To ſay the Relation of 4 *to* 7, is a particular and determinate Compariſon, whereas to ſay the Relation *betwixt* 4 and 7 is ambiguous, for this may be either the Relation of 4 to 7, or of 7 to 4. When the Relation of different Couplets are to be again compared, this Determination is abſolutely neceſſary.

(2) *Of* Geometrical *Relation.*

If we compare two Numbers ſo as to conſider how often the Antecedent contains or is contain'd in the Conſequent, the Relation in that View is called *Geometrical*, and the Quote of the greater Term divided by the leſſer, (which ſhews the *how oft* required) is called the *Geometrical Ratio*, or Exponent of the Geometrical Relation of the Antecedent to the Conſequent. *Ex. gr.* If we compare 4 to 12, the Geometrical Ratio is 3, ſignifying this Relation of 4 to 12, *viz.* that it is contain'd in it 3 times; and reciprocally, the Ratio of 12 to 4, is 3, ſignifying the Relation of 12 containing 4, 3 times. Again; the Ratio of 4 to 19, or 19 to 4, is $4\frac{3}{4}$, ſignifying this Relation, *viz.* of 4 being contain'd in 19, or 19 containing 4 four times, and $\frac{3}{4}$ Parts of it. Of two equal Numbers, the Geometrical Ratio is always 1, expreſſing a Ratio of Equality.

Geometrical Relation is alſo diſtinguiſhed into two kinds, one of Equality and another of Inequality; and the laſt again into two Species, *viz.* a Relation of *containing*, when the Antecedent is greateſt, and of *being contained*, when the Antecedent is leaſt; both which having the ſame Number for their Exponent, it muſt be applied differently according to the different Caſes. But the Compariſon is clearly and certainly determined in the ſame manner as is already explain'd in Arithmetical Relation, *viz.* by putting the Antecedent before the Conſequent, with the Word *To* betwixt them; thus, to ſay the Relation of 4 to 12, is determinate and certain; and in this Caſe the Ratio 3 ſignifies that 4 is contain'd 3 times in 12: And if you call it the Ratio of 12 to 4, it ſignifies a Relation of *Containing*.

But again *obſerve*; There is another way of conceiving the Geometrical Relation of a leſſer Number to a greater, whereby the Exponents of the two reciprocal Relations will be different: Thus, in comparing a leſſer to a greater Number, we may conſider what Part or Parts it is of the greater; and then the Fraction expreſſing this is the Geometrical Ratio: For Example; the Ratio of 4 to 12 is $\frac{1}{3}$; and ſo by this Method the Ratios, or Exponents of the Reciprocal different Relations of two Numbers, will always be different, and of themſelves determine the Quality or Species of the Relation which they expreſs, *i. e.* whether

whether the Antecedent is greater or lesser, and that whether the Numbers compared are known or not: Thus, the Antecedent being greater, the Ratio will always be a whole or mixt Number; so the Ratio of 12 to 4 is 3, or of 14 to 4 is $3\frac{1}{2}$; but the Antecedent being least, it will be a proper Fraction; so the Ratio of 4 to 12 is $\frac{1}{3}$, and of 4 to 14 is $\frac{2}{7}$. Now by the common Rules we find what Fraction the Antecedent is of the Consequent, by dividing that Term by this; thus, if they are both Integers, the Quote and Fraction sought is got by making the Antecedent Numerator, and the Consequent Denominator, (which is perhaps not in its lowest Terms, but that is no matter;) so 4 is $\frac{4}{14}$, or $\frac{2}{7}$ of 14. And if they are not both Integers, then we proceed by the Rules given in Division of Fractions; so the Ratio of 4 to $13\frac{2}{3}$ is $\frac{12}{41}$, for $13\frac{2}{3} = \frac{41}{3}$, by which dividing 4, the Quote is $\frac{12}{41}$, which expresses what Fraction 4 is of $13\frac{2}{3}$ from the Nature of Division.

Again; Since the Fraction expressing the Relation of a lesser to a greater, is equal to the Quote of the Antecedent divided by the Consequent; therefore, according to the Distinction of the reciprocal Relations of two Numbers, now explain'd, the Ratio is in all Cases the Quote of the Antecedent divided by the Consequent, expressing how oft the Antecedent, when it is the greater Term, contains the Consequent; or, when it is least, what Part or Parts, *i. e.* what Fraction it is of the Consequent. And this was the Method of the *Antients* in explaining Geometrical Relation: But this we may reduce to a more uniform Notion, whereby the same general Definition comprehends both the mutual Relations of Inequality, and the Ratios they have to the same Number, thus: All Quotes are Fractions, and may be expressed fractionally; for by making the Dividend Numerator, and the Divisor Denominator, when they are both Whole Numbers, that is a Fraction equivalent to the Quote; and if they are not both Integers, yet being divided by the common Rule, the Quote will come out in Form of a Fraction (tho' in some Cases it may be equal to a Whole Number). And hence the Relation of a greater to a lesser may be conceived under a like View with that of a lesser to a greater, above explained, *viz.* as being equal to a certain Number of Parts of the Consequent; and the improper Fraction expressing this is the Ratio, found out the same way as in the other Case. For Example; The Ratio of 7 to 5 is $\frac{7}{5}$, and of 7 to $2\frac{2}{5}$ it is $\frac{35}{12}$, for $2\frac{2}{5} = \frac{12}{5}$: From which we may deduce this general Definition of a Geometrical Relation, *viz.* That it is *the Relation of the Antecedent being equal to a certain Part or Parts,* i. e. *a certain Fraction proper or improper of the Consequent*; which Fraction is the Ratio, expressing the Relation of a lesser to a greater when it is a *proper* Fraction, but of a greater to a lesser when it is improper. Hence, lastly, we may reason about these Ratios under the Notion of Quotes or Fractions indifferently, these being in effect the same: And the great use of this is, that from the Doctrine of Fractions we can easily deduce the Theory of Geometrical Relation and Proportion, as you'll find afterwards done.

Now as to these different Ways of conceiving Geometrical Relation, it is indifferent as to the Truth of the Science built upon it, which we chuse; for they answer equally true to all the Ends and Purposes of comparing the Relations of Numbers; because they depend so upon one another, that in comparing the Relations in two different Couplets, if the Relations taken one Way are equal, they will be so taken the other way: Thus, if the Antecedent is greatest, the Ratio is the same either way; for both the Methods of taking it coincide, and it is either a Whole or Mixt Number, or their Equivalent improper Fra-

ction: If the Antecedent is leaſt, then the Ratio taken the one way is the reciprocal Quote or Fraction of the Ratio taken the other way, and conſequently two Ratios being equal the one way, they are ſo the other way alſo, becauſe the Reciprocals of equal Fractions or Quotes are alſo equal. Thus, for Example; If the Ratios of 10 to 14, and of 15 to 21, taken the one way, are both equal, *viz.* $1\frac{2}{5} = \frac{7}{5}$, then alſo muſt the Ratios taken the other way in both be $\frac{5}{7}$; or alſo thus, as $\frac{14}{10} = \frac{21}{15}$ is the Ratio the one way; ſo alſo will $\frac{10}{14} = \frac{15}{21}$ be the Ratio taken the other way (from *Lem.* 6. *Ch.* I. of Fractions.)

I have explain'd both theſe Methods of conceiving and expreſſing the Geometrical Relation of Numbers, becauſe you'll meet with both in other Books on this Subject; but chiefly becauſe I find it convenient to follow ſometimes the one Method and ſometimes the other; and ſometimes for Variety and Illuſtration to apply both Methods.

COROLLARIES.

1. Comparing theſe different Notions of a Geometrical Ratio, with the common Operations of multiplying and dividing, this will evidently follow, *viz.* That the Ratio of two Numbers taken the firſt way, (which is always a whole or mixt Number) is that Number by which the leſſer Term being multiplied, or the greater divided, (which ſoever of them is the Antecedent) gives the other Term; and taken the ſecond way, (which may be either a whole or mixt Number, or a proper Fraction) it is that Number by which the Antecedent being divided, or the Conſequent multiplied, (which ſoever of them is the greateſt) gives the other: Or alſo thus; By whoſe Reciprocal the Antecedent being multiplied, or the Conſequent divided, gives the other. If this is not evident, it may be made ſo thus:

If the Ratio is taken the firſt way, the Reaſon of this is manifeſt from the mutual Proof of Multiplication and Diviſion Thus: If A is leſs than B, and if $B \div A = d$, which is the Ratio; then $Ad = B$, and $Ad \div d$, that is, $B \div d = A$. Again; If the Ratio is taken the other way, ſuppoſe A is the Antecedent, the Ratio of A to B is this Quote $\frac{A}{B}$, and its reciprocal $\frac{B}{A}$. Now if A, B are both Integers, $\frac{A}{B}$ and $\frac{B}{A}$ are really fractional Expreſſions in Terms; and from the Rules of Fractions, it is plain that if the Antecedent A is divided by $\frac{A}{B}$, the Quote is B; or if B, the Conſequent is multiplied by it, the Product is A. Again; Take the reciprocal Ratio $\frac{B}{A}$, and the Reverſe of the former Operations produce the ſame Effect. Thus: A multiplied by $\frac{B}{A}$ produces B, and B divided by it quotes A: all which is evident from the common Rules. But again: If A, B are not both Integers, then ſuppoſe the Quote $\frac{A}{B}$, taken in its own proper Terms, is this fractional Expreſſion $\frac{n}{o}$ (for every Quote is either a Fraction proper, or improper, which is expreſſible fractionally) then is $A = \frac{n}{o}$ of $B = B \times \frac{n}{o} = B \div \frac{o}{n}$. Again; Since $A = \frac{n}{o}$ of B, therefore $B = \frac{o}{n}$ of $A = A \times \frac{o}{n} = A \div \frac{n}{o}$, wherein you ſee all the Parts of the Rule.

Example in Numbers. Thus: The Ratio of 12 and 4 taken the firſt way is 3; then $12 \div 3 = 4$, and $4 \times 3 = 12$. Again; Make 4 the Antecedent, and 7 the Conſequent; the Ratio taken the ſecond way is $\frac{4}{7}$; then $4 \div \frac{4}{7} = 4 \times \frac{7}{4} = 7$, and $7 \times \frac{4}{7} = 7 \div \frac{7}{4} = 4$.

2. *Hence*

2. *Hence again:* Having one Number given, and the Ratio betwixt it and another, we can easily find that other; providing also, that if the Ratio is a whole or mixt Number, it be determined whether the greater or lesser Term is sought. And if it is a proper Fraction, let it be determined whether the Number sought is the Antecedent or Consequent. *Exam.* Let there be given one Number 8, and 3 the Ratio betwixt it and another Number, to find that other Number. If 8 is the greater Term of the Relation, or if you call it the Antecedent, then $8 \div 3 = 2\frac{2}{3}$ is the Number sought; or if it is the lesser Term or Consequent, then is $8 \times 3 = 24$, the Term sought. Again; If 8 is the given Term, and $\frac{3}{4}$ the Ratio, then 8 being the Antecedent, $8 \div \frac{3}{4} = 8 \times \frac{4}{3} = \frac{32}{3} = 10\frac{2}{3}$ is the Consequent; but if 8 is the Consequent, then $8 \times \frac{3}{4} = 8 \div \frac{4}{3} = 6$ is the Antecedent.

Observe, To prevent such tedious Names as *Arithmetical* and *Geometrical Ratios,* we shall hereafter use the Word *Difference* for the first, and *Ratio* for the other.

III. *Of* PROPORTION.

When the same Kind of Relation, Geometrical or Arithmetical, is consider'd in each of two or more Couplets of Numbers, whose Terms are also similarly compared, *viz.* the lesser to the greater, or the greater to the lesser in each; then if these similar Relations are equal, *that is,* if their Exponents are equal Numbers, this Equality is call'd *Proportion*; which we may therefore more briefly define thus, *viz.* Proportion is the Equality of the *Differences* or *Ratios* of two or more Couplets of Numbers, whose Terms are similarly compared. And then the Numbers stated in Order as they are compared, the Antecedent of each Couplet before its Consequent are said to be proportional Numbers, or *Proportionals,* Arithmetical or Geometrical, according as we consider *Differences* or *Ratios.* *Exam.* 3, 4, and 5, 6 are arithmetically proportional; because their Antecedents are both least, and their Differences are equal, *viz.* 1; also these 7, 3, and 5, 9, the Antecedents being both greatest, and the Difference in each being 4.

Again; These 3, 6, and 4, 8 are geometrically proportional; the Antecedent in each being the lesser, and the Ratio equal, *viz.* 2 taken the one way, and $2\frac{1}{2}$ taken the other: Also these 9, 2, and 18, 4, whose Ratio taken either way is $4\frac{1}{2}$

Universally: If $A - B = C - D$, then A, B, C, D are arithmetically proportional: and if $\frac{A}{B} = \frac{C}{D}$, then A, B, C, D, are geometrically proportional.

SCHOLIUMS.

1. Among Arithmetical Proportionals, 0 may be one Term. *Exam.* 0, 2, and 3, 5 have the same Difference, *viz.* 2. But all the Terms of Geometrical Proportionals must be real Quantities.

2. The two Numbers compared are ordinarily written with some Mark betwixt them, (the first written being the Antecedent, and the other the Consequent); and two Couplets, whose similar Relations are equal, (*i. e.* which are proportional) are written with some different Mark betwixt them: and it being convenient to distinguish the Marks of Arithmetical and Geometrical Relation, and their Equalities or Proportion, these distinguishing Marks I shall make thus: Betwixt two Numbers compared Arithmetically, (or in respect of Differences) I shall set a Point, thus, 3.5; and betwixt two such Couplets, whose Difference is equal, a Colon, thus, 3.5 : 4.6. Again; Betwixt two Numbers compared Geometrically, (or in respect of Ratios) set a Colon, thus, 3 : 4; and betwixt two such Couplets, whose similar Ratios are equal, a double Colon, thus, 3 : 4 :: 6 : 8; and then these Marks

ſave the Trouble of telling what kind of Compariſon is made of the Numbers. *Univerſally*: Theſe expreſs Arithmetical Proportionals, $a.b:c.d$; and theſe Geometrical, $a:b::c:d$; and then we read or expreſs the Proportionality thus, *viz.* In Arithmeticals we ſay, As much as the one Antecedent a exceeds, or wants of its Conſequent b; ſo much does the other Antecedent c exceed, or want of its Conſequent d; and when it is ſo betwixt four Numbers, they are Arithmetically proportional. In Geometricals we ſay, As oft as $\bar{a}$ contains, or is contain'd in b; ſo oft c contains, or is contained in d. Or alſo thus: What Fraction (proper or improper) a is of b; the ſame, or equal, Fraction is c of d.

Obſerve alſo, That Proportionality may be read thus more generally, *viz.* As a to b, ſo is c to d, Arithmetically or Geometrically; underſtanding this to ſignify what is above explain'd. And indeed the word Proportion, or Proportionality, is no more than a Word contriv'd to expreſs more briefly the Equality of Relation explain'd; ſo to ſay that four Numbers are proportional, is only ſaying all in one Word what muſt be ſaid in a great many, if explained at large, as has been now done.

The Diſtinction then betwixt Differences or Ratios, and Proportion, is this: A Difference or Ratio ariſes from the Compariſon of two or more ſimilar Relations, whoſe Exponents being equal make Proportion; which therefore can't exiſt without at leaſt four Terms (*i. e.* two Couplets); but obſerve, that the ſame Number may be antecedent in one Couplet, and conſequent in another; and therefore there may be Proportion, where there are but three different Numbers, if the one is twice taken: So theſe 3.4 5 are Arithmetically proportional, for $3.4:4.5$; and theſe Geometrically, $3:6:12$, for $3:6::6:12$.

It is to be obſerv'd too, that ſometimes the word Proportion is uſed for Ratio; ſo we ſay the Proportion of 3 to 6 for the Ratio of 3 to 6.

IV. *Of* Proportion, *Conjunct and Disjunct.*

When two or more Couplets are proportional, but ſo as they have no common Term (*i. e.* the Antecedent of none of them is the ſame Number as the Conſequent of another), they are called *Disjunct* Proportionals; as theſe Arithmeticals, $2.3:4.5:6.7$: and theſe Geometricals, $1:2::3:6::4:8$. But if the Conſequent of the firſt Couplet is the ſame Number as the Antecedent of the ſecond, and ſo on, the Conſequent of every Couplet the Antecedent of the next, theſe are called *Conjunct* (or continued) Proportionals; as theſe, $1.2:2.3:3.4$: and theſe, $1:2::2:4::4:8$. But in this Caſe the common Term need not be twice written; for it is enough to ſet all the different Terms in one continued Rank or Series, according to their Order of Compariſon; thus, $1.2.3.4.$ or $1:2:4:8$; wherein it is underſtood that every Term is compared to the following; ſo that each is taken both as an Antecedent and a Conſequent, except the firſt, which is only Antecedent, and the laſt, which is only Conſequent. Such Series of Numbers are alſo called Arithmetical and Geometrical *Progreſſions.*

V. *Of* Proportion, *Direct and Reciprocal.*

When the Ratios of two Couplets are ſimilar and equal, the Proportion is called *Direct,* as in theſe, $1.2:3.4$: and theſe, $1:2::4:8$. But if they are diſſimilar, yet ſo as the one is equal to the Reciprocal of the other, the Proportion is called *Reciprocal*; as in theſe, $1.2:4.3$, and in theſe, $1:2::6:3$.

Scholium. This Diſtinction may be applied either to Arithmeticals or Geometricals, tho' it is commonly applied only to the laſt; as to which *obſerve,* That, according to *Definition* V. Proportion is no other thing than what is here called *Direct,* the other proceeding merely from the inverting one of the Couplets of a Direct or Proper Proportion; So that to ſay $1:2::6:3$ are proportional reciprocally, is no more than to ſay that

theſe Numbers are proportional, if you invert any one of the Couplets, thus, 2 : 1 : : 6 : 3; this Diſtinction is therefore of no uſe in diſcovering or applying any Properties of Numbers; but it has taken riſe from the Circumſtances of ſome particular kind of mixt practical Queſtions; in which, tho' the Numbers are ſtated in the proper Order of Proportion, yet the Proportion is ſaid to be reciprocal from a certain Conſideration of the Order of the Terms as they lie in the Subject of the Queſtion, of which you'll ſee Examples in the proper Place.

VI. *Of* Proportion, *Harmonical.*

If four Numbers are ſuch, taken in a certain Order, that the firſt has the ſame Ratio to the fourth, as the Difference of the firſt and ſecond has to the Difference of the third and fourth, theſe Numbers are ſaid to be in *Harmonical* Proportion. Example: 9, 12, 16, 24, are harmonically proportional, *viz.* 9 : 24 : : 12 — 9 : 24 — 16, (or 9 : 24 : : 3 : 8) but if the firſt is to the fourth, as the Difference of the third and fourth to the Difference of the firſt and ſecond, it is called *Contra-harmonical* Proportion; as 2, 8, 4, 6, where 2 : 6 : : 6 — 4 : 8 — 2 (*i. e.* 2 : 6 : : 2 : 6.).

SCHOLIUMS.

1. This kind of Proportion may exiſt alſo betwixt three Numbers, one of which is compared as a middle Term to both the Extremes; that is, if the firſt has the ſame Ratio to the third, as the Difference of the firſt and ſecond to the Difference of the ſecond and third; or as the Difference of the ſecond and third to that of the firſt and ſecond; for ſuch coincide with the former by taking the middle Term twice. Example: 3, 4, 6, are Harmonical, becauſe 3 : 6 : : 4 — 3 : 6 — 4 (*viz.* 3 : 6 : : 1 : 2); and theſe Contra-harmonical, 3, 5, 6; becauſe 3 : 6 : : 6 — 5 : 5 — 3 (*viz.* 3 : 6 : : 1 : 2). And if in a Series of Numbers every three Terms are Harmonical, it is an Harmonical Series or Progreſſion, (*i.e.* a Series of Conjunct or Continued Harmonicals.) *Example:* 10, 12, 15, 20, 30, 60, wherein every three Terms next together are Harmonical. And when of four Terms the two middle are different, that may be call'd Disjunct Harmonical Proportion.

2. But this conſiderable Difference is to be obſerv'd betwixt this kind of Proportion and the former, *viz.* That here in a Series of continued Harmonicals every four Terms adjacent will not alſo be Harmonical, as they are Geometrical or Arithmetical in theſe kinds of Progreſſion. *Example:* Tho' 10, 12, 15, are Harmonical, and alſo 12, 15, 20; yet 10, 12, 15, 20, are not ſo.

3. If we take the Word *Proportion* ſtrictly, according to the general Definition, for an Equality of ſimilar Relations betwixt two Numbers, then it muſt be own'd there is no new Species of Proportion here; for there is no new kind of Relation in any two Terms, but only a mixt Compariſon betwixt two Numbers and their Differences from others; and the Proportion conſtituted is really a Geometrical one, or an Equality of Ratios; not indeed of the given Numbers, but of the Ratio of the Extremes, compared to the Ratio of the Differences of theſe Extremes with the middle Terms: But tho' this is not ſtrictly a Proportion among the given Numbers immediately compared together, yet it is certainly a new and Complex Relation (diſtinct from the immediate Arithmetical or Geometrical Relation betwixt the ſeveral Couplets of Numbers themſelves), to which alſo the Name *Proportion* is applied.

This Denomination of *Harmonical* ariſes from *Muſick*; becauſe the Relations or Proportions of Sounds that make Harmony are found to be of the Kind here defin'd, of which you ſhall find a more particular Explication afterwards.

AXIOMS.

AXIOMS.

Obſerve, The following *Axioms* are expreſſed ſo as to regard Geometrical Relations only; and the Ratio is to be underſtood as the Quote of the Antecedent divided by the Conſequent: But by putting the Word *Difference* in place of Ratio, they are alſo applied to Arithmetical Relations or Differences.

Axiom I. Two or more equal Numbers (integral or fractional) A and B, have all the ſame Ratio to the ſame Number D, and D has the ſame Ratio to equal Numbers. Alſo theſe Numbers, A, B, *&c.* are equal, which have all the ſame Ratio to the ſame Number D, or to which the ſame Number D has the ſame Ratio. Again; Two equal Numbers have the ſame Ratio to any other two alſo equal; ſo if $A=B$, and $C=D$, then $A:C::B:D$; this being the ſame as $A:C::A:C$.

Coroll. If $A:B::C:D$, and $L:M::N:O$, then $\frac{A}{B}:\frac{L}{M}::\frac{C}{D}:\frac{N}{O}$; for $\frac{A}{B}=\frac{C}{D}$, and $\frac{L}{M}=\frac{N}{O}$.

II. Two different Numbers, A and B, have different Ratios to the ſame Number D; and the greater or leſſer of the two, A and B, has the greater or leſſer Ratio to D. Again; The ſame Number D has a different Ratio to two different Numbers A and B, and it has reciprocally the leſſer or greater Ratio to the greater or leſſer of the two Numbers A, B. Alſo that is the greater or leſſer of two Numbers A, B, which has the greater or leſſer Ratio to one Number D; and that one of two Numbers, A, B, is the greater or leſſer to which the ſame Number D has the leſſer or greater Ratio.

Coroll. To the ſame three Numbers, A, B, C, there canont be two different Numbers that make a 4th Proportional; nor to two Numbers, two that are each a third Proportional.

III. If each of two Ratios is equal to a third, they are equal to one another. *Example:* If $A:B::C:D$, and $A:B::F:G$, then is $C:D::F:G$.

Coroll. If one Ratio is equal to another, and this equal to a 3d, and this 3d to a 4th, and ſo on; the firſt Ratio is equal to the laſt, and each of them equal to each; ſo that the Terms of any two of them are proportional Numbers. *Example*: If $A:B::C:D::E:F::G:H$, then is $A:B::G:H$.

IV. Two equal Ratios, $A:B$ and $C:D$, are both equal to, or both greater, or both leſſer than any third Ratio $G:H$. But *obſerve,* That the Reverſe holds only when they are both equal to $G:H$, *i. e.* they are then only equal to one another; for they may be both greater or leſſer, and yet not equal; there being Degrees of Inequality, but none of Equality.

V. Two equal Ratios are ſtill equal in whatever Order they are taken: So that of two Couplets that make Proportion, it is indifferent which of them is ſet firſt or laſt. *Example:* If $A:B::C:D$, then alſo $C:D::A:B$.

VI. All Ratios of Equality are equal Ratios; ſo $A:A::B:B::C:C$, *&c.* but equal Ratios are not always Ratios of Equality; and theſe things muſt be carefully diſtinguiſhed: For a Ratio of Equality is the Ratio betwixt two equal Numbers; but two Ratio's may be equal, tho' they are both Ratio's of Inequality, *i. e.* of unequal Numbers.

General COROLLARIES.

I. *For* Arithmetical *Proportion.*

I. In *Addition*; If two Numbers are added together, then o, the two Numbers and the Sum are arithmetically proportional. *Example:* $2+4=6$, and $0.2:4.6$. Universally, $0.a:b.a+b$.

II. In *Subtraction*; o, the Difference Subtractor and Subtrahend, are arithmetically proportional. *Example:* $12-9=3$, and $0.3:9.12$: Universally, $0.a-b:b.a$.

II. *For* Geometrical *Proportion.*

III. In *Multiplication*; 1 is to any one of the Factors in the same Ratio as the other Factor is to the Product, *i. e.* these four are geometrically proportional. *Example:* $3\times4=12$, and $1:3::4:12$, or $1:4::3.12$: for the Product 1 contains the one Factor as oft as the other expresses or contains Unity. Universally, $1:a::b:ab$, or $1:b::a:ab$.

IV. In *Division*; The Divisor or Quote is to Unity as the Dividend is to the other, (*i e.* these four are geometrically proportional). Example: $18\div3=6$, and $6:1::18:3$, or $3:1::18:6$. Universally, $A\div d=q$, then $d:1::A:q$, or $q:1::A:d$; for d is contain'd in A, q times, or q in A, d times; that is, as oft as q or d expresses or contains Unity.

V. The Numerator of a simple Fraction is to the Denominator, as the Fraction (or Quantity expressed by it) is to Unity. Example: $\frac{2}{3}:1::2:3$. Universally, $\frac{a}{n}:1::a:n$; for a contains $\frac{a}{n}$ Parts of n, and $\frac{a}{n}$ signifies $\frac{a}{n}$ Parts of 1.

Observe, This is but a particular Case of the last; for, call $\frac{a}{n}=q$, and then $q:1::a:n$. Here the Terms a, n, are always integral; but in the other the Dividend and Divisor, a, d, may be integral or fractional. Again; 1 is to any Fraction as the Denominator to the Numerator; $1:\frac{a}{b}::b:a$; for $1\div\frac{a}{b}=\frac{b}{a}$. Hence, lastly, 1 is a mean Proportional betwixt any Fraction and its Reciprocal; so $\frac{a}{b}:1::1:\frac{b}{a}$.

VI. Of two equal Fractions, their Numerators and Denominators are geometrically proportional. *Example:* $\frac{2}{3}=\frac{4}{6}$, and $2:3::4:6$; also $2:4::3:6$; for because $\frac{2}{3}=\frac{4}{6}$, therefore $\frac{2}{4}=\frac{3}{6}$, (*Lem.* 6. *Chap.* 1. *B.* II.) Hence $2:4::3:6$. *Universally,* if $\frac{a}{n}=\frac{r}{s}$, then $a:n::r:s$, and $a:r::n:s$; for equal Fractions being equal Quotes, or Ratios, their Terms are in Proportion; it being the same thing to say that $\frac{a}{n}=\frac{r}{s}$, as to say that these are proportional $a:n::r:s$; the Equality of these Quotes or Fractions constituting the Proportion.

VII. Two Fractions having a common Denominator, are proportional with their Numerators. *Example:* $\frac{2}{5}:\frac{3}{5}::2:3$; for the Quote of $\frac{2}{5}$ by $\frac{3}{5}$ is the Quote of 2 by 3, *viz.* $\frac{2}{3}$.

$\frac{2}{3}$. *Universally*, $\frac{a}{n} : \frac{b}{n} :: a : b$. For $\frac{a}{n} \div \frac{b}{n} = \frac{a}{b}$, (*Cor.* 1. to the Rule for Division of Fractions.)

VIII. Two Fractions having a common Numerator, are proportional with their Denominators taken in a reverse Order. *Example*; $\frac{2}{5} : \frac{2}{7} :: 7 : 5$; or $\frac{a}{n} : \frac{a}{r} :: r : n$; for $\frac{a}{n} \div \frac{a}{r} = \frac{r}{n}$, or $r \div n$, by Division of Fractions; for by the general Rule the Quote is $\frac{ar}{an}$, which reduces to $\frac{r}{n}$

Observe, That in this Case the Fractions are said to be reciprocally proportional with their Denominators; because the one Fraction is to the other, as the Denominator of the last to that of the first; and is one Example of a true and direct Proportion, said to be reciprocal, because of the reverse Order in taking the Denominators, with respect to the Order in which the Fractions are taken.

IX. Any Fraction is to another in the same Ratio, as the Product of the Numerator of the first by the Denominator of the other, is to the Product of the Denominator of the first by the Numerator of the other, (*i. e.* as the new Numerators, when the Fractions are reduced to a common Denominator.) *Example*; $\frac{2}{3} : \frac{5}{7} :: 14 : 15$; or $\frac{a}{n} : \frac{r}{s} :: as : nr$; because the Quote of these two Products, or new Numerators, is the Quote of the two Fractions by Division of Fractions; and the Quotes are the Ratios.

X. Unity is a Geometrical Mean betwixt any Number and its Reciprocal. *Example*: $3 : 1 :: 1 : \frac{1}{3}$, and $\frac{2}{3} : 1 :: 1 : \frac{3}{2}$. *Universally*: $A : 1 :: 1 : \frac{1}{A}$, and $\frac{A}{B} : 1 :: 1 : \frac{B}{A}$.

XI. A Fraction compounded of two Fractions, or the Product of two simple Fractions, is to any of them as the Numerator of the other is to its Denominator. Thus, $\frac{A}{B}$ of $\frac{C}{D}$, or $\frac{AC}{BD} : \frac{A}{B} :: C : D$, and $\frac{AC}{BD} : \frac{C}{D} :: A : B$. In Numbers $\frac{2}{3}$ of $\frac{5}{7}$, or $\frac{10}{21} : \frac{2}{3} :: 5 : 7$.

Hence any square Fraction is to its Root, as the Numerator to the Denominator of the Root. Thus $\frac{A^2}{B^2} : \frac{A}{B} :: A : B$. In Numbers, $\frac{4}{9} : \frac{2}{3} :: 2 : 3$.

XII. Two Fractions are as their alternate Fractions. Thus; $\frac{A}{B} : \frac{C}{D} :: \frac{A}{C} : \frac{B}{D}$, the common Ratio being $\frac{AD}{BC}$. In Numbers, $\frac{2}{3} : \frac{5}{7} :: \frac{2}{5} : \frac{3}{7}$.

XIII. Any two Numbers are in the same Ratio with their Reciprocals taken reciprocally. *Example*: $2 : 3 :: \frac{1}{3} : \frac{1}{2}$; and $\frac{2}{3} : \frac{4}{7} :: \frac{7}{4} : \frac{3}{2}$. *Universally* in Integers, $A : B :: \frac{1}{B} : \frac{1}{A}$; and in Fractions $\frac{A}{B} : \frac{C}{D} :: \frac{D}{C} : \frac{B}{A}$, the common Ratio being $\frac{AD}{BC}$.

And

And obſerve, That ſince the Quote of any Number divided by another, is reducible to a fractional Expreſſion; alſo ſince the Reciprocal of that Quote is the Quote of the ſame Numbers, changing the Diviſor into the Dividend and this into that, which we call the reciprocal Quote; therefore the Quotes of any two Couplets of Numbers are in the ſame Ratio as the reciprocal Quotes taken reciprocally; which we may expreſs generally thus: $A \div B : C \div D :: D \div C : B \div A$: for let $A \div B = \frac{a}{b}$, and $C \div D = \frac{o}{n}$; then is $B \div A = \frac{b}{a}$, and $D \div b = \frac{n}{o}$, (by what is ſhewn in Diviſion of Fractions) and by what is now ſhown $\frac{a}{b} : \frac{o}{n} :: \frac{n}{o} : \frac{b}{a}$.

XIV. Any Number whatever is to any of its Multiples or Fractions in the ſame Ratio as any other Number is to its like Multiple or Fraction. This is the immediate Conſequence of the Definition; for that Likeneſs proceeds from, or conſtitutes an equal Ratio, and may be repreſented in any of theſe Forms, *viz.* $A : An : B : Bn$; or $A : A \div n :: B : B \div n$, as has been ſhewn in Diviſion of Fractions, where n is the Ratio; and according as it repreſents a whole Number or a Fraction, ſo will An, and $A \div n$ repreſent a Multiple or Fraction of A. And if $\frac{n}{m}$ is a Fraction, proper or improper, then this Truth may alſo appear thus: $A : \frac{n}{m}$ of $A :: B : \frac{n}{m}$ of B: for $\frac{n}{m}$ of $A = A \div \frac{m}{n}$, and $\frac{n}{m}$ of $B = B \div \frac{m}{n}$, as has been ſhewn in Diviſion of Fractions; and ſo $\frac{m}{n}$ is the Ratio, by which the Antecedents being divided give the Conſequents.

XV. Any two Numbers whatſoever are in the ſame Ratio, or proportional with any their like Multiples or Fractions; *i. e.* as the Products or Quotes of theſe Numbers multiplied or divided by any the ſame Number. Thus: $A : B :: An : Bn$; or $A : B :: A \div n : B \div n$; which Proportionality follows from this, That the like Fractions or Multiples of any two Numbers whatſoever, are the ſame Fractions one of the other as theſe Numbers are, (*Cor.* 5, 6. *Lem.* 2. *Chap.* 1. *B.* II.) and like or equal Fractions conſtitute equal Ratios.

XVI. The Quotes of the ſame Number divided ſeparately by any two different Diviſors are in the reciprocal Ratio of the Diviſors. Thus: $A \div n : H \div m :: m : n$; for whatever kind of Numbers A, n, m repreſent, $A \div n$, $A \div m$ repreſent certain Fractions into which theſe Quotes will reſolve; and $n \div A$, $m \div A$ repreſent the Reciprocal of theſe Fractions. Hence $A \div n : A \div m :: m \div A : n \div A$, (*Cor.* 13.) and $m \div A : n \div A :: m : n$, (by the laſt); therefore $A \div n : A \div m :: m : n$, (*Ax.* 3.)

XVII. Any the like Fractions of any two Numbers, are to one another in the ſame Ratio as any other like Fractions of the ſame two Numbers. Thus: $\frac{a}{b}$ of $A : \frac{a}{b}$ of $B :: \frac{c}{d}$ of $A : \frac{c}{d}$ of B; becauſe the Ratio of each of theſe Antecedents to its Conſequent, is that of $A : B$, (*Cor.* 15.); therefore they are equal, (*Ax.* 3.)

XVIII. Any two different Fractions of the ſame Number are to one another in the ſame Ratio as the ſame two Fractions of any other Number. Thus, $\frac{a}{b}$ of $A : \frac{c}{d}$ of $A :: \frac{a}{b}$ of $B : \frac{c}{d}$ of B. The Reaſon of this is $\frac{a}{b}$ of $A = A \div \frac{b}{a}$, and $\frac{c}{d}$ of $A = A \div \frac{d}{c}$, (as has been ſhewn in Diviſion of Fractions) and $A \div \frac{b}{a} : A \div \frac{d}{c} :: \frac{d}{c} : \frac{b}{a}$ [*Cor.* 16.]. For the

the ſame Reaſons, $\frac{a}{b}$ of B $=$ B $\div \frac{b}{a}$, and $\frac{c}{d}$ of B $=$ B $\div \frac{d}{c}$: alſo B $\div \frac{b}{a}$: B $\div \frac{d}{c}$ :: $\frac{d}{c}$: $\frac{b}{a}$. Hence, (by *Ax.* 3.) A $\div \frac{b}{a}$: A $\div \frac{d}{c}$ :: B $\div \frac{b}{a}$: B $\div \frac{d}{c}$; that is, $\frac{a}{b}$ of A : $\frac{c}{d}$ of A :: $\frac{a}{b}$ of B : $\frac{c}{d}$ of B.

XIX. If one Number A is divided by another B, and the Quote q be divided or multiplied by another n, this laſt Quote or Product is the ſame Number that will be found by taking the Quote or Product of A divided or multiplied by the ſame n, and dividing it by B. Thus, if $A \div B = q$, then $A n \div B = q n$, and $\overline{A \div n} \div B = q \div n$. The Reaſon is eaſy: For we ſhall ſuppoſe that $A n \div B = Q$, then are q and Q Like Fractions of A and A n, becauſe of the ſame Diviſor B, and therefore $A : A n :: q : Q$, [*Cor* 15.] But $A : A n :: q : q n$ [*Cor.* 14.], therefore $Q = q n$ (*Ax.* 2. *Cor.*): And again, ſuppoſing $\overline{A \div n} \div B = Q$, then $A : A \div n :: q : Q$; but $A : A \div n :: q : q \div n$; hence $Q = q \div n$.

XX. If one Number A is divided by another B, and the Quote q be divided or multiplied by another n, this laſt Quote or Product is the ſame that will be found by taking reverſly the Product or Quote of B, multiplied or divided by the ſame n, and by it dividing A. Thus; if $A \div B = q$, then $A \div B n = q \div n$; alſo $A \div \overline{B \div n} = q n$: For ſuppoſe $A \div B n = Q$, then $q : Q :: B n : B$ [*Cor.* 16.]; but $B n \div n = B$, therefore $q : Q :: B n : B n \div n$; alſo $q : q \div n :: B n : B n \div n$ [*Cor.* 14.]. Hence $Q = q \div n$ (*Ax.* 2.).

XXI. From theſe two laſt we ſee again evidently, that if any Diviſor and Dividend are equally multiplied or divided, the Quote made of theſe new Numbers is the ſame as that made of the Numbers multiplied or divided: Thus $A \div B = A n \div B n$.

General SCHOLIUM.

It has been frequently obſerved, that ſuch an Expreſſion as this, $\frac{A}{B}$, does not repreſent directly and immediately a Fraction in Terms, unleſs A and B do both repreſent Integers; yet $\frac{A}{B}$ may very well expreſs the Quote of A divided by B, tho' it's but a general and indeterminate Expreſſion thereof; for it is in effect no more than a Sign or Mark for theſe Words, *The Quote of* A *divided by* B, and expreſſes the Quote only in the ſame indeterminate manner as theſe Words do. But now from what has been ſhewn in *Coroll.* 19. and 20. we have learn'd this very remarkable thing, *viz.* How any Quote, tho' it's only expreſſed in this general Form, may be multiplied or divided; *i.e.* how another Expreſſion of the ſame kind may be found from the given Terms, equal to the Product or Quote of the given Quote by any given Number: For which this is the Rule, *viz.* Multiply or divide the given Dividend, and apply to the Product or Quote the given Number, and you have the Product or Quote ſought. Again, reciprocally, divide or multiply the given Diviſor, and apply the Product or Quote to the given Dividend, and you have alſo the Product or Quote ſought. Thus, for Example: $\frac{A}{B} \times n = \frac{A n}{B} = \frac{A}{B \div n}$, and $\frac{A}{B} \div n = \frac{A \div n}{B} = \frac{A}{B n}$, taking theſe as general Expreſſions of Quotes: For ſuppoſe $\frac{A}{B} = q$, it's ſhewn that $q n = A n \div B$, (*Coroll.* 19.) $= A \div \overline{B \div n}$ (*Coroll.* 20.): Alſo that $q \div n = \overline{A \div n} \div B$ (*Coroll.* 19.) $= A \div B n$ (*Coroll.* 20.); wherefore taking theſe other Forms which expreſs the

the ſame things, it is $\frac{A}{B} \times n = \frac{An}{B} = \frac{A}{B \div n}$, and $\frac{A}{B} \div n = \frac{A \div n}{B} = \frac{A}{Bn}$. *Obſerve* alſo, That theſe Expreſſions are right, whether n repreſent an Integer or Fraction; ſince to multiply or divide by a Fraction is a Mixt Operation of Multiplication and Diviſion; ſo that in this Caſe An, $A \div n$, Bn, $B \div n$, do all repreſent the Effect of a Mixt Operation, whoſe Parts are both juſtly applied to the Diviſor or Dividend, to repreſent the Multiplying or Dividing the Quote or Fraction $\frac{A}{B}$.

Upon which Rule this is to be *obſerved,* That Quotes expreſſed in this general manner are multiplied and divided the ſame way as Fractions are, (which are Quotes of Integers, and therefore but a particular Caſe of this general one) doing the ſame with the Diviſor and Dividend as with the Denominator and Numerator of a Fraction.

Hence follows, That all the Rules of Operations in Fractions are truly applicable to Quotes expreſſed in this general Form; becauſe all the Reaſons of theſe Operations in Fractions have been demonſtrated from the Conſideration of the Equi-multiplication and Diviſion of the Fraction, by equi multiplying or dividing its Numerator and Denominator; which is the whole Foundation of the Demonſtrations of the Rules for multiplying and dividing Fractions. Then as to the Addition and Subtraction of Fractions, the Demonſtration takes in alſo the Conſideration of *Lemma* 2. and 3. *Ch.* 1. *Book* II. which are of the ſame Uſe in the Addition and Subtraction of Quotes; becauſe all Quotes are ſuch Fractions of the Dividend as the Reciprocal of the Diviſor expreſſes (as has been frequently mention'd), which I ſhall here very briefly apply. In the firſt Place, Two Quotes are reduced to a common Diviſor the ſame way as two Fractions to a common Denominator: thus $\frac{A}{B}$ and $\frac{C}{D}$ are reduced to theſe, $\frac{AD}{BD}$ and $\frac{BC}{BD}$; becauſe the Equi-multiplication of the Diviſor and Dividend makes an equal Quote (*Coroll.* 21.) and $\frac{AD}{BD} + \frac{BC}{BD} = \frac{AD+BC}{BD}$: For whatever the Diviſor BD is, the Quote is that Fraction of the Dividend expreſſed by the Reciprocal of the Diviſor; and the Sum of the Like Fractions of two Numbers is the Like Fraction of the Sum of theſe Numbers (*Lemma* 2. *Ch.* 1. *B.* II.). Again; $\frac{AD}{BD} - \frac{BC}{BD} = \frac{AD-BC}{BD}$ (*Lemma* 3. *Ch.* 1. *Book* II.)

We muſt alſo *obſerve,* That beſides theſe common Operations, all the other Properties of Fractions whoſe Reaſon depends upon Multiplication and Diviſion of their Terms, are the ſame way applicable to Quotes; particularly, all theſe relating to Geometrical Proportion of their Terms, explained in the preceding *Corollaries*: To which remember to join theſe three Properties of equal Fractions in *Lem.* 6. *Ch.* 1. *Book* 2, *viz.* That theſe Fractions or Quotes being equal $\frac{A}{B} = \frac{C}{D}$, then, 1°, $AD = BC$; 2°. $\frac{A}{C} = \frac{B}{D}$; 3°. $\frac{B}{A} = \frac{D}{C}$.

Now the Uſe of all that has been explained concerning theſe general and indeterminate Expreſſions of Quotes, is in the Demonſtration of the following Theory of Proportion, eſpecially the Geometrical Kind; for to make a general and compleat Theory, we muſt conſider both Integers and Fractions; and where any Theorem is true univerſally, whether applied to Integers or Fractions, then to make the Demonſtration alſo univerſal, and yet as ſimple as poſſible, it's neceſſary to make any Letter, A, or B, repreſent any Number, Integer or Fraction, indifferently; and then all the Properties of proportional Numbers depending upon the Conſideration of their Ratios (which are Quotes; ſo the Ratio of A to

B is the Quote of A divided by B, expreſſed $\frac{A}{B}$, or $A \div B$), and the Demonſtrations depending upon the Multiplication and Diviſion of theſe Quotes; therefore, by ſhewing that the Operations of Fractions and their other Properties are the ſame in Ratios or Quotes thus expreſſed, we learn an univerſal Method of Demonſtration (ſo far as the Multiplication and Diviſion of the Ratios or Quotes are concerned) whether the Ratio in the given Terms is a real Fraction, as when A, B are both Integers, or if it's only looked upon as a more general Expreſſion of a Quote, as when A, B, are both, or one of them, Fractions; and indeed we ſhall find that as equal Ratios or Quotes, and equal Fractions, are in effect the ſame thing; ſo the Properties of Fractions already explained, applied to Quotes, do contain all the Truths hereafter propoſed concerning proportional Numbers, or they may be deduced from them.

It's true indeed that there are various Methods of demonſtrating this Theory; one of which is independent of the things now explained concerning Quotes, and is perhaps the moſt ſimple and eaſy in many Caſes; yet other Methods are beautiful, and deſerve to be conſidered, as they ſerve to enlarge our Ideas and Knowledge: But beſides, for different Propoſitions different Methods are convenient; and as different Perſons may 'tis probable be pleaſed, ſome with one, ſome with another way of repreſenting and demonſtrating the ſame Truth, ſo ſome will be pleaſed with a Variety of Ways; therefore the Things I have explained concerning Quotes were neceſſary; and I have accordingly uſed Demonſtrations which ſuppoſe the Knowledge of them: And as I have uſed different Methods for different things, ſo I have alſo uſed a Variety of Demonſtrations for ſome particular Truths, where it could be done without being tedious. To conclude then, When you find in the following Theory any Quote thus repreſented, $\frac{A}{B}$ (or $A \div B$) and the Product or Quote of this by any other, as n, or $\frac{r}{n}$ (whatever Kind theſe repreſent), performed and expreſſed as if $\frac{A}{B}$ or $\frac{r}{n}$ were real Fractions; as thus, $\frac{A}{B} \times n = \frac{An}{B}, = \frac{A}{B \div n}$: Or $\frac{A}{B} \times \frac{r}{n} = \frac{Ar}{Bn}$. Alſo $\frac{A}{B} \div n = \frac{A \div n}{B} = \frac{A}{Bn}$; or $\frac{A}{B} \div \frac{r}{n} = \frac{An}{Br}$. Remember that all this is the Application of what has been here explained.

Of the ſeveral Kinds of Ratios, *as they were diſtinguiſhed by the* Ancients.

Before I end this Chapter, I ſhall give you an Account of the ſeveral Kinds into which Ratios were diſtinguiſhed by the Ancients; and which are yet upon occaſion made uſe of by ſome.

I. *Ratios* are either,

1. *Multiple,* when the Antecedent contains the Conſequent a certain Whole Number of times, without a Remainder; *i. e.* when it is a Multiple of it. *Exam.* 4 to 1, or 12 to 3.

2. *Super-Particular,* When the Antecedent contains the Conſequent once, and 1 over; as 4 to 3, or 5 to 4.

3. *Super-Partient,* When the Antecedent contains the Conſequent once, and a Remainder greater than 1; as 5 to 3.

Obſerve, That as the ſame Relation may be betwixt different Terms; ſo 5 : 3 : : 10 : 6. Theſe two Definitions ſuppoſe the loweſt Terms of the Ratio: (Which is by ſome called the *Exponent* of the Relation, tho' I apply this to the Quote or Fraction made of any two Terms in the ſame Relation, becauſe they are all equal; leaving them to be diſtinguiſhed

guiſhed by the loweſt Terms, and ſuch as are not ſo); For otherwiſe two Numbers may appear to be a Super-partient Ratio, which are Super-particular; ſo 4 : 3 :: 8 : 6; but 4 to 3 is Super-particular, and conſequently ſo muſt 8 to 6; and yet this appears Super-partient, unleſs we reſtrain the Definitions to the loweſt Terms.

4. *Multiple Super-particular*; When the Antecedent contains the Conſequent oftner than once with 1 over; as 5 : 2, or 13 : 4.

5. *Multiple Super-partient*; When the Antecedent contains the Conſequent oftner than once, and a Remainder greater than 1; as 12 : 5, or 18 : 7.

Again; They diſtinguiſhed the Ratios of the leſſer to the greater, by prefixing the Word *Sub* inſtead of *Super* in the preceding Names; thus, *Sub-Multiple*, as 1 : 4; *Sub-particular*, as 3 : 4; *Sub-partient*, as 3 : 5, and ſo on.

Alſo, They had particular Names for the ſeveral Species or Sub-diviſions of each Kind. So the Multiple are either double, triple, *&c.* the Super-particular were called *Seſqui-altera*, as 3 : 2; *Seſqui-tertia*, as 4 : 3, and ſo on; ſetting the Word *Seſqui* before the Name of the leſſer Term. For the *Super-partient*, they put the Name of the Number by which the Antecedent exceeds the Conſequent betwixt the Words *Super* and *Partient*, and the leſſer Term of the Ratio laſt of all; thus, 5 : 3 is called *Superbis partiens tertias*. But theſe Names are uſeleſs and troubleſome, ſince we can much eaſier and more plainly expreſs the Ratio by its Terms.

II. I ſhall next ſhew you a certain regular Method and Order, in which all poſſible Examples of each of theſe Kinds of Ratios may be found, and in their loweſt Terms.

1. For *Multiple* Ratios make 1 the leſſer Term, and any other Number from 2 *in infinitum*, the greater.

2. For *Super-particular* Ratios; Make any Number greater than 1 the leſſer Term, and the next greater Number make the greater Term; that is, compare every Term of the natural Series of Numbers from 2; as 2, 3, 4, 5, 6, *&c.* to the next.

3. For *Super-partient* Ratios; Take any Number above 2 for the leſſer Term, and to it add any leſſer Number which is not an aliquot Part of it, nor the Multiple of any aliquot Part of it greater than 1, that Sum is the greater Term: Thus you may find all the Super-partient Ratios which can have any aſſumed Number for the leſſer Term, ſuppoſing you can find all the aliquot Parts of any Number; as you'll learn in *Book* V. *Chap.* 1. The *Reaſon* of this is, That if n is not an Aliquot of a, nor the Multiple of any aliquot Part greater than 1, then a, and $a+n$ cann't have a common Meaſure (but 1); for if they have, then becauſe it meaſures a and $a+n$, it muſt alſo meaſure n; and conſequently n is either an aliquot Part or Multiple of ſome aliquot of $\bar{a}$, contrary to Suppoſition. Laſtly, becauſe n is leſs than $\bar{a}$, yet greater than 1; therefore $a+n$ to a is *Super-partient*. *Example*: 7 $(=4+3)$: 4.

4. The *Multiple*-Super-particular and Super-partient Ratios are eaſily found from theſe Rules.

III. I ſhall end this with another Obſervation upon the Dependence of the different Kinds of Ratios upon one another, and upon the Ratio of Equality.

1. Take any two equal Numbers, as 1 : 1, one of them compared to the Sum of both, is in double Ratio, or as 1 to 2. Again; the leſſer of theſe compared to the Sum of both is triple Ratio; as 1 : 3. By going on ſo, you'll have all the Multiple Ratios.

2. Of all the Species of Multiple Ratios, compare the greater Term to the Sum of both, you have all the Species of Super-particulars: Thus, from 1 : 2 comes 2 : 3, from 1 : 3 comes 3 : 4, and ſo on.

3. Of all the Super-Particulars, compare the leſſer to the Sum, you have all the double Super-particulars: So from 2 : 3 comes 2 : 5, from 3 : 4 comes 3 : 7. *&c.* Again; From the double Super-particulars come the Triple, as from 2 : 5 comes 2 : 7; and from the triple comes the Quadruple, and ſo on.

4. Of

4. Of all the Super-particular Ratios, compare the greater to the Sum, and they are Super-partient, and they are particularly of that Kind in which the Excess and the lesser Term are Super-particular; thus from $3:2$ come $5:3$, whose Exponent is $1\frac{2}{3}$; from $4:3$ come $7:4$, whose Exponent is $1\frac{3}{4}$. Or if you take these two Series, *viz.* 5, 7, 9, *&c.* still increasing by 2, and this, 3, 4, 5, *&c.* increasing by 1, compare their Terms in Order, and you have all these Super-partient Ratios, *viz.* $\frac{5}{3}.\frac{7}{4}.\frac{9}{5}.\frac{11}{6}.\frac{13}{7}.$ *&c.*

5. From the Super-partient Ratios proceed their Multiples the same way as from the Super-particulars.

The *Reasons* of all these things are obvious.

CHAP. II.

Of Arithmetical Proportion.

Observe: *To shorten writing, I mark the Words* Arithmetical Proportion, *or* Arithmetically Proportional, *by this Character,* :*l*; *and* Continued Arithmetical Proportion *by this* ∸ *l*.

§. 1. *Containing the more general Doctrine common to both* Conjunct *and* Disjunct Proportion.

PROB. I. *Three Numbers given, to find a 4th* :*l*.

Example 1.

Given. Sought.

2. 5 : 7. 10.

Operation.

$5-2=3$. then $7+3=10$.

Example 2.

Given. Sought.

9. 7 : 4. 2.

Operation.

$9-7=2$. then $4-2=2$.

RULE. TAKE the Difference of the first and second, and either add it to the third, (as in *Exam.* 1.) or subtract it from it, (as in *Exam.* 2.) according as the Antecedent is less or greater than the Consequent, (*i.e.* the first Term than the second) the Sum or Difference is the fourth sought. So universally to these, $a.b.c$, a 4th :*l* is $b-a+c$, (if b is greatest) or $c-\overline{a-b}=c-a+b$.

The Reason is plainly contain'd in the Definition of :*l*.

COROL. Since 3 Numbers :*l* are in effect the same Case as 4 Numbers, by taking the middle Term twice; therefore a 3d :*l* to two given Numbers is found by taking the Difference of the first and second, and adding it to, or subtracting it from the second. Which is also deduced from the same Definition as naturally as the former Case. *Exam.* Given 5. 8, the third :*l* is 11; or given 8. 5, the third :*l* is 2.

Universally to these a, b, a 3d is $b-a+b=2b-a$; or $b-\overline{a-b}=b-a+b=2b-a$.

SCHO-

SCHOLIUMS.

1. When a fourth :*l* is found, the four Terms will either be conjunct or disjunct, according as the Difference of the 2d and 3d is equal, or not, to that of the first and second; which is not considered in this *Problem*, nor in any of the following Propositions, where four Terms are mention'd.

2. A third or fourth :*l* is possible in all Cases, if the first Term is less than the second, but not if it is greater; for then the Difference may be greater than the second or third, and so cannot be subtracted; as in these 5.2, to which a third is impossible in real and positive Numbers; but if we bring in what are call'd, by the Algebraists, negative Numbers, or Numbers conceiv'd as less than nothing, then it is possible. For *Example:* To 5 and 2 I find a third, thus; Take the Difference $5 - 2 = 3$, and because it is greater than 2, and can't be subtracted from it, take $3 - 2 = 1$, and conceive this as it were subtracted from 0; then is $0 - 1$ the third Term sought, and the three stand thus, $5.2.0 - 1$. It is true indeed there is no such thing in Nature as a Number less than 0, and so this will be presently looked upon as chimerical. All I shall say here is, that for the common Applications it is of no Use; and for the Use that is made of it in Algebraical Reasonings and Calculations, I leave you to seek it in Books of that kind; yet thus much I shall shew you here, *viz.* that such an Expression answers in general to the Definition of Arithmetical Proportion, which requires only that the Difference of 5.2, and of $2, 0 - 1$ be equal: and from the Nature of Subtraction this will be so, if that Difference (which is 3) added to $0 - 1$ make 2, which it does; that is, $0 - 1 + 3 = 2$. For if to 0 we add 3, the Sum is 3; from which taking one, the Remainder is 2, *i. e.* $0 + 3 - 1 = 2$; which is the same thing in effect as $0 - 1 + 3$; for the Signs $+, -$ signify only the adding the one, or subtracting the other from some first Quantity; and we may begin with either of these Operations, unless that first Quantity, as here 0, make the Subtraction really impossible; and then we begin with the Addition.

THEOREM I.

Example:
$a.b:c.d.$
$3.5:7\ 9.$
and
$3+9=5+7=12.$

If four Numbers are :*l*, thus, $a.b:c.d$, the Sum of the Extremes is equal to the Sum of the middle Terms, $a+d = b+c$.

DEMON. If the Difference were 0, the thing would be evident; and if it is not so, yet the greater Extreme d exceeds its adjacent Mean c, as much as the lesser Extreme a wants of its adjacent Mean b; whence the Truth propos'd is manifest.

Or *thus*; Let any Whole be represented by $a+d$; if you take any Quantity from d so as the Remainder be c, and add as much to a, so as the Sum be b, it is certain that, $b+c = a+d$, because we have added and subtracted equally from $\overline{a+d}$.

Or from *Prob.* 1. thus: To these three a, b, c, a fourth :*l* is $\overline{a-b+c}$, or $b-a+c$; which added to a, the Sum is plainly $b+c$.

Or, *lastly*, Any four Numbers :*l* may be represented thus; $a.\overline{a+d}:b\ \overline{b+d}$. In which the *Theorem* is manifest, *viz.* $a+b+d = a+d+b$, being the same three Numbers on both Sides.

The *Reverse* of this *Theorem* is also true, *viz.* that of four Numbers taken in a certain Order, if the Sums of the Extremes and Means are equal, these Numbers are :*l*; that is, $a.b:c.d$ are :*l*, supposing $a+d = b+c$.

DEMONSTRAT. By equal Subtraction of a from both Sides, it is $d = b+c-a$. *Again*; By equal Subtraction of c, it is $d-c = b-a$; wherefore $a.b:c.d$, their Differences being equal.

Again;

Again; The contrary of this *Theorem* and its *Reverſe* is alſo true, *viz.* if four Numbers are not :*l*, the Sums of the Extremes and Means are not equal: for if this were, the Numbers would be : *l* (by the Reverſe); and if the Sums are not equal, the Numbers are not :*l*; for if they were, theſe Sums would be equal; by the *Theorem*.

Scholium. To ſave needleſs Repetition, I would have it obſerv'd once for all, That where-ever any *Theorem* and its *Reverſe* are true, the Contraries are alſo true; for they plainly follow from the other. Of which we may make this *general Demonſtration:* Let A and B repreſent two Propoſitions; then ſuppoſe that if A is true, B is alſo true; and reverſly, if B is true, A is alſo true: Their Contraries are alſo true, *i. e.* if A is not true, neither is B; for if this were, the other would be ſo alſo, (by the Reverſe of the firſt); and if B is not true, neither is A; for if this were, ſo would be the other, by the firſt. Now in all that follows, both in Arithmetical and Geometrical Proportions, where the Reverſe of a *Theorem* is true, I ſhall mention it, and, if reed be, demonſtrate it; if it is not true, I ſhall ſhew it by one *Example*, which will be ſufficient to make this appear; and therefore I ſhall never trouble you with mentioning the Contraries.

COROLLARIES.

I. If three Numbers, $a.b.c$ are :*l*, the Sum of the Extremes is equal to twice the Mean, or this is the half of that; *i. e.* $a+c=2b$: For by repeating the Mean thus, $a.b:b.c$, it is the ſame with the Caſe of four Numbers; or this Caſe may be alſo repreſented thus, $a.\ a+d.\ a+2d$; wherein the Truth propos'd is manifeſt; for the Sum of the Extremes is $2a+2d=2\times\overline{a+d}$. Or, by *Cor.* to *Prob.* 1. where we ſee that a 3d :*l* to a, b is $2b-a$, conſequently $2b-a+a=2b$.

$a.b\ c$
$2.5.8$
and
$2+8=2\times5=10.$

The *Reverſe* of this is alſo true; for if $a+c=2b$, then $b-a=c-b$; whence $a.b.c$ are :*l*.

II. Hence we have got another Rule for finding a third or fourth :*l*. *viz.*

1. For a third. Subtract the firſt from double the ſecond, and the Remainder is the third. *Example:* To theſe 3.7. a third is 11. Thus; $2\times7=14$, then $14-3=11$.

2. For a fourth. Subtract the firſt from the Sum of the ſecond and third, the Remainder is the fourth. *Example:* To theſe 12.5.8, a 4th is 1. Thus $5+8=13$, then $13-12=1$.

Problem II.

Of four Numbers :*l*, $a.b:c.d$, having the two Extremes a, d, and one Mean, to find the other.

Example:
$a.b:c.d$
$3.5:7.9$
Operation:
$3+9=12$, and
$12-5=7$, or
$12-7=5$.

Rule. From the Sum of the Extremes ſubtract the given Mean, the Remainder is the Mean ſought. Thus, $\overline{a+d}-b=c$.

Demon. This follows from the laſt *Theorem*, and is evident in this Repreſentation, $a.\overline{a+d}:b.\overline{b+d}$; wherein $a+b+d-b=a+d$.

Scholium. We may ſolve this Problem thus, *viz.* Take the Difference of the given Mean and its adjacent Extreme, and add or ſubtract it with the other Extreme, as the Caſe requires.

Prob-

PROBLEM III.

To find an Arithmetical Mean, b, *betwixt two given Numbers,* a *and* c.

RULE. Take half the Sum of the given Numbers, it is the Mean sought. Thus, $b = \frac{a+c}{2}$.

Example: $a=4.\ c=14.$ $b=9=\frac{4+14}{2}=\frac{18}{2}$.

DEMONSTR. This follows from the *Coroll.* last *Theorem*, or may be made evident thus; $a.\ a+d.\ a+2d$ are three Numbers $\div l$, and the Sum of the Extremes is $2a+2d$, whose half is $a+d$ the middle Term.

SCHOLIUM. We may solve this Problem thus; Take half the Difference of the given Numbers, add it to the lesser, or subtract it from the greater, and you have the Mean sought. The Reason is obvious.

THEOREM II.

IF four Numbers are $:l,\ a.b:c.d$, they will be so alternately, thus, $a.c:b.d$.

Example: $2.6:3.7$ $2.3:6.7$

DEMONSTR. This is clear from *Theor.* 1, because the middle Terms being still the same, only in a different Order, their Sum is the same; and being equal to the Sum of the Extremes, hence they are $:l$, also in the alternate Order.

Or we may demonstrate the *Theorem* thus; Since $a.b:c.d$, then $a+d=b+c$, *Theor.* 1. take away a and also b from both Sides, and we have $d-b=c-a$; whence $a.c:b.d$.

LEMMA.

Part I. If two Numbers are added to, or subtracted from other two orderly, *i. e.* the lesser of the one with the lesser of the other, and the greater with the greater: The Sums will differ by the Sum, and the Differences by the Difference of the Differences of the given Numbers.

Example 1. 6.9 added to 5.7 the Sums are $6+5=11$, and $9+7=16$; then $16-11=5=9-6\ (=3)+7-5\ (=2)$.

Example 2. 5.7 subtracted from 8.13, the Remainders are $8-5=3$, and $13-7=6$; then $6-3=3=13-8\ (=5)-7-5\ (=2)$.

DEMONSTR. Let $a\ a+d$ and $b.\ b+c$ represent any given Numbers; their Sums, taken as proposed, are $a+b$, and $a+d+b+c$, whose difference is plainly $d+c$ the Sum of the given Differences. Again; Their Differences are $a-b$ and $a+d-b-c$, which we may also express $\overline{a-b}+\overline{d-c}$, or $\overline{a-b}-\overline{c-d}$, according as d or c is greatest; in which Expressions the Difference from $a-b$ is $d-c$, or $c-d$, the Difference of the given Differences d and c: For if d is greater than c, we must take it $\overline{a-b}+\overline{d-c}$, exceeding $\overline{a-b}$ by $d-c$: And if c is greater than d, we take $\overline{a-b}-\overline{c-d}$, wanting of $\overline{a-b}$, the Difference $c-d$.

Part II. If two Numbers are added to, or subtracted from other two in a contrary Order, (*viz.* the lesser with the greater) the Sums differ by the Differences, and the Differences by the Sums of the first Differences.

Ex. 1.	Diff.	*Ex.* 2.	Diff.	Universally:	Diff.
4 . 7	3	8 . 15	7	$a \;.\; a+d$	d
6 . 2	4	7 . 3	4	$b+c \;.\; b$	c
10 . 9	1	1 . 12	11	Sums $a+b+c \;.\; a+b+d$	$c-d$. or $d-c$.
				Diffs. $a-b-c \;.\; a+d-b$	$d+c$
				Or $b+c-a \;.\; b-a-d$	$c+d$.

Demonstr. For the Sums, the thing is plain in the Universal *Example* annex'd. For the Differences it may be a little further explained, thus; $a+d-b$ is plainly greater than $a-b-c$, and their Difference is $\overline{a+d-b}-\overline{a-b-c}=a+d-b-a+b+c=d+c$. Again; $b+c-a$ is greater than $b-a-d$, and their Difference is $\overline{b+c-a}-\overline{b-a-d}=b+c-a-b+a+d=c+d$.

Coroll. *to the* 1st *Part*. If two Numbers are equally multiplied or divided, the Products or Quotes differ by the like Multiple or aliquot Part of the given Difference: Because Multiplying and Dividing is but a repeated Adding and Subtracting. Or it may also appear thus; $an-bn=\overline{a-b}\times n$, and $\frac{a}{n}-\frac{b}{n}=\frac{a-b}{n}$.

Theorem III.

If four Numbers :*l* are added to, or subtracted from other four also :*l*, in order, the lesser with the lesser, and greater with the greater Term in the respective Couplets, the Sums or Differences are also :*l*, with a Difference which is the Sum or Difference of the given Differences (*Exam.* 1. and 2.) But if they are added or subtracted in a contrary Order, the Sums and Differences are also :*l*; but the Sums have a Difference equal to the Difference, and the Differences have a Difference equal to the Sum of the given Differences (*Exam.* 3. and 4.).

Example 1.	Diff.	*Example* 2.	Diff.	*Example* 3.	Diff.	*Example* 4.	Diff.
2 . 4 : 3 . 5	2	6 . 4 : 9 . 7	2	3 . 5 : 8 . 10	2	14 . 13 : 12 . 11	1
5 . 6 : 9 . 10	1	5 . 2 : 4 . 1	3	12 . 9 : 4 . 1	3	7 . 9 : 3 . 5	2
7 . 10 : 12 . 15	3	1 . 2 : 5 . 6	1	15 . 14 : 12 . 11	1	7 . 4 : 9 . 6	3

Demonstr. The universal *Reason* of both Parts of this *Theorem* is plainly contained in the preceeding *Lemma*: For the four Numbers being :*l*, the Sums or Differences are in the same Difference, which by the *Lemma* is according to what's here proposed.

COROLLARIES.

1. If four Numbers :*l*, or any Series $\div l$ are equi-multiplied or divided, the Products or Quotes are also :*l* or $\div l$, with a Difference which is the like Product or Quote of the given Difference: Thus, If $a\,.\,b:c\,.\,d$, then $an\,.\,bn:cn\,.\,dn$, and $\frac{a}{n}\,.\,\frac{b}{n}:\frac{c}{n}\,.\,\frac{d}{n}$.

2. If two Series $\div l$ are orderly added to or subtracted from one another, the Sums or Differences are $\div l$, with a Difference that is the Sum or Difference of the given Differences: And if they are added or subtracted in a contrary Order, they are also $\div l$; but the Sums differ by the Difference, and the Differences by the Sum of the given Differences.

Example

Exam. 1.	Diff.	*Exam.* 2.	Diff.	*Exam.* 3.	Diff.	*Exam.* 4.	Diff.
3. 5. 7. 9.11	2	3.6.9.12.15	3	1. 3. 5. 7. 9	2	11.13.15.17.19	2
2. 6.10.14.18	4	2.4.6. 8.10	2	14.11. 8. 5. 2	3	8. 7. 6. 5. 4	1
5.11.17.23.29	6	1.2.3. 4. 5	1	15.14.13.12.11	1	3. 6. 9.12.15	3

§. II. *Of Arithmetical* Progreſſion.

Obſerve, *By the* Diſtance *of one Term of a Series from another, is meant, The Number of Terms from the one excluſive to the other incluſive; or, including both, it's the Number of Terms leſs* 1.

Problem IV. *To raiſe an* Arithmetical Series *from a given Number, with a given Difference.*

Rule. Add the given Difference to, or ſubtract it from, the given Number; the Sum or Remainder is the 2d Term: To or from which add or ſubtract the ſame Difference, and you have the 3d Term; and thus you may go on to any Number of Terms, increaſing; but decreaſing, you can go no farther than till the laſt Term found is equal to or leſs than the Difference.

Demonstr. The *Reaſon* of this *Rule* is plainly contained in the Definition of Arithmetical Progreſſion.

Example 1. Given, the 1ſt Term 3, and common Difference 2; the Series is 3 . 5 . 7 . 9 . 11 . 13 . *&c.*

Example 2. The 1ſt Term 20, and Difference 4; a decreaſing Series is 20 . 16 . 12 . 8 . 4.

SCHOLIUMS.

1. An Arithmetical Progreſſion can be raiſed, increaſing *in infinitum,* from any given Number, but not decreaſing; for by continual leſſening of any Number you muſt at laſt exhauſt the whole.

2. If the two firſt Terms of a Series be called $a . b$, the whole may be repreſented thus, $a . b . 2b - a . 3b - 2a . 4b - 3a$. *&c.* for theſe are the Expreſſions that ariſe by finding a 3d $: l$ to $a . b$, and a fourth to $a . b$, and the Term laſt found, and ſo on (by *Prob.* 1.) Where *obſerve,* That the Number multiplying b is the Diſtance of that Term from a. But again;

3. If we call the leſſer Extreme of a Series a, the greater l, and the common Difference d; then, by the Method of the preſent *Problem,* it's manifeſt that the Increaſing Series from a is repreſented thus, $a . a+d . a+2d . a+3d$. &c. to $. l$. and the Decreaſing Series thus, $l . l-d . l-2d$. &c. to a: from which Repreſentations theſe *Corollaries* are evident.

COROLLARIES.

1. If the leſſer Extreme of a Series is the common Difference, the other Terms are the ſeveral Multiples of that Difference. Thus: If $a = d$, then $a, a+d, a+2d,$ *&c.* is the ſame as this, $d, 2d, 3d,$ *&c.* Wherefore the Series of the Multiples of any Number is an Arithmetical Progreſſion.

2. If o is the firſt Term, the ſecond is the common Difference; and all the following are the Series of the Multiples of that Difference: Thus; $0, d, 2d, 3d,$ *&c.*

3. Be-

3. Betwixt o and any Multiple of a Number d, as nd, there can be put as many Arithmetical Means as the Multiplier less 1, or $n-1$; the common Difference being that Number d; as is manifest in this Series o, d, $2d$, $3d$.

4. Every Arithmetical Progression, whose first Term is not o, has its several Terms equal to the Sums of the several Terms of such a Series, added to the first Term of the given Series. Thus; In a, $a+d$, $a+2d$, *&c.* they are equal to the Sums of a added to each Term of this Series o, d, $2d$, *&c.*

5. As any Progression may be thus represented, a, $a+d$, $a+2d$, *&c.* a being the lesser Extreme, and d the common Difference, it is manifest that the Multiplier of d in every Term expresses the Distance of that Term from the first. And hence it follows immediately, that

6. Every Term is equal to the Sum of the lesser Extreme, and such a Multiple of the Difference d, whose Multiplier is the Distance of that Term from a. And *reversly*, the lesser Extreme is equal to the Difference of any greater Term, and such a Multiple of the common Difference, whose Multiplier is the Distance of that Term from the lesser Extreme. *Thus:* Let the Distance of any Term from a be m, that Term is $a+md$; which if we call l, then is $l=a+md$; and reversly, $a=l-md$.

Scholium. Because the greater Extreme of a Series is, by what's now shown, equal to $a+md$, or $a+\overline{n-1}\times d$, (m being the Distance of the Extremes $=n-1$, the Number of Terms less 1) and every Term below having the common Difference once less contained in it, or multiplied by a Number less by 1 than its Multiplier in the preceding greater Term; therefore a decreasing Series which, when the greater Extreme is called l, we have seen represented thus, *viz.* l, $l-d$, $l-2d$, *&c.* may also be represented thus, $a+md$, $a+\overline{m-1}\,d$, $a+\overline{m-2}\,d$, *&c.* Or thus; $a+\overline{n-1}\,d$, $a+\overline{n-2}\,d$, $a+\overline{n-3}\,d$, *&c.* (because $m=n-1$) going on so till the Multiplier of d be equal to 1; and then we have $a+d$, the Term next a.

7. But again more universally: From the same Expression of a Series, it's manifest that the Difference of the Multipliers of d in any two Terms expresses the Distance of these two Terms; for in every Term ascending, the Difference is taken once more than in the preceding; and therefore from any Term to any other, it's as many times oftner taken in the greater than in the lesser, as their Distance expresses, *i. e.* the Difference of the Multipliers of d is their Distance. And hence it follows immediately, that

8. Any Term of an Arithmetical Progression is equal to the Sum or Difference of any other Term; and such a Multiple of the common Difference, whose Multiplier is the Distance of these Terms. Also the Difference of any two Terms is equal to such a Multiple of the common Difference, whose Multiplier is the Distance of these Terms. For any Term being expressed $a+md$, (m the Distance of this Term from a) any greater Term must have a Multiple of d, whose Multiplier exceeds m by the Distance of these Terms (by the last); so that Distance being r, the greater Term is $a+\overline{m+r}\,d$. But $\overline{m+r}\,d=md+rd$. Hence $a+\overline{m+r}\,d=a+md+rd$. And reversly, $a+\overline{m+r}\,d-rd=a+md$; and $a+\overline{m+r}\,d-a-md=rd$. Or this Truth may also be deduced from *Cor.* 6. Thus; Any Part of a Series, *i.e.* from any Term to any other, is still an Arithmetical Progression, whereof these two Terms are the Extremes; which being called a, L, and their Distance m, it's shewn that $a=\mathrm{L}-md$, and $\mathrm{L}=a+md$; and hence, *lastly*, $\mathrm{L}-a=md$.

Exam. In this Series, 3. 5. 7. 9. 11. 13. 15. If we compare 5 and 13, whose Distance is 4; then is $5=13-8$ ($=2+4$) and $13=5+8$; also $13-5=8$.

Scholium. The immediate Use and Application of this last Truth we have in the Solution of these Problems.

1. Having

(1.) Having one Term of a Series, and the *common Difference*, to find a Term at any Diſtance from the given one, without finding all the intermediate ones; the Solution of which is plainly contained in this *Corol.* and is this; Multiply the common Difference by the given Diſtance, the Sum or Difference of this Product, and the given Term, is the Term ſought; ſo in the preceding *Exam.* $13 = 5 + 8$ $(= 2 \times 4)$ and $5 = 13 - 8$.

(2.) Having any two Terms and their Diſtance, to find any other Term at any Diſtance from either of the given Terms; which is ſolved thus: Take the Difference of the given Terms, and divide it by the given Diſtance, the Quote is the *common Difference*: By which we can find a Term at any Diſtance from either of the given Terms by the Method of the preceding.

(3.) Of theſe three things, *viz.* the *common Difference*, the Diſtance of any two Terms, and the Difference of theſe two Terms, having any two, the third Term may be found; for any Term being called a, d the common Difference, and m the Diſtance of any greater Term from a, that Term is $= a + md$; and the Difference of a, and $a + md$, is md; which if we call M, then (1.) If d, m are given, M $(= md)$ is alſo known. (2.) If M and m are given, d is alſo known; for it is $= M \div m$. (3.) If M and d are given, m is known; for it is $= M \div d$.

9. The Sum of the Extremes (or of any other two Terms) of an Arithmetical Progreſſion, is equal to twice the leſſer added to the Product of their Diſtance and *common Difference*. Thus; The leſſer Extreme a, the greater l, the Diſtance m, and the *common Difference* d; I ſay, $a + l = 2a + md$; for $l = a + md$, by *Coroll.* 6. wherefore $a + l = a = a + md = 2a + md$.

Scholium. We ſhall apply this to ſome particular Progreſſions, as,

(1.) Suppoſe the leſſer Extreme 1, and the *common Difference* 2, as in this Series 1 : 3 : 5 : 7, *&c.* (which we call the natural Series of odd Numbers) the Sum of the Extremes is always double the Number of Terms; which if we call n, then $a + l = 2n$; for the leſſer Extreme being 1, its double is 2, equal to the *common Difference*; therefore $a + l = 2a + md = 2 + 2m = 2 \times \overline{1 + m} = 2n$, becauſe $n = 1 + m$.

(2.) Suppoſe the leſſer Extreme is the *common Difference*, the Sum of the Extremes is equal to the Product of the *common Difference*, and Number of Terms $+ 1$. Thus; Since $a = d$, then $a + l = 2a + md = 2d + md = d \times \overline{2 + m} = d \times \overline{n + 1}$, becauſe $m = n - 1$; and therefore $m + 2 = n - 1 + 2 = n + 1$. Or ſee it thus; Such a Series is a, $2a$, $3a$, *&c.* na, and $a + na = a \times \overline{n + 1}$.

Yet more particularly, if the leſſer Extreme and Difference is 2, as in this Series 2 : 4 : 6 : 8 : 10, *&c.* (which we call the natural Series of even Numbers); then is $a + l = 2 \times \overline{n + 1} = 2n + 2$; that is, double the Number of Terms $+ 1$; or the Sum of 2, and twice the Number of Terms.

10. The Difference of the Extremes, (or of any two Terms) is equal to the Product of their Diſtance by the common Difference; for $l = a + md$, therefore $l - a = md$.

Theorem V.

If there are two Arithmetical Series having the ſame common Difference, any two Terms in the one are $::$, or have the ſame Difference with any two in the other, taken at the ſame Diſtance.

Exam. In this Series, 2. 4. 6. 8. 10. 12; and this, 5. 7. 9. 11. 13. 15. Theſe are $::$ *viz.* 2. 8 : 5. 11; and theſe 4. 12 : 7 : 15.

Demon.

Demon. In each Pair the greater contains the leſſer, and the *common Difference* taken the ſame Number of Times, (*i. e.* the ſame Multiple of *d*) becauſe of the equal Diſtances.

Theorem VI.

Any two Terms of an Arithmetical Series are : *l*, with any other 2 taken at the ſame Diſtance.

Exam. In this Series, 3.5.7.9.11.13.15; theſe are : *l*, 3.7 : 11.15; and 3.9 : 5.11.

Again; If you take any 3 or more Terms equally diſtant from one another, they make a Progreſſion, or continued Series; ſo in the preceding Series, 3.9.15, or 3.7.11.15.

Demon. The Reaſon is the ſame as in the laſt *Theorem*.

COROLLARIES.

1. If from the Sum of any two Terms you take any other Term, the Remainder is equal to a Term of the Series as far diſtant from one of the Terms added on the one hand, as the Subtractor is from the other of them on the other hand: For ſince any four Terms are : *l*, whereof the two leſſer are at the ſame Diſtance as the two greater, (and conſequently the leaſt and that next to the greateſt at the ſame Diſtance as the greateſt and that next to the leaſt,) therefore, from the Sum of the two middle Terms of theſe four, any of the Extremes being ſubtracted, the Remainder is the other Extreme. Or from the Sum of the Extremes, one of the Means being taken, leaves the other Mean; whence the *Coroll.* is manifeſt.

Obſerve again: If the Subtractor is one of the Extremes, *i. e.* lies on the ſame hand of (or is leſſer or greater than) either of the Terms added, the Remainder will be the other Extreme, *i. e.* will lie on the oppoſite hand of (or be contrarily greater or leſſer than) either of the Terms added; and conſequently the Diſtance of the Remainder from the Subtractor will be the Sum of the Diſtances of both the Terms added from the Subtractor, (or from the Remainder it ſelf, which is at the ſame Diſtance): But if the Subtractor is one of the Means, *i. e.* lies betwixt (or is leſs than one, and greater than the other of) the Terms added, the Remainder will be the other Mean, or lies alſo betwixt the Terms added; and conſequently its Diſtance from the Subtractor is the Difference of the Diſtances of the Terms added from the Subtractor, (or from the Remainder, which is the ſame Diſtance.

Exam. In this Series, A.B C.D.E.F.G.H, it's true that $C + F - A = H$. Becauſe A.C : F.H, which is as far from F on the one hand, as A is from C on the other hand; and as far from A, as the Sum of the Diſtances of C and F from A. Again; $A + H - C = F$, which is as far from H on the one hand, as C is from A on the other; and as far from C, as the Difference of the Diſtances of A and H from C.

2. If any Term of a Series is doubled, and from that double another Term ſubtracted, the Remainder is a Term of the ſame Series, as far diſtant from the Term doubled on the one hand, as the Subtractor is on the other; and conſequently the Remainder is twice as far from the Subtractor, as the Term doubled is. Hence reverſly, the half of the Sum of any two Terms is equal to a Term in the middle of the Terms added, if there is ſuch a middle Term. But however, this is true, That the Term which is in the middle betwixt any two Terms is the half of their Sum.

So in the preceding Series, $2D - B = F$, and $D = \frac{B + F}{2}$, becauſe B.D.F are ÷ *l*.

3. Again; More univerſally, If any Term of a Series ÷ *l* is multiplied by any Number, and from the Product be ſubtracted the Product of another Term by a Multiplier leſs than the former by 1; the Remainder is a Term of the ſame Series, whoſe Diſtance from

that Term whoſe Multiple is ſubtracted is equal to the Product of theſe two, *viz.* the Number which multiplied the Term whoſe Multiple is the Subtrahend, and the Diſtance of the two given Terms from one another. The *Reaſon* will be plain from this *Example*: Let any two Terms of a Series be A . B; a Series continued from theſe is A . B . 2 B — A . 3 B — 2 A . 4 B — 3 A. *&c.* (*Schol.* Prob. 1. *&c.*) But by the two preceding *Coroll.* each of theſe Terms is a Term of any Series to which A, B, can belong; ſince 2 B — A is a 3d :*l* to A . B, and each of the reſt a 4th to A B, and the preceding. And here it's evident that each Term is twice, thrice, *&c.* as far from A as B is, according to the Multiplier of B.

4. If any three or more Terms are added together, and from the Sum be taken the Product of a Term leſſer or greater than any of them, multiplied by a Number 1 leſs than the number of Terms added, the Remainder is a Term of the Series whoſe Diſtance from the Term whoſe Multiple is ſubtracted, is equal to the Sum of the Diſtances of the Terms added, from the ſame.

The *Reaſon* will be plain from the *Theorem*, thus: Let any Term of a Series be A, and any other two, both greater or both leſſer than A, be B, C, then $B+C-A$ is a Term of the Series as far from A, as the Sum of the Diſtances of B and C from A (by *Coroll* 1.) Add another Term D, the Sum is $B+C+D-A$, from which ſubtract A, the Remain. is $B+C+D-2A$; which is a Number found in the manner propoſed, and (by *Cor.* 1.) is a Term of the Series as far diſtant from A, as the Sum of the Diſtances of the two Terms added, *viz*, $\overline{B+C-A}$ and D, which is the Sum of the Diſtances of the three Terms $B+C+D$. It's manifeſt that by adding another Term continually, and ſubtracting A at every Step, the ſame thing will ſtill hold true: For at every Step there will be one Term more added, and A once more ſubtracted; ſo that the Multiplier of A will be ſtill 1 leſs than the number of Terms added.

Or we may ſee this Truth in another manner without the *Theorem*, thus: Any Term of a Series may be called A, and if the common Difference of the Series be d, then all the Terms above A are $A+d$. $A+2d$. $A+3d$. *&c.* Suppoſe any three or more of theſe Terms are added together, and let n repreſent the Number of Terms added; alſo let m repreſent the Sum of all the Numbers which multiply d (*i. e.* the Sum of the Diſtances of the ſeveral Terms added, from A), it's manifeſt that their Sum will be $nA+md$: from which ſubtract $\overline{n-1}\times A=nA-A$, the Remainder is $nA+md-nA+A=A+md$; which by the nature of an Arithmetical Series is ſuch a Term of a Series whoſe leſſer Extreme is A, and the Difference d, as that its Diſtance from A is equal to m, the Sum of the Diſtances of the Terms added. If the Series is A . $A-d$. $A-2d$, *&c.* the Demonſtration will proceed the ſame way.

SCHOLIUM. The immediate Uſe and Application of theſe *Corollaries* is in the Solution of the following Problems.

(1.) To find any Term of a Series having its Diſtance from the 1ſt Term, alſo the 1ſt Term, and any 2 other, the Sum or Difference of whoſe Diſtances from the 1ſt Term is equal to the Diſtance of the Term ſought: The Solution of which is plainly contained in *Coroll.* 1. and need not be repeated.

2. To find any Term of a Series having the 1ſt, and any 3 or more others, the Sum of whoſe Diſtances from the 1ſt is equal to the Diſtance of the Term ſought; as in *Coroll.* 4

3. To find any Term of a Series, having the 1ſt, and any other whoſe Diſtance from it is an aliquot Part of the Diſtance of the Term ſought; as in *Coroll.* 1. or 2.

4. To find any Term of a Series, having the firſt, and another whoſe Diſtance from the 1ſt is double the Diſtance of the Term ſought.

Obſerve, If the Term ſought is betwixt the Terms given, but not in the very middle, you have a Rule for ſolving this in *Prob.* 1. *Cor.* 8. *Sch.* 2.

THEOR.

Theorem VII.

In any Arithmetical Series, the Sum of the Extremes is equal to the Sum of any two mean Terms equally diſtant from the reſpective Extremes, (*i. e.* the leſſer Mean from the leſſer Extreme, and the greater from the greater; or contrarily) and to the double of the middle Term, where the Number of Terms is odd; *that is*, theſe Sums are all equal, *viz.* that of the Extremes, and of every two mean Terms equally diſtant from the Extremes; and the double of the middle Term, when the Number of Terms is odd.

Exam. In this Series, 3. 6 9. 12. 15. 18. 21. 24. 27. Theſe are equal, $3+27=6+24=9+21=12+18=2\times15=30$.

Demon. This follows eaſily from the preceding, compared with *Theor.* 1. For the Terms, whoſe Sums are here ſaid to be equal, are :l by the preceding; and the Sums of the Extremes and Means, or double the middle Term, are equal by *Theor* 1. *Cor.* 1. Thus, A. B. C. D. E. F. G being a Series :l, theſe are :l, A. B : F. G; hence $A+G=B+F$. Again; A. C : E. F, and $A+F=C+E$; and C. D E being :l, $C+E=2D$.

Or we may ſhew this Truth by another Repreſentation, as in the Margin; wherein A is the leſſer Extreme, m the Diſtance of the Extremes, and d the *common Difference*; ſo that $A+md$ is the greater Extreme, (*Cor.* 6. *Prob.* 4.) And the ſame Series being continued from A, and $A+md$. Which we ſuppoſe carried equally on, *i. e.* to half of the Number of Terms, if that is even; and to the middle Term incluſive in both, if the Number of Terms is odd. You ſee, that as in the increaſing Series d is once more added in every Step; ſo in the decreaſing one, it is once more ſubtracted; and conſequently the Sums of the correſpondent Terms in the two Series muſt ſtill be equal to the Sum of the Extremes, *viz.* $2A+md$. For any Whole being compoſed of two Parts, if we take away from the one, and add as much to the other, the whole, or Sum of theſe Parts continues ſtill the ſame; ſo by conſtantly adding d to A, and ſubtracting it from $A+md$, the Sum remains equal.

$$\left.\begin{array}{ll} \textit{Leſſer Extr.} & \textit{Greater.} \\ A & +A+md \\ A+d & +A+\overline{m-1}\,d \\ A+2d & +A+\overline{m-2}\,d \\ A+3d & +A+\overline{m-3}\,d \\ \textit{\&c.} & \textit{\&c.} \end{array}\right\}=2A+md.$$

Or it may be more ſimply repreſented by making l the greater Extreme, and ſubtracting d continually from it; thus, A. $A+d$. $A+2d$, &c. $l-2d$. $l-d$. l, carrying each Part from A and l equally on, as before; where the ſame Truth is manifeſt from the ſame Principle of equal Addition and Subtraction.

SCHOLIUMS.

1. When a Series has an even Number of Terms, there are two Terms which we call the *Two middle Terms*; and then the *Theorem* may be expreſſed thus; The Sum of the two middle Terms, and of every two equally diſtant are equal: And we may ſee the ſame Truth alſo in this Repreſentation, &c. $m-2d$. $m-d$. m. n. $n+d$. $n+2d$. &c. Increaſing and decreaſing from the two middle Terms m, n.

2. Where the Series has an odd Number of Terms (*i e.* a middle Term equally diſtant from both Extremes), then we may expreſs the *Theorem* thus; The Double of the middle Term, and the Sums of every two Terms equally diſtant from it, are equal; and it may be repreſented thus; &c. $m-2d$. $m-d$. m. $m+d$. $m+2d$. &c.

3. *Obſerve* alſo, That the Sum of any two Terms in a Series is equal to the Sum of any other two equally diſtant from the former two reſpectively; becauſe the four are :l. Alſo Double of any Term is equal to the Sum of any two equally diſtant from it; or, Any Term is equal to the Half Sum of any equally diſtant from it.

4. *Again*,

4. *Again*: When a Series has an even Number of Terms, tho' the two middle Terms are not in the continued Ratio of all the reſt above and below, yet the Sum of the Extremes, and of every two Terms equally diſtant from them, will ſtill be equal; for the four are :/ at leaſt disjunctly, becauſe of the *common Difference* and *equal Diſtance*.

General SCHOLIUM.

In every Arithmetical Progreſſion theſe five things are conſiderable, *viz.* The two Extremes, the Common Difference, the Number of Terms, and the Sum: From which a Variety of Problems ariſe; whereof thoſe are the chief and moſt uſeful, in which are given any three of theſe things, to find the other two; and theſe I ſhall next explain.

The Uſe of the Symbols *employed in the following* Problems.

$a =$ The leſſer Extreme. $s =$ Sum of the Series. $n =$ Number of Terms.
$l =$ The greater Extreme. $d =$ Common Difference. $m = n - 1 =$ The Diſtance of the Extremes.

PROBLEM V.

Given the Extremes a, l, and Number of Terms n, To find the Difference d, and Sum s.

RULE 1. For d; The Difference of the Extremes divided by the Number of Terms leſs 1 Quotes of the common Difference ſought, thus, $d = \frac{l - a}{n - 1}$.

Example: $a = 3 . l = 15$ and $n = 7$. then is $d = 2 = \frac{15 - 3}{7 - 1} = \frac{12}{6}$, as in the Series 3 . 5 . 7 . 9 . 11 . 13 . 15.

DEMONSTR. In *Cor.* 10. *Probl.* IV. it's ſhewn that $d \times \overline{n - 1} = l - a$. and dividing equally by $n - 1$, it is $d = \frac{l - a}{n - 1}$.

2. For s; multiply the Sum of the Extremes by the Number of Terms, and take half of the Product, it's the Sum: Thus $s = \frac{\overline{a + l} \times n}{2}$.

Example: $a = 3 . l = 15 . n = 7$. then is $s = 63 = \frac{\overline{3 + 15}\ (= 18) \times 7}{2} = \frac{126}{2}$, as in the preceding Series.

DEMONSTR. 1. If the Number of Terms is *even*, (*i.e.* a Multiple of 2.) then the Sums of the Extremes, and of every Pair of Means equally diſtant from the Extremes, are equal (*Theor.* 7.) But all theſe equal Sums together make the total Sum; and it's evident there are as many of theſe equal Sums as the half Number of Terms; ſince each Sum takes in two Terms; therefore the Sum of the Extremes (or any one of theſe equal Sums) being multiplied by the Number of Terms, produces double the total Sum, and conſequently its half is the Sum ſought.

2. If the Number of Terms is odd (or not a Multiple of 2), then there is a middle Term, and an equal Number on each hand, which Number is plainly the half of $n - 1$, (the middle Term being excluded) and the Sum of all, excluding the middle, is by the former Reaſoning, $\frac{\overline{a + l} \times \overline{n - 1}}{2}$: But the middle Term is $= \frac{a + l}{2}$ (*Theor.* 7.) which

being

being added to the Sum of the other Terms, makes the Total, *viz.* $\frac{\overline{a+l}\times\overline{n-1}}{2}+\frac{a+l}{2}$ $=\frac{\overline{a+l}\times\overline{n-1}+\overline{a+l}}{2}=\frac{\overline{a+l}\times\overline{n-1+1}}{2}=\frac{\overline{a+l}\times n}{2}$.

Or the whole *Demonſtration* may be made, without diſtinguiſhing whether the Number of Terms is even or odd, thus; Any Series increaſing may be repreſented; $a\,.\,a+d\,.\,a+2d$, &c. and the ſame Series decreaſing may be taken thus, $l\,.\,l-d\,.\,l-2d$, &c. And theſe two repreſenting the ſame Series, only in a different Order, have equal Sums; and therefore the Sum of both together is double the Sum of either. Again; It's obvious, that adding their correſponding Terms, the Sums are conſtantly the ſame, *viz.* $a+l$. and the Number of them being n, therefore the total of theſe equal Sums is $\overline{a+l}\times n=2S$, whence $s=\frac{\overline{a+l}\times n}{2}$.

$a\,.\,a+d\,.\,a+2d\,.\,\&c.=s$
$l\,.\,l-d\,.\,l-2d\,.\,\&c.=s$
$a+l\,.\,a+l\,.\,a+l\,.\,\&c.\ 2s=$
$\overline{a+l}\times n$. hence $s=\frac{\overline{a+l}\times n}{2}$

Scholium. In a Series of an even Number of Terms we may take the Sum of the Extremes, and multiply by half the Number of Terms; or half the Sum of the Extremes by the Number of Terms; for all theſe ways make the ſame Sum; thus, $\frac{\overline{a+l}\times n}{2}=$ $\overline{a+l}\times\frac{n}{2}=\frac{a+l}{2}\times n$.

COROLLARIES.

1. Betwixt any two different Numbers a, l, we can put any Number of Arithmetical Means; becauſe l being greater than a, and n greater than 1, it follows, that $d=\frac{l-a}{n-1}$ is always poſſible, however great n be. But then alſo *obſerve*, That betwixt two Integers theſe Means will not in every Caſe be Integers. And to make them all ſo, the greateſt Number of Means cannot be a greater Number than the Difference leſs 1 betwixt the Extremes, or $l-a-1$. The Reaſon is plain; for to the leſſer Term a, there can be ſucceſſively joined as many Units as are in the Difference betwixt l and a; and when the laſt of theſe Units is added, the Sum will be equal to l; conſequently the preceding Sums are in Number equal to $l-a-1$; and becauſe they differ all by 1, therefore they make with the Extremes a and l an Arithmetical Progreſſion; and the greateſt that can be in Integers, becauſe the Difference is the ſmalleſt, *viz.* 1.

2. The Sum of the natural Series, 1.2.3.4, *&c.* is equal to half the Product of the laſt Term multiplied into the next greater Number, becauſe that next is the Sum of the Extremes, and the greateſt Extreme is the Number of Terms: So $1+2+3+4+5=5\times 6\div 2$.

3. In any Series whoſe Number of Terms is odd, the Sum is equal to the Product of the Number of Terms multiplied into the middle Term; becauſe the middle Term is half the Sum of the Extremes.

4. From this *Prob.* and *Cor.* 9. *Prob.* 4. compared, theſe things follow, *viz.*

(1.) The Sum of that Series, whoſe leſſer Extreme is 1, and the Difference 2; as 1.3.5.7. *&c.* (which is the natural Series of odd Numbers) is equal to the Square of the Number of Terms. For in this Caſe, $a+l=2n$ (*Cor.* 9. *Prob.* 4.) and hence $\frac{a+l}{2}=n$, and $s=\frac{\overline{a+l}\times n}{2}=nn$.

(2.) The

(2.) The Sum of any Series, whose lesser Extreme and Difference are equal, is equal to half the Product of these Factors, *viz.* the common Difference, and the Sum of the Number of Terms and its Square; thus, $s=\frac{dn^2+dn}{2}$. For in this Case, $a+l=dn+d$ (*Cor.* 9. *Prob.* 4.) therefore $\overline{a+l}\times n=dn^2+dn$. Hence $s=\frac{\overline{a+l}\times n}{2}=\frac{dn^2+dn}{2}$. Again; Particularly if $a=d=2$; as in this Series, 2. 4. 6. 8, *&c.* (which makes the natural Series of even Numbers; for it comprehends all the Multiples of 2.) The Sum is equal to the Sum of the Number of Terms and its Square; thus, $s=n^2+n$; for $dn^2+dn=2n^2+2n$; therefore $s=\frac{dn^2+dn}{2}=\frac{2n^2+2n}{2}=n^2+n$.

PROBLEM VI.

Having the Extremes $a, l,$ and Difference d; to find the Number of Terms n, and Sum s.

RULE 1. For n. Divide the Difference of the Extremes by the common Difference, the Quote is the Number of Terms less 1: Thus, $n-1=\frac{l-a}{d}$; therefore $n=\frac{l-a}{d}+1=\frac{l-a+d}{d}$; *i. e.* to the greater Extreme add the *common Difference*, and from the Sum take the lesser Extreme, and divide the Remainder by the *common Difference*, the Quote is the Number of Terms.

Exam. $a=3$, $l=15$, $d=2$; then is $n=2=\frac{15+2-3}{2}=\frac{14}{2}$.

DEMON. By *Cor.* 10. *Prob.* 4. $d\times\overline{n-1}=l-a$; and dividing each by d, it is $n-1=\frac{l-a}{d}$. Whence $n=\frac{l-a}{d}+1=\frac{l-a+d}{d}$.

2. For s. apply $a. l. n.$ By *Prob.* 5. thus, $s=\frac{\overline{a+l}\times n}{2}$. Or without finding n, apply the given Numbers, thus: Add the Difference of the Squares of the Extremes, to the Product of the Sum of the Extremes, by the common Difference; this Sum divided by double the *common Difference*, the Quote is the Sum sought; thus, $s=\frac{\overline{a+l}\times d+l^2-a^2}{2d}$.

Exam. $a=3$, $l=15$, $d=2$; then is $s=63=\frac{\overline{3+15}\times 2+225-9}{2\times 2}=\frac{36+216=252}{4}$.

DEMON. Instead of n take its Equal above found, $\frac{l-a+d}{d}$; and substitute this instead of n in the other Rule, *viz.* $s=\frac{\overline{a+l}\times n}{2}$, and it is $s=\frac{a+l}{2}\times\frac{l-a+d}{d}=\frac{\overline{a+l}\times d+l^2-a^2}{2d}$; for $\overline{a+l}\times\overline{l-a+d}=al-a^2+ad+l^2-al+ld=al-al+ld+ad+l^2-a^2=ld+ad+l^2-a^2=\overline{a+l}\times d+l^2-a^2$.

PROBLEM VII.

Having one of the Extremes a or l, with the Number of Terms n, and Difference d; to find the other Extreme l or a, and the Sum.

Rule 1. For l or a: Multiply the Difference by Number of Terms less 1; add the Product to the given Extreme if it's a, but subtract if it's l; the Sum or Remainder is the other Extreme. Thus, $l = d \times \overline{n-1} + a$, and $a = l - d \times \overline{n-1}$.

Exam. $a=3$, $n=7$, $d=2$; then is $l=15=2\times 6+3$.

Demon. This Rule is expresly contained in *Coroll. 6. Probl.* IV.

2. For s apply a, l, n, by *Prob.* 5. Or without finding n proceed thus: Multiply the given Extreme by double the Number of Terms; to the Product add (if the given Extreme is a), or subtract (if it's l) the Product of the common Difference multiplied by the Difference betwixt the Number of Terms and its Square: the half of this Sum or Difference is the Sum sought. Thus; $s = \frac{2na + d \times \overline{nn-n}}{2} = \frac{2nl - d \times \overline{nn-n}}{2}$, or $\frac{2nl - dnn + dn}{2}$.

So in the preceding, $s = 63 = \frac{2\times 7\times 3 + 2\times \overline{49-7}}{2}$, when $a=3$ is given.

Demon. Take a or l, as they are expressed in the first Part, *viz.* $l - d\times\overline{n-1}$ for a, and $a + d\times\overline{n-1}$ for l. Substitute these instead of a, l in this Rule, $s = \frac{\overline{a+l}\times n}{2}$. Thus, Since $a = l - d\times\overline{n-1}$, then $a + l = l - d\times\overline{n-1} + l = 2l - d\times\overline{n-1}$, and $\overline{a+l}\times n = 2ln - d\times\overline{n-1}\times n = 2ln - d\times\overline{nn-n}$. Consequently, $s = \frac{\overline{a+l}\times n}{2} = \frac{2ln - d\times\overline{nn-n}}{2}$; which is the Rule when l is given. Again; Since $l = a + d\times\overline{n-1}$, then $a + l = 2a + d\times\overline{n-1}$, and $\overline{a+l}\times n = 2an + d\times\overline{nn-n}$. Hence, $s = \frac{\overline{a+l}\times n}{2} = \frac{2an + d\times\overline{nn-n}}{2}$, the Rule when a is given.

Scholium. In the preceding Problems, we have found d by means of a, l, n. Also n by means of a, l, d, and a by means of d, n, l. And here it may be useful to observe a Mistake of a very considerable Author, Tacquet, who gives Rules for finding d, or n, or a, by means of only 2 of the given things in the preceding Problems. His Rules are these: d is the integral Part of this Quote $\frac{l}{n-1}$, and a is what remains over the Division. Again; $n-1$ is the integral Part of this Quote $\frac{l}{d}$, and a what remains over the Division. Which Rules he founds upon this, that $l = a + d\times\overline{n-1}$. Whence he concludes, that $\frac{l}{d} = n-1$, and a remaining over; also that $\frac{l}{n-1} = d$, and a remaining. But you'll easily perceive that these Conclusions are not true universally, and can be so only in a particular Case, *viz.* when a is less than d, or than $n-1$; yet he delivers them as general Rules without Limitation, at least without mentioning it, if he did observe any. The Reason and Necessity of this Limitation I thus prove: If $d\times\overline{n-1}$ is divided by either Factor d, or $n-1$, the Quote will be the other of them; and if to the Product $d\times\overline{n-1}$ we had first added another Number a less than the Divisor, the Quote would be the same as before, and the Remainder would be the Number added. But if a the Number added, is equal to, or greater than the Divisor, it's plain that the Quote would be greater than the

2 other

other Factor, because it's contained in the Number added; and the Remainder would not be the Number added, but less, because it must be less than the Divisor, which here is less than the Number added. Take this particular *Example*: 5 . 7 : 9 : 11 : 13; whose greater Extreme 13 being divided by 4 ($=n-1$) the Quote is 3 and 1 over; yet the common Difference of the Series is 2. and the lesser Extreme 5.

What can be done by means of two things given, you'll learn afterwards.

PROBLEM VIII.

Having the Extremes, a, l, and Sum s; to find the Number of Terms n, and Difference d.

RULE 1. For n divide double the Sum by the Sum of the Extremes, the Quote is the Number of Terms, thus; $n=\frac{2s}{a+l}$.

Example: $a=3$. $l=15$. $s=63$. then is $n=7=\frac{2\times 63}{3+15}=\frac{126}{18}$.

DEMONSTR. By *Probl.* V. $s=\frac{\overline{a+l}\times n}{2}$; hence, multiplying both by 2, it is $2s=\overline{a+l}\times n$; then, dividing both by $a+l$, it is $n=\frac{2s}{a+l}$.

2. For d apply a . l . n. by *Probl.* V. Or without finding n, work thus; Take the Squares of the Extremes, and divide the Difference of these Squares by the Difference betwixt the Sum of the Extremes, and double the given Sum, the Quote is the common Difference; thus, $d=\frac{l^2-a^2}{2s-a-l}$.

Example: $a=3$. $l=15$. $s=63$; then is $d=2=\frac{15\times 15-3\times 3}{2\times 63-3-15}=\frac{225-9}{126-18}$.

DEMON. Instead of n, take its Equal $\frac{2s}{a+l}$, and put in the Rule of *Prob.* V. *viz.* $d=\frac{l-a}{n-1}$ thus, $\frac{2s}{a+l}-1=\frac{2s-a-l}{a+l}$, therefore $n-1=\frac{2s-a-l}{a+l}$, and $\frac{l-a}{n-1}=\overline{l-a}\div\frac{2s-a-l}{a+l}=\frac{\overline{l-a}\times\overline{a+l}}{2s-a-l}=\frac{l^2-a^2}{2s-a-l}$.

PROBLEM IX.

Having one Extreme a, or l, with the Sum s, and Number of Terms n, To find the other Extreme l or a, and the Difference d.

RULE 1. For l or a, divide double the Sum by the Number of Terms, and from the Quote subtract the given Extreme, the Remainder is that sought. Thus, $l=\frac{2s}{n}-a$, and $a=\frac{2s}{n}-l$. Or thus: From double the Sum take the Product of the Number of Terms and given Extreme, and divide the Remainder by the Number of Terms, the Quote is the Extreme sought; thus, $l=\frac{2s-an}{n}$, and $a=\frac{2s-}{n}$.

Example: $a=3$. $s=63$. $n=7$. then is $l=\frac{2\times 63-3\times 7}{7}=15$; and if $l=15$ is given, then $a=3=\frac{2\times 63-7\times 15}{7}$.

DEMON.

Demonstr. By *Probl.* V. $s=\frac{\overline{a+l}\times n}{2}$: Hence $2s=\overline{a+l}\times n$, and $\frac{2s}{n}=a+l$. and laſtly, $a=\frac{2s}{n}-l=\frac{2s-ln}{n}$, and $l=\frac{2s}{n}-a=\frac{2s-an}{n}$.

2. For d apply a, l, n, by *Probl.* V. or without finding n, do thus; Take the Difference betwixt Double the Sum and Double the Product of the Number of Terms by the given Extreme; divide this by the Difference betwixt the Number of Terms and its Square; the Quote is the Difference ſought; thus, $d=\frac{2s-2an}{nn-n}=\frac{2ln-2s}{nn-n}$.

Example: $a=3$. $n=7$. $s=63$. then is $d=2=\frac{2\times 63-2\times 3\times 7}{7\times 7-7}=\frac{126-42}{49-7}$.

Demonstr. By the preceding Rule $a=\frac{2s-ln}{n}$: Subſtitute this inſtead of a in the *Rule* of *Probl.* V. *viz.* $d=\frac{l-a}{n-1}$; thus, $l-a=l-\frac{2s-ln}{n}=\frac{ln-2s+ln}{n}=\frac{2ln-2s}{n}$. Therefore $\frac{l-a}{n-1}=\frac{2ln-2s}{n}\div\overline{n-1}.=\frac{2ln-2s}{nn-n}$, the Rule for d when l is given.

Again; $l=\frac{2s-an}{n}$: hence $l-a=\frac{2s-an}{n}-a=\frac{2s-2an}{n}$, therefore $d=\frac{l-a}{n-1}=\frac{2s-2an}{n}\div\overline{n-1}=\frac{2s-2an}{nn-n}$.

Coroll. We learn here how to find the Sum of the Extremes by means of the Sum of the Series, and the Number of Terms; thus, $a+l=\frac{2s}{n}$, as we ſee above; for $a=\frac{2s}{n}-l$, and $l=\frac{2s}{n}-a$. ſo that $a+l=2\times\frac{2s}{n}-a-l$. whence $a+l=\frac{2s}{n}$.

Problem X.

Having the Sum of the Series s, the Difference d, and Number of Terms n, To find the Extremes a, l.

Rule. By the Sum and Number of Terms find the Sum of the Extremes, as in the laſt *Corollary*, viz. $a+l=\frac{2s}{n}$; then by d and n find the Difference of the Extremes, *viz.* $l-a=d\times\overline{n-1}$ (*Cor.* 10. *Prob* IV.) Laſtly, having the Sum and Difference of the Extremes, find the Extremes thus: To the half of their Sum add half the Difference, the Sum is the Greater Extreme; And from the half Sum take the half Difference, the Remainder is the Leſſer Extreme; thus, $l=\frac{a+l}{2}+\frac{l-a}{2}=\frac{s}{n}+\frac{d\times\overline{n-1}}{2}$; for $a+l=\frac{2s}{n}$; therefore, $\frac{a+l}{2}=\frac{s}{n}$, and $l-a=d\times\overline{n-1}$, and $\frac{l-a}{2}=\frac{d\times\overline{n-1}}{2}$. Then $a=\frac{a+l}{2}-\frac{l-a}{2}=\frac{s}{n}-\frac{d\times\overline{n-1}}{2}$: which Expreſſions being reduced to a more ſimple Form, they are equal to theſe, *viz.* $l=\frac{2s+dnn-dn}{2n}$, and $a=\frac{2s-dnn+dn}{2n}$; in which Terms alſo the *Rule* may be expreſſed.

Exam-

Example: $s=63 . n=7 . d=n$, then is $a=3=\frac{63}{7}-\frac{2\times6}{2}=9-6$, and $l=15=\frac{63}{7}+\frac{2\times6}{12}=9+6$.

DEMONSTR. The last Part of the *Rule* only wants to be demonstrated; and because it is a general Truth of frequent Use, I shall put it by itself in the Form of a

LEMMA.

The half Sum of two Numbers added to their half Difference makes the greater of the two, and their Difference makes the lesser of the two.

Example: The Sum of two Numbers being 18, and their Difference 12, the Greater of them is $15=9+6$; and the Lesser is $3=9-6$.

DEMONSTR. The two Numbers being a, l, the half Sum is $\frac{a+l}{2}$, and half Difference $\frac{l-a}{2}$, then is $\frac{a+l}{2}-\frac{l-a}{2}=\frac{a+l-l+a}{2}=\frac{2a}{2}=a$, and $\frac{a+l}{2}+\frac{l-a}{2}=\frac{a+l+l-a}{2}=\frac{2l}{2}=l$.

PROBLEM XI.

Having one Extreme a, or l, with the Sum s, and Difference d, To find the other Extreme l or a, and Number of Terms n.

RULE 1. For the Extreme sought: If it's the Lesser, a, then take the Square of the Greater Extreme (l^2), to which add the Product of the Extreme by the common Difference (dl); and to this Sum again add the 4th Part of the Square of the Difference, $\left(\frac{dd}{4}\right)$ from which Sum subtract the Product of twice the Difference by the given Sum of the Series ($2ds$), out of which Remainder, *viz.* $l^2+dl+\frac{dd}{4}-2ds$, extract the Square Root; to which Root add half the common Difference, or $\frac{d}{2}$; the Sum is the Lesser Extreme; thus, $a=\overline{l^2+dl+\frac{dd}{4}-2ds}\big|^{\frac{1}{2}}+\frac{d}{2}$.

For the Greater Extreme; To the Square of the Lesser add the Product of the Sum into twice the Difference; and to this Sum again add the 4th Part of the Square of the Difference; from which Sum subtract the Product of the given Extreme and Difference; then take the Square Root of the Remainder, from which take half the Difference; the Remainder is the Greater Extreme; thus, $l=\overline{a^2+2ds+\frac{dd}{4}-ad}\big|^{\frac{1}{2}}-\frac{d}{2}$.

DEMONSTR. By *Prob.* 7. $l=a+dn-d$, and by *Prob.* 8. $n=\frac{2s}{a+l}$; substitute this for n in the former, and it is $l=a+\frac{2sd}{a+l}-d$, which being reduced is $\frac{aa+al+2sd-ad-dl}{a+l}=l$. Multiply both by $a+l$, and it is $aa+al+2ds-ad-dl=l^2+al$. Subtract al from both, and it is $aa+2ds-ad-dl=l^2$. Add dl to both, it is $aa+2ds-ad=l^2+dl$. wherefore (by *Prob.* 6. *Ch.* II. §. 2. *Book* III.) $l=\overline{aa+2ds+\frac{dd}{4}-ad}\big|^{\frac{1}{2}}-\frac{d}{2}$. which is the Rule for l.

Again;

Again; By *Prob.* VII. $a=l-dn+d$, and by *Prob.* VIII. $n=\frac{2s}{a+l}$; therefore $a=l-\frac{2ds}{a+l}+d$; whereby we come to this, $a^2+al=al+l^2-2ds+ad+dl$: Take away al, and alſo ad from both, then $a^2-ad=l^2-2ds+dl$, and (by *Prob.* 6. *Ch* II. §. 2. *Book* III.) $a=\overline{l^2+dl+\frac{dd}{4}-2ds}\big|^{\frac{1}{2}}+\frac{d}{2}$, the Rule for a.

Example: $a=3$. $s=63$. $d=2$. then is $l=15$. for $a^2=9$. $2ds=2\times2\times63=252$. $\frac{dd}{4}=1$. $ad=6$. ſo that $a^2+2ds+\frac{dd}{4}-ad=9+252+1-6=262-6=256$. whoſe Square Root is 16. from which take $\frac{d}{2}=1$; The Remainder is $15=l$.

2. For the Number of Terms, n: If a is given, then from twice the leſſer Extreme, $(2a)$ take the Difference (d); divide the Remainder $(2a-d)$ by the Difference: Then take the 4th Part of the Square of this Quote, to which add the Quote of twice the Sum $(2s)$ divided by the common Difference $\left(\frac{2s}{d}\right)$; Out of that Sum $\left(\frac{2s}{d}+\frac{1}{4}\right.$ of the Square of $\left.\frac{2a-d}{d}\right)$ extract the Square Root; from which take half of the firſt Quote $\left(viz.\ \frac{1}{2}\text{ of }\frac{2a-d}{d}\right)$; the Remainder is the Number of Terms ſought: Thus, for Brevity, call $\frac{2a-d}{d}=b$; then is $n=\overline{\frac{2s}{d}+\frac{b^2}{4}}\big|^{\frac{1}{2}}-\frac{b}{2}$.

If l is given, Take the Sum of twice l and d, which divide by d, and call this Quote $b\left(=\frac{2l+d}{d}\right)$; from the $\frac{1}{4}$ of the Square of this, $\left(viz.\ \frac{bb}{4}\right)$ take $\frac{2s}{d}$ and extract the Square Root of the Remainder *viz.* $\overline{\frac{bb}{4}-\frac{2s}{4}}\big|^{\frac{1}{2}}$. Take the Sum or Difference of this Root, and $\frac{b}{2}$, and one or the other is the Number of Terms: Which Rule is expreſſed thus, $n=\frac{b}{2}\pm\overline{\frac{b^2}{4}-\frac{2s}{d}}\big|^{\frac{1}{2}}$.

Demon. By *Prob.* 6. $n=\frac{l-a+d}{d}$, and (by *Prob.* 9.) $l=\frac{2s-an}{n}$. Hence $l-a+d=\frac{2s-an}{n}-a+d=\frac{2s-2an+nd}{n}$. Conſequently $n=\left(\frac{l-a+d}{d}\right)=\frac{2s-2an+nd}{nd}$. Multiply both by nd, and it is $nnd=2s-2an+nd$; add $2an$, and ſubtract nd from both, and we have $nnd+2an-nd=2s$. Divide each Member by d, and it is $n^2+\frac{2a-d}{d}\times n=\frac{2s}{d}$. And calling $\frac{2a-d}{d}=b$, it is $n^2+bn=\frac{2s}{d}$. Whence (by *Prob.* 6. *Chap.* 2. §. 2. *B.* III.) we have $n=\overline{\frac{2s}{d}+\frac{b^2}{4}}\big|^{\frac{1}{2}}-\frac{b}{2}$, the Rule for n when a is given.

Again; By *Prob.* 9. $a=\frac{2s-ln}{n}$. Hence $l-a=l-\frac{2s-ln}{n}=\frac{ln-2s+ln}{n}=\frac{2ln-2s}{n}$, then $l-a+d=\frac{2ln-2s}{n}+d=\frac{2ln-2s+dn}{n}$, as above; therefore $n\ (=\frac{l-a+d}{d}$

as

as above) $= \frac{2ln - 2s + dn}{nd}$. Multiply both by nd, and it is $nnd = 2ln - 2s + dn$; add $2s$, and then ſubtract nnd from both Sides, and we have $2s = 2ln + dn - nnd$. Divide all by d, and it is $\frac{2s}{d} = \frac{2l+d}{d} \times n - n^2$. Call $\frac{2l+d}{d} = b$, then it is $\frac{2s}{d} = bn - n^2$; and (by *Prob.* 6. *Chap.* 2. §. 2. *Book* III.) $n = \frac{b}{2} \pm \overline{\frac{b^2}{4} - \frac{2s}{d}}\Big|^{\frac{1}{2}}$, the Rule for n when l is given.

SCHOLIUM. Theſe Rules are tedious both in the Inveſtigation and Application: But there is another Method of ſolving the Problem, which tho' it is only by Trials, yet it proceeds directly and certainly to the Anſwer; and is rather eaſier than the former Work, and therefore I ſhall here explain it.

Another RULE.

By *Prob.* 1. raiſe a Series from the given Extreme and Difference; and take the Sum of the Series gradually as it riſes, continuing this Operation till the Sum is equal to the given Sum; and the Series ſo raiſed will ſhew both the Number of Terms, and the Extreme ſought; which is the laſt Term found in the Series.

The Reaſon of this Rule will be obvious from one *Example*. Suppoſe $a = 4$, $d = 3$, $s = 91$. In the annext Operation, you ſee in the upper Line the given Extreme 4, and the Difference 3 continually added. In the ſecond Line are the Terms of the Progreſſion formed by that continual Addition; and in the third Line are the ſeveral Sums of the preceding Series taken continually from the Beginning, by adding the next Term to the preceding Sum. Whence we ſee in the preſent *Example*, that the Extreme ſought is 22, and the Number of Terms 7.

$$\begin{array}{ccccccc} 4 + & 3 + & 3 + & 3 + & 3 + & 3 + & 3 \\ 4. & 7. & 10. & 13. & 16. & 19. & 22 \\ 4. & 11. & 21. & 34. & 50. & 69. & 91 \end{array}$$

Obſerve, That the Tedioufneſs of this Method, when the Number of the Terms is great, may be relieved by the following Means, *viz.* Take any Number for n at a gueſs, (in which to prevent being too wide of the Truth, have a regard to the given Numbers); then by this Number n, with the given d and a or l, find the other Extreme, and the Sum; and if this Sum differs from that given, begin at the Extreme laſt found, and raiſe a Series, increaſing or decreaſing, as the Caſe requires, till you find a Sum equal to the given one. For *Example*: Suppoſe $a = 3$, $d = 4$, $s = 406$; I gueſs $n = 12$, and hereby find $l = 47$ $(= 4 \times 11 + 3)$ and $s = 300$ $(= \overline{47 + 3} \times 6)$; which being leſs than 406, I begin at 47, and adding the Difference 4 till the Sum is equal to 406, I find that this happens upon adding that Difference twice, *i. e.* that two Terms more with the 47 make the Sum given; whence 'tis certain that 14 is the Number of Terms, (for there were 12 to bring it to 47, and 2 now added) and 55 the greater Extreme.

$$\begin{array}{ccc} 47 + & 4 + & 4 \\ & 51. & 55 \\ 300. & 351. & 406 \end{array}$$

SCHOLIUMS.

I. For the more convenient and ready Uſe of the laſt ſeven *Problems*, we ſhall put them all in a Table, that they may appear in one View; expreſſed ſimply by their Characters, whoſe Signification I ſhall repeat:

a = leſſer Extreme.	n = Number of Terms.	s = Sum of the whole Series.
l = greater Extreme.	d = the common Difference.	

TABLE *of the preceding Seven* Problems.

Probl.	Given.	Sought.	Solutions.
5	$a, l, n.$	$d, s.$	$d=\frac{l-a}{n-1}$. $s=\overline{\frac{a+l\,n}{2}}$.
6	$a, l, d.$	$n, s.$	$n=\frac{l-a+d}{d}$. s, as above; or thus, $s=\frac{ad+ld+l^2-a^2}{2d}$.
7	a or l . n . d	l or a . s	$l=a+d\overline{n-1}$. $a=l-d\overline{n-1}$. s, as above; or thus, $s=\frac{2an+dn^2-dn}{2}$, $s=\frac{2ln-dn^2+dn}{2}$
8	$a, l, s.$	$n.\ d.$	$n=\frac{2s}{a+l}$ d, as above; or thus, $d=\frac{l^2-a^2}{2s-a-l}$
9	a or l . n . s	l or a . d	$l=\frac{2s-an}{n}$, $a=\frac{2s-ln}{n}$ d, as in the 5th; or thus, $d=\frac{2s-2an}{n^2-n}$, $d=\frac{2ln-2s}{n^2-n}$
10	$d.n.s.$	$a.\ l.$	$a=\frac{2s-dn^2+dn}{2n}$. $l=\frac{2s+dn^2-dn}{2n}$
11	a or l . d . s	l or a . n	The Solution of this is by raiſing a Series from the given Extreme with the given Difference, till the Sum is equal to that given. Or thus, $a=\overline{l^2+dl+\frac{dd}{4}-2ds}\vert^{\frac{1}{2}}+\frac{d}{2}$. and $l=\overline{a^2+2ds+\frac{dd}{4}-ad}\vert^{\frac{1}{2}}-\frac{d}{2}$ Then $n=\overline{\frac{2s}{d}+\frac{b}{4}}\vert^{\frac{1}{2}}-\frac{b}{2}\left(\frac{2a-d}{d}=b\right)$ and $n=\frac{b}{2}\pm\overline{\frac{b^2}{4}-\frac{2s}{d}}\vert^{\frac{1}{2}}\left(\frac{2l+d}{d}=b\right)$

II. According as the given things in any of the preceding *Problems* are choſen or related to another, ſo will the *Problem* be poſſible or impoſſible: For any three Numbers taken at random will not make a poſſible *Problem* of each of them, (tho' of ſome it will); becauſe there are particular Relations, within certain Limits, which the given Numbers muſt have to one another in moſt of theſe *Problems*; ſo that they may be poſſible, *i. e.* ſo that the three given things may belong to the ſame Progreſſion. The Poſſibility or Impoſſibility will appear by applying the Rules; for if the given Numbers are inconſiſtent, one or more of the things ſought will be found impoſſible, by ſome Abſurdity that will appear in applying the given Numbers to one another according to the Rule.

But now if you require how to *invent* three Numbers conſiſtent with one another, to make *Data* for any of theſe *Problems*; it may be done either of theſe Ways; *viz.*

1. Take any two Numbers whatever for a, d, and any Integer greater than 1, for n; and by theſe three find l, s; and thus you have five things all belonging to one *Progreſſion*, out of which to chuſe any 3 for *Data* of a *Problem*. The Reaſon of this Rule is plainly thus: That from any Number a we may raiſe a Progreſſion with any Difference d to any Number of Terms n we pleaſe.

2. Or take any two Numbers for a, l ſo that a do not exceed l; and any Integer greater than 1 for n, and by theſe find d, s. The Reaſon of this is, that betwixt any two Numbers

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bers $a, l,$ any Number of Arithmetical Means may be placed, as has been ſhewn in *Corol.* to *Prob* 5.

3. Therefore if $a, n, d,$ or a, n, l are the Terms to be invented, we can find them by themſelves; and if any 2 of theſe 3, (as $a, n.$ a, d $n, d.$ $a, l.$ $n, l.$ $d, l.$) with any other Term [except that one Caſe d, l, s] are to be invented, we can find them without finding all the 5; yet one of the two things not required muſt be found: for we muſt take either $a, n, d,$ or $a, n, l,$ and by them find the other Term to be invented. But if $d,$ l and s are to be invented, we muſt find all the 5 by means of $a, n, d,$ or $a, n, l.$

But again, it may be required to invent the three things to be given in each of theſe *Problems,* without the Invention of any of the other two; which by the Rules now given cannot be done, except when $a, n, d,$ or a, n, l are to be given. For this you have Rules in what immediately follows, when it is poſſible to be done.

III In the preceding *Problems,* no leſs than three things are neceſſary to be known, to make each of them determinate to one certain Solution: But if we ſuppoſe only two of the five things to be given for finding the other three; then, of ſuch *Problems,* ſome will be indeterminate, and have an infinite Number of Solutions, *i. e.* we can find an endleſs Variety of Numbers for the three things ſought, which will all ſatisfy the *Problem:* Alſo of theſe indeterminate *Problems,* ſome will be abſolutely indeterminate as to ſome of the things ſought, ſo that any Number whatever may be aſſumed. But others of the things ſought [and in ſome of theſe *Problems* all the things ſought] muſt be taken within certain Limits, which nevertheleſs admit of an infinite Variety of Solutions. Again; Others of theſe *Problems,* wherein only two things are given, will be determinate to a certain Number of Solutions, according to the different Circumſtances and Relations of the two given things; for there will be one or more Solutions, as theſe Circumſtances differ.

Of all theſe Problems, *Indeterminate* or *Determinate,* there are juſt 10; becauſe there are juſt ſo many different Choices of two things to be found in five. Thus; the five things being $a, l, d, n, s,$ the Choices of 2, to make *Data* of a Problem, are theſe; $a, l.$ $a, d.$ $a, n.$ $a, s.$ $l, d.$ $l, n.$ $l, s.$ $d, n.$ $d, s.$ $n, s.$ Of which there are 6 that are *Indeterminate,* and 4 *Determinate.*

§. 3. *Containing* Problems *concerning* Arithmetical *Progreſſions, wherein two things only are given to find the other three.*

THAT I may deliver the Rules and Demonſtrations of the following *Problems* in the moſt ſimple and eaſy manner; and that you may underſtand them aright, take theſe few previous Explanations.

1. Tho' I have ſhewn in moſt of the following *Problems,* how to find, by means only of the two given things, any one of the three things ſought; you are not to underſtand it, as if all theſe three Rules were to be applied in the ſame Solution, *i. e.* as if three Numbers found, one by each of theſe Rules, might be taken for the Solution of the *Problem;* becauſe there being a Variety of Solutions for each of theſe three things, any one Solution for each of them will not make a Combination that can ſolve the *Problem;* for this plain Reaſon, *viz.* when any one right Number is taken for any one of the three unknown things, this with the two given things determine the other two things ſought, according to the preceding *Problems,* which have but one limited Solution; ſo that we cannot with any one Solution, for one of the things ſought, join any one of the Solutions for the other two things; theſe being now determin'd by the Solution which we have choſen for the former one, together with the two given things: therefore the particular Rules for the different things ſought, are to be underſtood only as Steps in ſo many different Methods of ſolving the ſame *Problem;* which are to be applied thus: By the two given things find any one of the

the unknown things, according to the Rule given for it; then take the thing now found, with the two given things; and by theſe three find the other two things ſought, by that one of the preceding *Problems* where theſe three things are given: But the Rules for theſe are ſet down along with the other.

2. As any three things taken at random could not make the preceding *Problems* poſſible, ſo neither here will any two Numbers make any of the following *Problems* poſſible; there being certain Limitations, in reſpect of one another, under which they muſt be taken in ſome *Problems*, tho' not in all. That we may not encumber the *Problems* with theſe things, I ſhall here explain theſe Limitations where they ought to be, and ſhew where there are none. Thus:

a has no Limitation, and may be any Number whatever, and even o.

l may be any Number whatever, if it's not leſs than a or d, nor greater than s; but has no Limitation with reſpect to n.

d may be any Number whatever not exceeding l or s, or it may alſo be o; but has no Limitation with reſpect to a or n.

n muſt be an Integer greater than 1; and tho', ſtrictly ſpeaking, there is no Progreſſion without three Terms, yet we ſhall here allow of two Terms as the ſmalleſt Progreſſion.

s may be any Number whatever, not leſs than a, or d, or l; but has no Limitation with reſpect to n.

The Reaſon of theſe things is obvious from the Nature of a Progreſſion, which may begin with o, or any Number, and proceed by any or no Difference; and may conſiſt of two only, or any other Number of Terms.

Now in all the following *Problems* it's ſuppos'd that the two given things are conſiſtent, according to theſe Directions; and for the Limitations expreſſed in the Rules for finding one of the things ſought, they do not only comprehend the Conditions now mention'd, but ſome of them contain more ſtrict ones, becauſe regard is to be had to the reſt of the five things. And in the Reaſon given for theſe Rules, the general Conditions now mention'd are frequently ſuppoſed; which therefore muſt be always in view, becauſe ſuch ſimple things need not be repeated, unleſs where there is any danger of Obſcurity.

3. As to the Method of inveſtigating the following particular Rules, I ſhall here give you a general Account of it, that I may ſave the repeating of the ſame things, as otherways would be neceſſary in the demonſtrating of each of them.

In the firſt place, they depend upon the preceding *Problems*, and are diſcovered thus: I take that one of theſe *Problems* in which are given the two given things of the preſent *Problem*, and that one of the three things ſought that I would now firſt find; then by the Rules of that *Problem* for finding the remaining two things, I diſcover what Limitations that one I would now find lies under with reſpect to the two given things, that theſe two remaining things may be poſſible; and then I conclude, that the thing I ſeek being taken within this Limitation, it belongs to the ſame Progreſſion with the two given things; and is therefore a true Solution. But more particularly: Suppoſe a, l are given, and I would find s; I go to the preceding *Prob.* 6. wherein a, l, s are given; and there the Rules for finding n, d are theſe, $n = \frac{2s}{a+l}$, and $d = \frac{l^2 - a^2}{2s - a - l}$. And here to make n poſſible, it's plain that $2s$ muſt be Multiple of $a+l$, becauſe n muſt be an Integer greater than 1; and the ſame Condition of s will make d poſſible; for ſince a does not exceed l, there is nothing to make it impoſſible, but $a+l$ being greater than $2s$; which it is not, if $2s$ be a Multiple of $a+l$. Therefore if s is taken ſo, as $2s$ be a Multiple of $a+l$, [or alſo if we take for s any Multiple of $a+l$; from whence certainly follows that $2s$ is a Multiple of $a+l$] any ſuch Number ſolves the Queſtion for s; that is, s ſo taken with a, l, belong to the ſame Progreſſion. For if they did not, n, d could not be found by their Means, according to ſuch Rules as have been diſcovered and demonſtrated upon that very Suppoſition, that all the five things do belong to the ſame Progreſſion. This

This manner of drawing the Concluſion you are to ſuppoſe in all the following Demonſtrations, which I ſhall never again repeat; but only ſhew you how that the Number ſought being taken according to the Rule, is conſiſtent with the Poſſibility of the remaining two things to be found. In doing of which, I have frequent Uſe for this *Principle*, *viz*. If one Number is equal to, greater or leſſer than another, any Multiple, or aliquot Part of the former is alſo equal to, greater or leſſer than the like Multiple or aliquot Part of the other; which being ſo very ſimple, it will be obvious in the Places where I uſe it, and therefore I ſhall not again repeat it.

We come now to the Problems, *whereof the firſt ſix are* Indeterminate, *and the other four* Determinate; *and mind, that the* Problems *here referred to, are thoſe of the preceding* §.

Prob. XII. *Given* a, l, *to find* n, d, s.

1. For n. Take any Integral Number greater than 1; you have the Reaſon of this in the preceding *Scholium* 1.

2. For d. Take any Number ſuch, that $\frac{l-a}{d}$ be an Integer, *i. e.* take any aliquot Part of $l-a$. For by *Prob.* 6. $n=\frac{l-a}{d}+1$, and $s=\frac{al+dl+l^2-a^2}{2d}$. Which are both poſſible, as d is taken; becauſe n being an Integer, $\frac{l-a}{d}$ muſt be ſo; and if we divide $l-a$ by any Integer, and call the Quote d, the ſame d dividing $l-a$ will return for a Quote the former Integer; therefore any Number d, which is an aliquot Part of $l-a$, makes $\frac{l-a}{d}$ an Integer. As for s, it requires only that l be not leſs than a, which is the general Condition.

3. For s, take any Multiple of $a+l$, or the half of any ſuch Multiple. For by *Prob.* 8. $n=\frac{2s}{a+l}$, and $d=\frac{l^2-a^2}{2s-a-l}$; which are both evidently poſſible, as s is limited.

Prob. XIII. *Given* n, d, *to find* a, l, s.

1. For a, take any Number, or even 0. The Reaſon is ſhewn already.

2. For l, take any Number not leſs than $d\overline{n-1}$. For by *Prob.* 7. $a=l-d\overline{n-1}$, and $s=\frac{2ln-dn^2+dn}{2}$. The former contains the very Conditions of the Rule, and the other requires only that $2ln+dn$ be greater than dn^2, or $2l+d$ greater than dn. But if l is at leaſt $=dn-d$, then is $2l+d=2dn-2d+d=2dn-d$; which is greater than dn, becauſe n is at leaſt 2.

3. For s, take any Number greater than $\frac{dn^2-dn}{2}$. For by *Prob.* 10. $a=\frac{2s-dn^2+dn}{2n}$, and $l=\frac{2s+dn^2-dn}{2n}$. Now if s is greater than $\frac{dn^2-dn}{2}$, then is $2s$ greater than dn^2-dn, and $2s+dn$ greater than dn^2; which evidently makes a and l poſſible.

Prob. XIV. *Given* a, n, *to find* l, d, s.

1. For l, take any Number not leſs than a; the Reaſon is ſhewn above.

2. For d, take any Number greater; the Reaſon is alſo ſhewn above.

3. For

3. For s, take any Number greater than an. For by *Prob.* 9. $l = \frac{2s - an}{n}$, and $d = \frac{2s - 2an}{n^2 - n}$. The Reaſon of the Limitation is plain.

Prob. XV. *Given* l, n, *to find* a, d, s.

1. For a, take o, or any Number not exceeding l. The Reaſon is ſhewn above.

2. For d, take any Number not exceeding $\frac{l}{n-1}$. For by *Prob.* 7. $a = l - d\overline{n-1}$, and $s = \frac{2ln + dn - dn^2}{2}$. In the former $d\overline{n-1}$ muſt not exceed l, nor conſequently muſt d exceed $\frac{l}{n-1}$. In the other, $2ln + dn$ muſt be greater than dn^2; hence $2l$ greater than $dn^2 - dn$, or $\overline{dn-1} \times n$; which is true, if d does not exceed $\frac{l}{n-1}$; for then $d\overline{n-1}$ does not exceed l, and conſequently $d\overline{n-1} \times n$ does not exceed ln; wherefore, laſtly, $2ln$ is greater than $d\overline{n-1} \times n$, as was required.

3. For s, take any Number leſs than ln, but not leſs than $\frac{ln}{2}$; for by *Probl.* IX $d = \frac{2ln - 2s}{n^2 - n}$, and $a = \frac{2s - ln}{n}$: The former requires only that s be leſs than ln, and the other that $2s$ be not leſs than ln, or s not leſs than $\frac{ln}{2}$.

Probl. XVI. *Given* a, d, *to find* n, l, s.

1. For n, take any Integer greater than 1.

2. For l, take any Number ſo that $\frac{l-a}{d}$ be integer; *i. e.* take any Multiple of d, and add a to it, the Sum is l; for by *Probl.* VI. $n = \frac{l-a}{d} + 1$, and $s = \frac{ad + ld + l^2 - a^2}{2d}$ The Reaſon of the Limitation is plain.

3. For s. This cannot be found without firſt finding ſome of the former; becauſe *Probl.* XI. whoſe *Data* are a, d, s, affords us nothing for this purpoſe.

Probl. XVII. *Given* n, s, *to find* a, l, d.

1. For a, take o, or any Number leſs than $\frac{s}{n}$; for by *Probl.* IX. $l = \frac{2s - an}{n}$ and $d = \frac{2s - 2an}{n^2 - n}$, which require only that $2an$ be leſs than $2s$, or an leſs than s, and, laſtly, n leſs than $\frac{s}{n}$.

2. For l, take any Number greater than $\frac{s}{n}$, but not exceeding $\frac{2s}{n}$; for by *Probl.* IX. $d = \frac{2ln - 2s}{n^2 - n}$, and $a = \frac{2s - ln}{n}$; the former requires that ln be greater than s, or l greater than $\frac{s}{n}$; the other that ln do not exceed $2s$, or l not exceed $\frac{2s}{n}$.

3. For

3. For d, Take any Number leſs than $\frac{2s}{n^2-n}$; for by *Prob.* X. $l=\frac{2s+dn^2-dn}{2n}$, and $a=\frac{2s-dn^2+dn}{2n}$, the laſt of which puts the narroweſt Limits upon d, *viz.* that $2s+dn$ be greater than dn^2, or $2s$ greater than dn^2-dn, and conſequently s greater than $\frac{dn^2-dn}{2}$; which is the Condition of the Rule.

Scholium. This *Problem* is in effect the ſame as this, *viz.* To divide a certain given Number (s) into a given Number of Parts (n), ſuch, that theſe Parts, from the leaſt to the greateſt, make a Progreſſion *Arithmetical.*

Probl. XVIII. *Given* a, s, *to find* l, n, d.

1. For l, Take any aliquot Part of $2s$, *i. e.* divide $2s$ by any Integer leſs than itſelf, ſo that the Quote be greater than a, and that when a is taken out of it the Remainder may not be leſs than a; that Remainder may be taken for l; and in order to this Solution begin with 2, and try all the Integers from that upwards, till you come to one which anſwers the Rule. The *Reaſon* is, becauſe by *Probl.* VIII. $n=\frac{2s}{a+l}$, and $d=\frac{l^2-a^2}{2s-a-l}$, of which the former plainly requires the Limitation of the Rule, and the other is evidently poſſible upon the ſame Conditions. And laſtly, *obſerve,* That there can be no more Solutions in this Method than there are Integers leſs than $2s$, which ſatisfy the Rule.

2. For n, Take any ſuch Integer that an be leſs than s, *i.e.* take any Integer greater than 1, but leſs than $\frac{s}{a}$; for by *Probl.* IX. $l=\frac{2s-an}{n}$, and $d=\frac{2s-2an}{n^2-n}$; which require only that $2an$ be leſs than $2s$, or an leſs than s, or n leſs than $\frac{s}{a}$. And *obſerve,* That here the Number of Solutions are determined to the Number of Integers that are leſs than $\frac{s}{a}$, and greater than 1.

3. For d, It cannot be found till ſome of the other two, l or n, is found.

Probl. XIX. *Given* l, s, *to find* a, n, d.

1. For a, Take any aliquot Part of $2s$, *i.e.* divide $2s$, by any Integer greater than 1, but leſs than itſelf, and ſuch alſo that the Quote be greater than l, and that when l is taken out of the Quote, the Remainder do not exceed l; that Remainder may be taken for a. The *Reaſon* of theſe Limitations is, that $n=\frac{2s}{a+l}$, and $d=\frac{l^2-a^2}{2s-a-l}$ (*Probl.* VIII.) and the Solutions are limited to the Number of Integers that ſatisfy the Rule, which it's plain cannot be infinite.

2. For n, Take any Integer greater than 1, and ſuch that n be greater than $\frac{s}{l}$, but leſs than $\frac{2s}{l}$, becauſe, by *Probl* IX. $a=\frac{2s-ln}{n}$, and $d=\frac{2ln-2s}{n^2-n}$: The firſt requires only that $2s$ be greater than ln, and conſequently $\frac{2s}{l}$ greater than n; the ſecond requires only that ln be greater than s, and conſequently n greater than $\frac{s}{l}$, ſo that the Number of Solutions is as many as the Integers greater than $\frac{s}{l}$ and leſs than $\frac{2s}{l}$.

3. For d, We cannot find it till a or n are firſt found.

Prob. XX. *Given* d, l, *to find* a, n, s.

Here s cannot be found, till a or n is found; and theſe may be both found together in one Operation. Thus: Take any Number not exceeding l, and ſuch alſo that it be equal to, or ſome Multiple of d, (by trying all the Multiples of d from $2d$, which will not exceed l); then is $n-1=1$, if the Number aſſumed is $=d$: But if it's a Multiple of d, the Multiplier is $=n-1$; and the Remainder, after the aſſumed Number is taken from l, is $=a$.

The Reaſon is, Becauſe by *Prob.* 7. $a=l-d\times\overline{n-1}$; where if $d\times\overline{n-1}=d$, then is $n-1=1$; otherwiſe, if $d\times\overline{n-1}=md$, then $n-1=m$.

Prob. XXI. *Given* d, s, *to find* n, a, l.

Here neither a nor l can be found till n be known; which may be found thus: Take any Integer greater than 1, and ſuch alſo that $nn-n$ be leſs than $\frac{2s}{d}$; for by *Prob.* 10. $a=\frac{2s+dn-dnn}{2n}$, and $l=\frac{2s+dnn-dn}{2n}$. By the firſt it follows, that dnn muſt be leſs than $2s+dn$; and taking dn from both, it follows that $dnn-dn$ muſt be leſs $2s$; and laſtly, (dividing both by d) that $nn-n$ muſt be leſs than $\frac{2s}{d}$. The other Part l is evidently poſſible, with the ſame Limitation.

General Corollary.

It's now manifeſt, how by theſe laſt ten *Problems* we can invent any three things that ſhall make any of the former ſeven *Problems* poſſible; and that after various manners, by taking any two of the three, under the general Conditions; and then with theſe two finding the other one, by that one of theſe laſt *Problems*, wherein theſe two things are given.

CHAP.

CHAP. III.

Of Geometrical *Proportion.*

§. 1. *Containing the more general Doctrine common to both* Conjunct *and* Disjunct *Proportionals.*

Obſerve, *In all that follows, I mark theſe Words,* Geometrical Proportion, *and* Geometrically Proportional, *by this* : : l; *and the Words* Continued Geometrical Proportion *by this* ∺ l.

Again; *When any Axioms are cited, you are to underſtand the* Axioms *at the End of* Chap. 1. *of this Book; and citing any of the general Corollaries there alſo explained, I mark them thus,* g. Cor.

PROBLEM I.

Having three Numbers given, to find a fourth, : : l.

RULE 1. FIND the Ratio of the firſt and ſecond Terms, either

1. By dividing the greater Term by the leſſer; and if the Antecedent is the leſſer, multiply; or if the greater, divide the third Term by that Ratio; the Product or Quote is the 4th ſought. Or,

2. Take the Quote of the firſt Term, divided by the ſecond, for the Ratio; and by it divide, or by its Reciprocal (*i. e.* the Quote of the ſecond Term divided by the firſt) multiply the third Term; the Quote or Product is the fourth ſought.

Exam. 1. To theſe 2 : 6 : : 5, a 4th is 15. For the Ratio of 2, 6 is 3, by which 5 multiply'd produces 15.

Exam. 2. To theſe 24 : 20 : : 6, a 4th is 5. For the Ratio of 24 to 20 is $1\frac{4}{20} = 1\frac{1}{5} = \frac{6}{5}$, and $6 \div \frac{6}{5} = \frac{30}{6} = 5$.

DEMON. The Reaſon of this Rule is plainly contained in the Definitions, and needs no farther Explication: or you may ſee it particularly in the *Coroll.* to the Definition of Geometrical Relation, which does in effect contain this *Problem*; for there it is ſhewn how to find a Number in any given Ratio to a given Number; and here the firſt and ſecond Terms contain the Ratio, in which the 4th Term ſought ought to be to the 3d. Or we ſhall ſet this once more before us in this univerſal Repreſentation of 4 Numbers : : *l*, viz. A : A*r* : : B : B*r*; wherein *r* being a whole or mixt Number, is the Ratio of A to A*r* in the one View, and its Reciprocal in the other; and it is manifeſt, that B*r* is the fourth ſought to theſe A : A*r* : : B; alſo A is a fourth to theſe B*r* : B : : A*r*, according to the Method of the foregoing Rule; which is therefore good.

RULE 2. Multiply the ſecond and third Terms together, and divide the Products by the firſt, the Quote is the fourth ſought.

So in *Exam.* 1. $15 = \frac{6 \times 5}{2} = \frac{30}{2}$; and in *Exam.* 2. $5 = \frac{20 \times 6}{24}$.

 Take

Take this other *Example:* To these $4:5::7$, the fourth is $8\frac{3}{4}=\frac{7\times 5}{4}$.

Universally: To these 3, $A:B::C$, a 4th is $\frac{BC}{A}$.

DEMON. The Reason of this Rule will easily appear from the preceding. For let the Ratio of $A:B$ be taken $\frac{A}{B}$, the Reciprocal of it is $\frac{B}{A}$; by which the 3d Term C being multiplied, the Product is the fourth sought by the preceding Rule: But this Product, according to the Rule of multiplying Fractions, is $\frac{BC}{A}$, if A, B, C are all Integers. But tho' they are not all Integers, yet it has been shewn in *Schol.* after *g. Cor.* 20. that the Quote of any two Numbers, B, A, is multiplied or divided by taking the Divisor and Dividend as the Numerator and Denominator of a Fraction, and applying the Rules for Fractions; therefore the Quote of $B\div A$ taken thus, $\frac{B}{A}$; and multiplied by C, produces this Expression, $\frac{BC}{A}$; which is according to the Rule, *viz.* $BC\div A$.

Or the Truth of this Rule we may see also in this Representation, $A:Ar::B:Br$; where it is plain, that $ArB\div A=Br$, and $ArB\div Br=A$. For if any Number is first multiplied by another, and the Product divided by the Multiplier, the Quote is necessarily the Number multiplied; which is evidently the Case here: for $Ar\times B$ is $=A\times rB$, and $A\times rB\div A=rB$.

COROL. Having two Numbers given, we may find a 3d by dividing the Square of the second Term by the first. Thus: To $2:6$, a third is $18=\frac{6\times 6}{2}$. For since $2:6::6:18$, then 18 is a fourth to $2:6::6$; which reduces this Case to the preceding.

THEOREM I.

IF four Numbers are $::l$, the Product of the two Extremes is equal to the Product of the two Means. And *reversly*; if these Products are equal, the four Numbers are $::l$.

Thus; If $A:B::C:D$, then $AD=BC$. In Numbers, $2:3::4:6$, and $2\times 6=3\times 4=12$.

DEMON. By the preceding *Problem*, $D=\frac{BC}{A}$; hence $DA=BC$.

Or thus: Since $A:B::C:D$, these Quotes are equal, *viz.* $\frac{A}{B}=\frac{C}{D}$; then the Products of the Divisor of each by the Dividend of the other are equal, (see *general Scholium* at the End of *Chap.* 1.) that is, $AD=BC$.

Or thus: $AD:BD::A:B$, (*g. Cor.* 15.) also $BC:BD::C:D$. But $A:B::C:D$; therefore $AD:BD::BC:BD$, (*Ax.* 3); hence $AD=BC$, (*Ax.* 1.)

Or lastly, Let $4::ls$ be thus represented, $A:Ar::B:Br$; then it is manifest that $ABr=ArB$.

For the *Reverse:* If $AD=BC$, then is $A:B::C:D$. For, by equal Division, $A=\frac{BC}{D}$; and again, $\frac{A}{B}=\frac{C}{D}$; hence $A:B::C:D$.

Or thus: $AD:BD::A:B$, and $BC:BD:C:D$; and since $AD=BC$, therefore $BC:BD::A:B$; hence $A:B::C:D$, (*Ax*, 3.)

Or

Or alſo thus: A 4th :: l to A, B, C is poſſible; ſuppoſe that to be N: *i. e.* A : B :: C : N; then is AN = BC (by the *Theor.*); but AD = BC (by Suppoſition); therefore AN = AD: Hence N = D; therefore D is the 4th :: l, that is, A : B :: C : D.

COROLLARIES.

1. If three Numbers are $\div l$, the Product of the Extremes is equal to the Square of the middle Term. Thus; A : B : C being $\div l$, $AC = B^2$. In Numbers, 2 : 4 : 8 are $\div l$, and $2 \times 8 = 4 \times 4$.

The *Reverſe* of this *Coroll.* is alſo true, *viz.* that if $AC = B^2$, then is A : B :: B : C.

2. If four Numbers are :: l, A : B :: C : D, the Product of all the four Terms is a ſquare Number, whoſe Root is the Product of the Extremes, or of the Means; theſe Products being equal. So AD = BC. Hence $AD \times BC = AD^2 = BC^2$.

Again; Three Numbers $\div l$ multiplied all together produce a Cube Number, whoſe Root is the middle Term. Thus: A . B : C being $\div l$, then $AC = B^2$, and $B^2 \times B = AC \times B = B^3$.

SCHOLIUMS.

1. As this *Theorem* is demonſtrated without the *Prob.* 1. ſo the 2d Rule of that *Problem* is an evident Conſequence of this *Theorem*: For, if A : B :: C : D; then $D = \frac{BC}{A}$, becauſe DA = BC.

2. All the general Corollaries relating to :: l, in the End of *Chap.* 1. may be moſt eaſily demonſtrated by this *Theorem*. For in all the Proportions there ſtated, we ſhall find this certain Mark of :: l, *viz* the equal Products of the Extremes and Means.

3. The Reverſe of this *Theorem* may be put in this Form, *viz.* If the Product of any two Numbers is equal to the Product of other two, theſe 4 are reciprocally :: l. Thus: If AD = BC, then A : B :: C : D. And here obſerve, that the Factors of theſe equal Products are ſaid to be reciprocally :: l; becauſe one of the two comparative Terms is taken out of the one Product, and the other out of the other Product.

PROBLEM II.

Of four Numbers :: l, having the two Extremes and one Mean, to find the other Mean; or to find one Mean in $\div l$ betwixt two Numbers.

RULE 1. Divide the Product of the Extremes by the known Mean, the Quote is the other.

DEMONSTR. This follows from the laſt *Theorem*; for if A : B :: C : D, then is AD = BC; and hence $AD \div B = C$, and $AD \div C = B$.

RULE 2. For one Geometrical Mean betwixt two Numbers, take the Square Root of the Product of the two Extremes.

DEMON. If A : B : C are $\div l$, then $AC = B^2$; hence $\overline{AC}^{\frac{1}{2}} = B$ (*Ax.* 1. *B.* III. *C.* I.)

Example: Betwixt 2 and 8 a Geometrical Mean is 4; for $2 \times 8 = 16$, and the Square Root of 16 is 4.

But if the Product of the Extremes has not a determinate Square Root, the Mean ſought is a Surd, or an infinite Series of decreaſing Quantities, as has been explain'd in *Book* III. So betwixt 2 and 7 the Mean is the Square Root of 14 = 3. 74, *&c.* which is continued *in infinitum*, according to the Method of Approximation, explained in *Book* III. *Ch.* I.

Theorem II.

If four Numbers are $::l$, $A:B::C:D$, they are so also *reversely*; that is, making the Consequents the Antecedents: Thus, $B:A::D:C$, or $D:C::B:A$.

In Numbers. If it be $2:3::4:6$, then it is $3:2::6:4$.

Demonstr. This follows evidently from the *Definition*; for since A and C do equally contain, or are contained in B and D, then, reversely, B and D are equally contained in, or do equally contain, A and C; which is the nature of $::l$.

Or thus: Since $\frac{A}{B}=\frac{C}{D}$, then the reciprocal Quotes are equal (see the general *Schol.* at the End of *Chap.* I.) *viz.* $\frac{B}{A}=\frac{D}{C}$, *i.e.* $B:A::D:C$.

Or also thus: $\frac{B}{B}=\frac{D}{D}$, and $\frac{A}{B}=\frac{C}{D}$; therefore, (by *Ax.* 1.) $\frac{B}{B}:\frac{A}{B}::\frac{D}{D}:\frac{C}{D}$. Again; $\frac{B}{B}:\frac{A}{B}::B:A$, and $\frac{D}{D}:\frac{C}{D}::D:C$ (general *Coroll.* 15.); therefore $B:A::D:C$ (*Ax.* 3.)

Or the same Truth appears simply in this Representation, *viz.* $A:Ar::B:Br$; whence $Ar:A::Br:B$, the Ratio being still the same.

Or lastly; It follows from the equal Product of the Extremes and Means; for all the Change made by reversing the Terms is, that the Extremes are become the Means, and the Means become the Extremes, and the Product of the Extremes and Means are still equal, which makes $::l$.

Theorem III.

If four Numbers are $::l$, $A:B::C:D$, they are so also *alternately*; that is, comparing the two Antecedents to one another, and the two Consequents: Thus, $A:C::B:D$.

In Numbers, if $3:5::6:10$, then $3:6::5:10$.

Demonstr. These Quotes being equal, $\frac{A}{B}=\frac{C}{D}$, the alternate Quotes are also equal, $\frac{A}{C}=\frac{B}{D}$ (see general *Schol.* at the End of *Chap.* 1.); hence $A:C::B:D$.

Or thus: A and C are the same Fractions, proper or improper, of B, D; but the same Fractions of two Numbers are the same Fractions of one another as these Numbers are (*Coroll.* 6. *Lemma* 2. *Chap.* I. *Book* II.), *i.e.* A is the same Fraction of C as B is of D, and Like or Equal Fractions make equal Ratios: Therefore $A:C::B:D$.

Or we may reason thus, $\frac{A}{B}:\frac{C}{B}::A:C$ (gen. *Coroll.* 15.); also $\frac{C}{D}:\frac{C}{B}::B:D$ (gen. *Coroll.* 16.); but $\frac{A}{B}=\frac{C}{D}$, therefore $\frac{A}{B}:\frac{C}{B}::B:D$; hence, lastly, $A:C::B:D$.

Or represent four Numbers $::l$ thus, $A:Ar::B:Br$; then it is plain, that $A:B::Ar:Br$ (by gen. *Coroll.* 15.)

Or, lastly, This does also follow from the equal Product of Extremes and Means, whose Factors are not changed, except in their Order, which does not alter the Product, it being still $AD=BC$.

COROLLARIES.

1. Of four Numbers $::l$, $A:B::C:D$, if A, B, are lesser than C, D, or these lesser than those, the two lesser are Like Fractions of the two greater; because $A:C::B:D$; or, reversly, $C:A::D:B$.

2. If

2. If three Numbers are given, to find a 4th :: l, and if the first is an aliquot Part or Multiple of the third, and that this can be easily discerned, then the 4th will be more easily found by making the 3d Term the 2d, and applying the first *Rule* of *Probl.* I.

Example: To these, $4:7::12$, a 4th is 21; for 4 is the third Part of 12, therefore I multiply 7 by 3.

3. If 4 Numbers are :: l, $A:B::C:D$, then if A is lesser, greater, or equal to C, B is also greater, lesser, or equal to D, because $A:C::B:D$.

THEOREM IV.

IF four Numbers are :: l, $A:B::C:D$, they are so also *compoundly*; i. e. the Sums of the Antecedents and of the Consequents are proportional with each Antecedent and its Consequent: Thus, $A+C:B+D::A:B$, or $::C:D$. Also the Sums of each Antecedent and its Consequent are proportional with the two Antecedents, or the two Consequents. Thus $A+B:C+D::A:C$, or $::B:D$.

Example: $3:5::6:10$, and $9:15::3:5$, also $8:16::3:6$.

DEMONSTR. The Antecedents A, C, are Like Fractions (proper or improper) of their Consequents, B, D (by the Nature of Ratios); but the Sum of the Like Fractions of two Numbers is the Like Fraction of the Sum of these Numbers, (*Lem.* 2. *Chap.* I. *Book* 2.) that is, $A+C$ is the same Fraction of $B+D$, as A is of B, or also C of D; and the same Fraction is the same Ratio; therefore $A+C:B+D::A:B::C:D$. For the second Part, *viz.* $A+B:C+D::A:C::B:D$, this follows from the same Principle; having first alternated the given Numbers, thus, $A:C::B:D$.

But we may demonstrate this otherwise, thus, $\frac{A}{B}=\frac{C}{D}$, and $\frac{B}{B}=\frac{D}{D}$; hence $\frac{A}{B}+\frac{B}{B}=\frac{C}{D}+\frac{D}{D}$; and by the Addition of these Quotes, considered as if they were Fractions (see gen. *Schol.* at the End of *Ch.* I.) it is $\frac{A+B}{B}=\frac{C+D}{D}$; hence $A+B:C+D::B:D$, which is the one Part; and for the other, since $\frac{A}{C}=\frac{B}{D}$, and $\frac{C}{C}=\frac{D}{D}$; hence $\frac{A+C}{C}=\frac{B+D}{D}$, and $A+C:B+D::C:D$, or $::A:B$.

Or also from the equal Products of Extremes and Means, $\overline{A+C}\times B=\overline{B+D}\times A$; for $\overline{A+C}\times B=AB+CB$, and $\overline{B+D}\times A=AB+AD$ (either by *Lem.* 3. *C.* V. *B.* I. if B and A are Integers; or by *Lem.* 2. *Ch.* I. *Book* II. if they are Fractions): But $AD=BC$ (*Theor.* 1.) hence $AB+CB=AB+AD$; *i. e.* $\overline{A+C}\times B=\overline{B+D}\times A$; wherefore $A+C:B+D::A:B$ (by the Reverse of *Theor.* 1.)

Or see it in this Representation, $A:Ar::B:Br$; hence it is plain that $A+B:Ar+Br\ (=\overline{A+B}\times r)::A:Ar$, the common Ratio being r; or $A+Ar\ (=\overline{1+r}\times A):B+Br\ (=\overline{1+r}\times B)::A:B$, these being equally multiplied in the first Pair by $\overline{1+r}$: Or in both Cases you see an equal Product of Extremes and Means.

COROLLARIES.

1. If there are ever so many Couplets of Numbers in the same Ratio, the Sum of any Number or all of the Antecedents is to the Sum of the same Number of the Consequents, as any of the Antecedents to its Consequent. Thus, if $a:A::b:B::c:C::d:D$. &c. then $a+b+c+d:A+B+C+D::a:A::$ &c. For $a+b:A+B::a:A$, or $c:C$, or $d:D$; hence again, $a+b+c:A+B+C::c:C$, or $d:D$, and again, $a+b+c$

$+d : A+B+C+D :: d : D$. Or it follows fimply from the fame general Principles as the *Theorem*, viz. The Sum of the Like Fractions of any Numbers, however many there be of them, is the Like Fraction of the Sum of thefe Numbers.

2. If four Numbers $::l$ are equally multiplied, the Products are alfo $::l$, and in the fame Ratio: Thus, if $A:B::C:D$ are multiplied by r, thefe are $::l$, $Ar:Br::Cr:Dr$; for Multiplying is only a repeated Addition: But this follows alfo from gen. *Coroll* 15. for $Ar:Br::A:B$, and $Cr:Dr::C:D$; hence $Ar:Br::Cr:Dr$.

3. If $A:B::C:D$, and $M:N::O:P$; and if the Ratio is the fame in both Ranks, *i. e.* $A:B::M:N$, and $C:D::O:P$, then the Sums of their correfponding Terms is alfo a Rank of $::l$'s; that is, $A+M.B+N::C+O:D+P$.

Obferve, The firft *Coroll.* is the fame thing in effect as this Propofition, *viz.* if two or more Numbers are compofed by addition of the fame number of Parts, thofe of the leffer Whole being leffer, compared one to one refpectively from the leaft to the greateft, than thofe of the greater Whole, and all in the fame Ratio to their Correfpondents in the other, Thefe *Wholes* are alfo in the fame *Ratio*.

THEOREM V.

If four Numbers are $::l$, $A:B::C:D$, they are fo *divifively*: Thus, the Difference of the Antecedents is to the Difference of the Confequents as each Antecedent to its Confequent, $A-C:B-D::A:B$, and alfo as $C:D$. Alfo, the Differences of each Antecedent and its Confequent are as each Antecedent or Confequent to the other, $A-B:C-D::A:C$, and alfo as $B:D$; and it is the fame thing if the Antecedents are leffer than their Confequents, or A leffer than C, and B than D; for then it is $C-A:D-B::A:B$, and $B-A:D-C::A:C$.

Example: $3:7::15:35$, and $12:28::3:7$; alfo $4:20::3:15$.

Demonstr. A, C, are Like Fractions of B, D, and (by *Lem.* 3. *Ch.* I. *Book* II.) the Difference of the Like Fractions of two Numbers is the Like Fraction of the Difference of thefe Numbers; that is, $A-C$ or $C-A$ is the fame Fraction of $B-D$ or $D-B$, as is A of B or C of D; therefore $A-C$ or $C-A:B-D$ or $D-B::A:B::C:D$. The fecond Part, *viz.* $A-B:C-D::A:C::B:D$, follows from the Alternation of the given Numbers.

Or we may demonftrate this *Theorem* thus: $\frac{A}{C}=\frac{B}{D}$, and $\frac{C}{C}=\frac{D}{D}$; hence $\frac{A-C}{C}=\frac{B-D}{D}$; that is, $A-C:B-D::C:D$, or $A:B$.

Or thus alfo; $\overline{A-C}\times B=AB-CB$, and $\overline{B-D}\times A=AB-AD$; but $AD=BC$, hence $AB-CB=AB-AD$; that is, $\overline{A-C}\times B=\overline{B-D}\times A$, the Product of the Extremes equal to that of the Means. Hence $A-C:B-D::A:B$.

Or thus; $A:Ar:B:Br$, and $A-B:Ar-Br$ $(=\overline{A-B}\times r)$ $::A:Ar$, the common Ratio being r. Alfo $Ar-A$ $(=\overline{r-1}\times A)$ $:Br-B$ $(=\overline{r-1}\times B)$ $::A:B$, thefe being equi-multiplied in the firft Pair by $\overline{r-1}$; or in both Cafes you fee an equal Product of Extremes and Means.

Scholium. This *Theorem* is the fame in effect as this Propofition, *viz* If any two Numbers are in the fame Ratio to one another as the Parts taken away, the Parts remaining are alfo in the fame Ratio. Thus, fuppofe A, B to be greater than C, D, then A, B, being confider'd as two Wholes, C, D are the Parts taken away, and $A-C$, $B-D$, are the

the Parts remaining. And the Propoſition is clearly expreſs'd in Signs thus, $A - C : B - D :: A : B :: C : D$; and if C, D are the greater, it is $C - A : D - B :: C : D :: A : B$.

We may alſo expreſs the Propoſition thus: If two Numbers are each the Sum of other two Numbers, or Parts, and if the one Whole is in the ſame Ratio to the other as one Part of the firſt Whole is to one Part of the other, then the other Parts are alſo in the ſame Ratio: Thus, A, B being the Wholes, C, and $A - C$ are the Parts of A, and D, $B - D$, the Parts of B; for $A - C + C = A$, alſo $B - D + D = B$.

The like *Corollaries* follow from this *Theorem* as from the laſt, by applying *Subtraction* and *Diviſion*, as we did there *Addition* and *Multiplication*: To which we may add the following

COROLLARIES.

1. Of four Numbers, A, B, C, D, if it is $A + C : B + D :: A : B$, or as $C : D$; but let it not be affirm'd to be both as $A : B$ and alſo as $C : D$, yet this will follow, that $A : B :: C : D$, and conſequently that $A + C : B + D$ is both as $A : B$ and $C : D$; For $A + C$ and $B + D$ are two Wholes, which being ſuppoſed in the ſame Ratio as any one of the Parts, $A : B$, the other two Parts are alſo in the ſame Ratio, by this *Theorem* and *Scholium*; that is, $A : B :: C : D$, and hence alſo $A + C : B + D :: A : B :: C : D$. From this again follows, that

2. If two Wholes are compoſed each of two Parts, and if the Parts of the one are not both in the ſame Ratio to the Parts of the other, the Wholes are in neither of theſe Ratios; for if they were in the Ratio of any one of them, they would be in the Ratio of both, and conſequently the Parts would be in the ſame Ratio, contrary to Suppoſition.

3. Of four Numbers, A, B. C, D, if $A - C : B - D :: A : B$, or $C : D$; but we don't ſay *and alſo as* $C : D$, then it will be $A : B :: C : D$, and conſequently $A - C : B - D :: A : B :: C : D$; for A, B, are here two Wholes, C and $A - C$ are the two Parts of A; and D, $B - D$ the Parts of B; but one of the Parts being in the Ratio of the Wholes, *viz.* $A - C : B - D :: A : B$ ſo are the other Parts, by this *Theorem*; that is, $A : B :: C : D$; and if $A - C : B - D :: C : D$, that is, if both the Parts of the one Whole are in the ſame Ratio to thoſe of the other, the Wholes are in the ſame Ratio, by *Theor.* IV. that is, $A : B :: C : D$.

THEOREM VI.

If four Numbers are $::l$, $A : B :: C : D$, they are ſo alſo *mixtly*; that is, comparing the Sums and Differences of the Antecedents and Conſequents; thus,

$$A + C : B + D :: A - C, \text{ or } C - A : B - D, \text{ or } D - B.$$
$$\text{Alſo } A + B : C + D :: A - B, \text{ or } B - A : C - D, \text{ or } D - C.$$

DEMONSTR. This follows from the two laſt; for the Sums or Differences here compared are in the ſame Ratio of one of the Antecedents to its Conſequent, *viz.* as $A . B$ in the firſt Part; or as the one Antecedent to the other, *viz.* $A : C$, in the ſecond Part.

SCHOLIUM. The three laſt *Theorems* may alſo be taken *reverſly* or *alternately*.

THEOREM VII.

If there are ever ſo many Ranks of four Numbers $::l$, and if any two of the comparative Terms (*i. e* the 1ſt and 2d, or 1ſt and 3d, or 3d and 4th, or 2d and 4th) are common to all the Ranks, then the other Couplets in every Rank are all in the ſame Ratio. Or if it's thus, *viz.* Two comparative Terms of the 1ſt Rank common to the 2d Rank,

2d Rank, and the remaining two of the 2d Rank common to the 3d Rank, and so on, then the other Couplets in each Rank will be $::l$.

Demonstr. All this is the simple and immediate Application of *Axiom* 3. and *Theorem* III. and needs no more Explication but a few *Examples*, where you may see the different Forms in which these things may appear.

EXAMPLES.

(1.)	(3.)	(5.)
If $A:B::C:D$, and $C:D::E:F$; then $A:B::E:F$.	If $A:B::C:D$, and $E:B::F:D$; then $A:E::C:F$.	If $A:B::C:D$, and $C:D::E:F$, and $E:F::G:H$; then $A:B::E:F::G:H$.
(2.)	(4.)	(6.)
If $A:B::C:D$, and $B:D::E:F$; then $A:C::E:F$.	If $A:B::C:D$, and $B:E::D:F$; then $A:C::E:F$.	If $A:B::C:D$, and $A:C::E:F$, and $G:H::A:C$; then $B:D::E:F::G:H$.

Theorem VIII.

If there are two Ranks of 4 Numbers $::l$, which have two comparative Terms common to both, the Sums or Differences of the Antecedents and Consequents of the two different Couplets are in the same Ratio with the Antecedent and Consequent of the common Couplet. Thus:

If $A : B::C : D$,
and $E : B::F : D$;
then $A+E:B::C+F:D$;
also $A-E:B::C-F:D$.

Demon. By the preceding $A:C::E:F$; whence $A+E$, or $A-E:C+F$, or $C-F::B:D$; or alternately, as in the Margin.

Again; Because $A+B:C+D::B:D$, and $E+B:F+D::B:D$; therefore these Proportions are also true, *viz.*

$A+E:C+F::A+B:C+D::E+B:F+B$,
and $A-E:C-F::A-B:C-D::E-B:F-B$.

Also any Couplet of the first Rank is $::l$ with any of the second; being all as $B:D$.

Again; It is also $A+E:B+B::C+F:D+D$. Since $2B:2D::B:D$.

Theorem IX.

If there are two Ranks of four Numbers $::l$, whereof the Extremes or Means of the one are the same as the Extremes or Means of the other; or if they are reversly the Means or Extremes of the other; then the remaining four Terms are reciprocally $::l$. *i. e.* make the remaining two of the one Rank the Extremes, and those of the other the Means, and these four are $::l$. Thus:

If $A:B::C:D$, and $E:B::C:F$; then $A:E::F:D$.	If $A:B::C:D$, and $A:E::F:D$; then $B:E::F:C$.	If $A:B::C:D$, and $B:E::F:C$; then $A:E::F:D$.	If $A:B::C:D$, and $E:A::D:F$; then $B:E::F:C$.

Demon. From the equal Products of Extremes and Means, it is $AD=BC=EF$; in which all these Conclusions are comprehended.

Theo-

THEOREM X.

IF four Numbers are $::l$, and if any two of the comparative Terms are equally multiplied or divided; or if the one Extreme or Mean is multiplied, and the other equally divided: or again, if the one Extreme is multiplied, and the other equally divided; and at the same time the one Mean multiplied by any other, or the same Number, and the other equally divided; the Proportionality still remains, tho' in the second Case the Ratio is changed; and will be also in the third Case, when two different Multipliers are employed. Thus: If $A:B::C:D$, then these Proportions follow, *viz.*

$A:B::Cn:Dn.$ $An:B::Cn:D.$	$A:B::\frac{C}{n}:\frac{D}{n}.$ $A:\frac{B}{n}::C:\frac{D}{n}.$	$An:B::C:\frac{D}{n}.$ $A:\frac{B}{n}::Cn:D.$	$An:Bn::Cn:Dn.$ $\frac{A}{n}:\frac{B}{n}::\frac{C}{n}:\frac{D}{n}.$
$An:Br::Cn:Dr.$ $\frac{A}{n}:\frac{B}{r}::\frac{C}{n}:\frac{D}{r}.$	$\frac{A}{n}:\frac{B}{n}::\frac{C}{r}:\frac{D}{r}.$ $An:Br::\frac{C}{r}:\frac{D}{n}.$	$\frac{A}{n}:Br::\frac{C}{n}:Dr.$	$An:\frac{B}{r}::Cr:\frac{D}{n}.$

Again; Instead of dividing them, we may apply the given Numbers as Divisors; which will make the following Proportions:

$\frac{n}{A}:\frac{n}{B}::\frac{n}{C}:\frac{n}{D}.$ $\frac{r}{A}:\frac{n}{B}::\frac{r}{C}:\frac{r}{D}.$	$\frac{n}{A}:\frac{n}{B}::D:C.$ $\frac{A}{n}:\frac{B}{r}::\frac{r}{D}:\frac{n}{C}.$	$An:\frac{r}{D}::Cn:\frac{r}{B}.$ $\frac{r}{A}:\frac{n}{B}::Dr:Cn.$

DEMON. In all these Conclusions, and many more that may be contrived of this Nature, the Truth of the Proportion is evident from the equal Product of the Extremes and Means; founded all upon this, that $AD=BC$, $nr=nr$, $\frac{A}{B}=\frac{C}{D}$, and $\frac{r}{n}=\frac{r}{n}$.

You'll find also other complex ways of arguing with proportional Numbers in the next Chapter.

§. 2. *Of* Geometrical *Progressions.*

Observe, *By the Distance of one Term of a Series from another, is meant the Number of Terms from the one exclusive to the other inclusive; or including both, it is the Number of Terms less* 1. *So if the Number of Terms is* n, *the Distance of the Extremes is* $n-1$.

PROBLEM III.

Having the first Term and Ratio to raise a Geometrical Series.

RULE. IF the given Ratio is a whole Number or mixt, then for an increasing Series multiply, and for a decreasing divide the first Term by the Ratio, the Product or Quote is the second Term; which multiplied or divided by the Ratio, gives the third Term, and so on. But if the Ratio is a proper Fraction, this of itself determines that the Series ought to increase, and we must multiply by the Reciprocal of the Ratio.

Example 1. First Term 2, Ratio 3, the increasing Series is $2:6:18:54$, *&c.* and the decreasing Series $2:\frac{2}{3}::\frac{2}{9}:\frac{2}{27}$, *&c.*

Example 2. Firſt Term 2, Ratio 4, the Series is 2 : 8 : 16 : 32, *&c.* Or thus, 2 : $\frac{1}{2} : \frac{1}{8} : \frac{1}{32}$, *&c.*

Demon. The Reaſon of this Rule is manifeſtly contained in the Definition of Geometrical Relation; ſee the *Coroll.* after it: for when the Ratio is a whole or mixt Number, which ſoever of the two Views there explained it is taken in, the Rule is good; becauſe for an increaſing Series, a whole or mixt Number is in the firſt View the Ratio, and in the ſecond it is the reciprocal Ratio; and by the *Coroll.* referred to, the firſt Term or Antecedent ought to be multiplied by this; and for a decreaſing Series, a whole or mixt Number is the Ratio in both Views, and Diviſion is the Rule. Laſtly, If the given Ratio is a proper Fraction, it is taken in the ſecond View, and neceſſarily infers an increaſing Series; and therefore the Antecedent multiplied by the reciprocal Ratio produces the Conſequent.

SCHOLIUMS.

1. If the Ratio is expreſſed by two Numbers ordered in a certain Compariſon, the one as Antecedent, and the other as Conſequent, then, that determines whether the Series decreaſes or increaſes; and accordingly, if we make a Fraction of the Antecedent ſet over the Conſequent, (which is the Ratio in the ſecond View) and by it divide, we ſhall raiſe an increaſing or decreaſing Series, acccording as the Ratio is a proper or improper Fraction. For *Exam.* If the Ratio is thus expreſſed, *viz.* the Ratio of 2 to 3, or 2 : 3; then dividing by $\frac{2}{3}$ makes an increaſing Series. But if it's the Ratio of 3 : 2, then dividing by $\frac{3}{2}$ makes a decreaſing Series.

2. Again: If the two firſt Terms of a Series are given, (which do contain the Ratio) then the Series is continued by the common Rules of finding a third or fourth $\div l$, *viz.* finding a third to the two given ones, and then a fourth to the ſame two, and the Term laſt found. And ſo a Series from $a : b$, whether increaſing or decreaſing, will be thus repreſented, $a : b : \frac{b^2}{a} : \frac{b^3}{a^2} : \frac{b^4}{a^3}$, *&c.* But when a Number diſtinct from the two firſt Terms is given for the Ratio, then

3. Any Geometrical Progreſſion, whether increaſing or decreaſing from a given Number a, may be clearly repreſented thus, $a : ar^2 : ar^3$, *&c.* adding ſtill an Unit at every Step to the Index of r: for according as r is ſuppoſed greater or leſſer than 1, ſo is the Series increaſing or decreaſing. If the Series increaſes, then is r the Ratio in one of the two Senſes explained, or the Reciprocal of it in the other. For it is the Quote of ar, the greater Term divided by a the leſſer; but the reciprocal Quote of a divided by ar, which is the Ratio in another View. Again; If the Series decreaſes, r is a proper Fraction, and is the Reciprocal of the Ratio taken in either View, which are in this Caſe the ſame, *viz.* the Quote of a divided by ar, which is $\frac{a}{ar} = \frac{1}{r}$, whoſe Reciprocal is r.

Wherefore, if we take the Ratio always under the Notion of the Quote of the Antecedent divided by the Conſequent; then r is to be underſtood as the Reciprocal of the Ratio; which makes the Series either increaſing or decreaſing, according as r is greater or leſſer than 1. And therefore tho' we call r the Ratio, yet if we underſtand it as the reciprocal Quote of the Antecedent divided by the Conſequent, there will be no Ambiguity or Hazard of Error; and hereby we ſhall ſave the Trouble of making different Repreſentations for increaſing and decreaſing Series. Yet in ſome Caſes it will be convenient to repreſent them differently; and then taking the Ratio always as the Quote of the greater Term

Term divided by the leſſer, which muſt be a whole or mixt Number; and calling that Ratio r, an increaſing Series will be thus repreſented, $a : ar : ar^2$, &c. and a decreaſing thus, $a : \frac{a}{r} : \frac{a}{r^2}$:, &c. Wherefore if we repreſent a Series in the firſt manner, it's ſuppoſed to be taken indifferently for increaſing or decreaſing, unleſs it's expreſly ſaid to be increaſing; but the other manner does always expreſs a decreaſing Series.

From theſe Rules and Expreſſions of Geometrical Series we have the following

COROLLARIES.

1. If the leſſer Extreme of a Series is the Ratio, then the other Terms are the ſeveral Powers of the Ratio, thus: $r : r^2 : r^3 : r^4$, &c. or $A : A^2 : A^3 : A^4$, &c. For here r or A is the Ratio, and $r \times r = r^2$; $r^2 \times r = r^3$, and ſo on: Whence it's plain, that the Series of the Powers of any Number is a continued Geometrical Progreſſion.

2. If 1 is the firſt Term of a Series, the ſecond Term is the Ratio; and all the ſucceeding Terms are the ſeveral Powers of the Ratio. Thus, $1 : r : r^2$, &c. for the Ratio of 1 to r being r, the ſecond Term muſt be r^2, the third r^3, and ſo on.

3. Betwixt 1, and any Power of a Number as r^n, there fall as many Geometrical Means as the Index of that Power leſs 1, *viz.* $n-1$, in the Ratio of 1 to the Root; for every Term after 1 is ſuch a Power of the ſecond, whoſe Index is equal to the Number of Terms after 1 to that Power; the thing is evident in the univerſal Series, $1 : r : r^2 : r^3 : r^4$, &c.

4 Every Geometrical Series, whoſe firſt Term is not 1, is equal to ſuch a Series multiplied by the given firſt Term; the Ratio of that Series being the ſame with that of the given Series. Thus: $A : Ar : Ar^2 : Ar^3 : Ar^4$:, &c. is no other than the Products of this Series, $1 : r : r^2 : r^3 : r^4$:, &c. multiplied by A.

5. As any Progreſſion may be thus repreſented, $A : Ar : Ar^2 : Ar^3$:, &c. it's manifeſt that the Index of the Ratio in every Term expreſſes the Diſtance of that Term from the firſt A: And hence it follows immediately, that

6. Every Term is equal to the Product of the firſt, by that Power of the Ratio whoſe Index is the Diſtance of that Term from the firſt; and reverſly, the firſt Term is equal to the Quote of any greater Term divided by that Power of the Ratio, whoſe Index is the Diſtance of that Term from the firſt. So, for *Example,* the firſt Term being A, the Ratio r, and the Diſtance of any Term from A being d, that Term is Ar^d; and if you call that Term L, then is $L = Ar^d$; and reverſly, $A = L \div r^d$.

SCHOLIUM. Becauſe the greater Extreme of a Series is, by what's now ſhewn, equal to Ar^d, or Ar^{n-1}, (A being the leſſer Extreme, r the common Ratio, and d the Diſtance of the Extremes equal to $n-1$, the Number of Terms leſs 1) and every Term below having the Ratio involved in it to one degree leſs than the preceding greater Term; therefore a decreaſing Series, which, when the greater Extreme is called l, is repreſented thus, $l : \frac{l}{r} : \frac{l}{r^2} : \frac{l}{r^3}$:, &c. may alſo be repreſented thus: $Ar^d : Ar^{d-1} : Ar^{d-2}$:, &c. Or thus, $Ar^{n-1} : Ar^{n-2} : Ar^{n-3}$:, &c. (becauſe $d = n-1$) going on ſo till the Index of r be equal to 1, and then we have Ar the Term next to A.

7 But again more univerſally: From the ſame Expreſſion of a Series, it's manifeſt that the Difference of the Indexes of the Ratio in any two Terms expreſſes the Diſtance of theſe two Terms; for in every Step aſcending, the Ratio is involved once more than in the preceding; and therefore from any Term to any other, it's as much oftner involved in the greater than in the leſſer, as their Diſtance expreſſes; *that is,* the Difference of their Indexes is their Diſtance; and hence it follows immediately, that

8. Any Term of a Geometrical Series is equal to the Product or Quote of any other lesser or greater Term multiplied or divided by such a Power of the Ratio, whose Index is the Distance of these Terms; and any Term divided by any other lesser Quotes such a Power of the common Ratio, whose Index is the Distance of these Terms. For any Term being expressed Ar^d, (d being the Distance of this Term from A) any greater Term must have a Power of r, whose Index exceeds d by the Distance of these Terms (by the last) so, that Distance being m, the greater Term is Ar^{d+m}; but $r^d \times r^m = r^{d+m}$; (*Th.6. Ch.1. B.3.*) consequently, $Ar^{d+m} = Ar^d \times r^m$; and reversly, $Ar^{d+m} \div r^m = Ar^d$, and $Ar^{d+m} \div Ar^d = r^m$. Or this Truth may be deduced from *Coroll. 6.* Thus: Any Part of a Series, *i. e.* from any Term to another is still a Geometrical Series, whereof these two Terms are the Extremes; which being called A, L, and their Distance d, it's shewn that $A = L \div r^d$, and $L = Ar^d$; and lastly, $L \div A = r^d$.

Exam. In this Series, 2 : 6 : 18 : 54 : 162 : 486. If we compare 6 and 162, whose Distance is 3; then is $6 = 162 \div$ Cube of 3, or 27; and reversly, $162 = 6 \times 27$; and lastly $162 \div 6 = 27$.

Schol. The immediate Use and Application of this last Truth we have in the Solution of these *Problems.*

(1.) Having any one Term of a Series and the Ratio, to find a Term at any Distance from the given one, without finding all the intermediate ones: The Solution of which is plainly contained in this *Coroll.* Thus: Take that Power of the Ratio, whose Index is the given Distance; and by it multiply or divide the given Term, and you have the Term sought above or below the given one. Thus: Any Term of a Series being multiply'd or divided by the fourth Power of the Ratio, gives a Term, which is the fourth above or below after the given one.

(2.) Having any two Terms of a Series and their Distance, to find any other Term at any Distance from either of the given ones; which is solved thus: Divide the greater by the lesser of the given Terms, the Quote is a Power of the Ratio, whose Index is the Distance of the given Terms. Suppose this Distance to be d, and extract the d Root of that Quote, (by the Methods explained, or referred to in *Book* III.) it is the Ratio sought: (all which is immediately contained in this *Coroll.*) Then having the Ratio, by it and any given Term we can find any other Term at any given Distance from the given Term, as in the preceding *Problem.*

Observe, For some Cases of this *Problem* there is a more easy Solution, which you'll find in *Theo.* XX see the 3d Article of the *Schol.* after the 4th *Coroll.* as particularly, suppose the Term sought is lesser or greater then either of the Terms given, and its Distance from the lesser if it's greater than either of them, or from the greater if it's lesser, is a Multiple of the Distance of the given Numbers.

(3.) Of these three things, *viz.* the common Ratio of a Series, the Distance of any two Terms, and the Ratio of these two Terms; having any two we can find the third: for any Term being called A, r the common Ratio, and d the Distance of any greater Term from A, that Term is Ar^d, and the Ratio of A to Ar^d is r^d; which if we call R, then (1.) if d, r, are given, R $(=r^d)$ is also known. (2.) If R and d are given, r is also known, *viz.* by extracting the d Root of $R = r^d$. (3. If r and R are given, d is also known, *viz.* by raising r to a Power which is equal to R $(=r^d)$ and the Index of it is d.

9. The Sum of the Extremes of a Geometrical Series is equal to the Product of the lesser Extreme multiplied into the Sum of 1, and such a Power of the Ratio whose Index is the Distance of the Extremes. Thus, A being the lesser, L the greater Extreme, and d the Distance of the Extremes; $A + L = A + Ar^d$, for $L = Ar^d$; and consequently $A + L = A + Ar^d = A \underline{+ r^d + 1}$. Again; Because any two Terms of a Series may be considered as Extremes of a lesser Series, the same Rule will be good for expressing their Sum. Thus,

Thus, one Term being Ar^d, and another greater being Ar^{d+m}, their Sum is $Ar^d + Ar^{d+m} = Ar^d \times \overline{r^m + 1}$.

(4.) The Difference betwixt the Extremes of a Series is equal to the Product of the lesser multiplied by the Difference of 1 and such a Power of the Ratio whose Index is the Distance of the Extremes: Thus, $L - A = Ar^l - A = \overline{r^l - 1} \times A$. The same way may the Difference of any two Terms be expressed; so, $Ar^{d+m} - Ar^d = \overline{r^m - 1} \times Ar^d$, in all which the Indexes express the Distances of the Terms from the lesser Extreme A.

Theorem XVIII.

The Sums or Differences of every two adjacent Terms in a Geometrical Series make also a Geometrical Series, and in the same Ratio. Thus, if A : B : C : D, &c. is a Geometrical Series, so is this also A + B : B + C : C + D, &c. and this, B — A : C — B : D — C, &c. or A — B : B — C : C — D, &c.

Example: 2 : 4 : 8 : 16 : 32, being a Geometrical Series, so are the Sums 6 : 12 : 24 : 48, and the Differences 2 : 4 : 8 : 16.

Demonstr. 1. For the Sums, It's true of any Series of four Terms, A : B : C : D. For A : B : : B : C; therefore A + B : B + C : : A : B. Again, B : C : : C : D, hence B + C : C + D : : B : C (*Theor.* IV.) wherefore A + B : B + C : : B + C : C + D; that is, A + B : B + C : C + D, are in the same Ratio of A : B, or B : C. By the same Reasoning, B + C : C + D : D + E will be found to be in the continued Ratio of B : C, or C : D, so that A + B : B + C : C + D : D + E, are in continued Progression. And so the Reason proceeds thro' the Series *in infinitum*.

2. For the *Differences*, It's true of any Series of four Terms A : B : C : D; for since A : B : : B : C, then (by *Theor.* V.) B — A : C — B : : A : B, or B : C; and since B : C : : C : D, then C — B : D — C : : B : C. Hence B — A : C — B : D — C, are in continued Progression, and so on to more Terms.

Or we may shew the Truth of this *Theorem* yet more simply, thus; $A : Ar : Ar^2 : Ar^3 : Ar^4$, &c. being a Geometrical Progression; so is $A + Ar : Ar + Ar^2 : Ar^2 + Ar^3 : Ar^3 + Ar^4$; and $Ar - A : Ar^2 - Ar$, &c. the continual Ratio of these being evidently r; for $\overline{A + Ar} \times r = Ar + Ar^2$, and so of the rest; also $\overline{Ar - A} \times r = Ar^2 - Ar$.

The *Reverse* of this *Theorem* is not true, for tho' the Sums of every two adjacent Terms of a Series are continuedly proportional, yet it will not always be true that that Series is so; for *Example:* 3 : 4 : 10 : 18 are not $\div l$; yet 3 + 4 : 4 + 10 : 10 + 18, or 7 : 14 : 28, are so.

Theorem XIX.

If there be two Geometrical Progressions in the same Ratio, then any two Terms in the one, and any two equally distant in the other, are in the same Ratio, or Proportional.

Example: 2 : 4 : 8 : 16 : 32; and 3 : 6 : 12 : 24 : 48, are in the same Ratio; hence 4 : 16 : : 3 : 12, or 4 : 32 : : 3 : 24.

Demonstr. The Reason is plain from the last; for of Terms at equal Distance the greater is always the Product of the lesser by that Power of the Ratio whose Index is the Distance; therefore the Ratio and Distance being equal, these Terms are in the same Ratio to one another, which Ratio is that Power of the common Ratio of their respective Series. Universally, let one Series be $a : ar : ar^2 : ar^3 : ar^4$; and another $b : br : br^2 : br^3 : br^4$. Then $ar : ar^3 : : br^2 : br^4$, the common Ratio here being r^2. Or thus, $ar^n : ar^{n+m} : : br^o : br^{o+m}$, the Ratio being r^m.

Coroll.

Coroll. Hence having any two Terms of a Series, and any Term of another Series which has the ſame Ratio with the former, we can find a Term of this other as far diſtant from the given one as the two given Terms of the other Series are. The Application is plain.

Theorem XX.

Any two Terms of a Geometrical Progreſſion are $::l$ with any other two equally diſtant; alſo any three or more equally diſtant are $\div l$; ſo in this, $2:4:8:16:32:64:128$, theſe are $::l$, $2:4::64::128$, and theſe, $2:16::8:64$: Again, theſe are $\div l$, $4:16:64$, and theſe $2:8:32:128$.

Demonstr. The *Reaſon* of this is the ſame as in the former *Theorem*, viz. the Equality of Diſtances which makes an Equality of Ratios betwixt the Numbers compared.

Corollaries.

1. If any two Terms are multiplied, and the Product divided by another Term, the Quote is equal to a fourth Term as far diſtant from one of the Terms multiplied on the one hand, as the Diviſor is from the other of them on the other hand For ſince any four Terms are $::l$, whereof the two leſſer are at the ſame Diſtance as the two greater, (and conſequently the leaſt and the next to the greateſt, at the ſame Diſtance as the greateſt and the next to the leaſt) therefore the Product of the two middle Terms of theſe four being divided by either of the Extremes, muſt quote the other; or the Product of the Extremes being divided by either of the Means, muſt quote the other; whence the *Corol.* is manifeſt.

Obſerve again, That if the Diviſor is one of the Extremes, *i e.* lies on the ſame hand of (or is leſſer or greater than) either of the Terms multiplied, the Quote will be the other Extreme; *i.e.* will lie on the oppoſite hand of (or be contrarily greater or leſſer than) either of the Terms multiplied; conſequently the Diſtance of the Quote from the Diviſor will be the Sum of the Diſtances of both the Terms multiplied from the Diviſor (or from the Quote itſelf, which is at the ſame Diſtance): But if the Diviſor is one of the Means, *i. e.* lies betwixt (or is leſs than the one and greater than the other of) the Terms multiplied; and conſequently the Diſtance of the Quote from the Diviſor will be the Difference of the Diſtances of the Terms multiplied from the Diviſor, or from the Quote itſelf, which is the ſame Diſtance.

Example: In this Series, $a:b:c:d:e:f:g:h$, it's true that $a:c::f:h$, where $cf \div a = h$, which is as far from f on the one hand, as a is from c on the other; and as far from a as the Sum of the Diſtances of c and f from a. Again; $ah \div c = f$, as far from h on the one hand, as c is from a on the other; alſo f is as far from c, as the Difference of the Diſtances of a and h from c.

Again; Becauſe it's the ſame thing in effect to multiply two Numbers, and divide their Product by another; or firſt to divide one of theſe two Numbers by the ſame Diviſor, and multiply the Quote by that other Number, for $\frac{bc}{a} = \frac{b}{a} \times c$, or $\frac{c}{a} \times b$. Therefore the laſt *Corollary* will be in effect the ſame if it's expreſſed in this manner, *viz.* if any Term of a Series is divided by another, and the Quote multiplied by a third Term, the Product is a Term as far diſtant from the Dividend or Multiplier on the one hand, as the other of theſe is from the Diviſor on the other hand; ſo that the Diſtance of the Product from the Diviſor will be the Sum of the Diſtances of the Dividend and Multiplier from the Diviſor, when the Diviſor is leſſer or greater than either of theſe; but if it lie between them, then the Diſtance of the Product from it will be the Difference of the Diſtances of the Dividend and Multiplier from the Diviſor; for $\frac{bc}{a} = \frac{b}{a} \times c$, which is now called a Product, is the very ſame Number that was before called a Quote; and what is here a Dividend

Dividend (*viz.* b or c) is the same as was before one of the Multipliers, (and is in effect a Multiplier here also) and the Divisor is the same as before.

2. If any Term of a Series is multiplied by itself, and the Product or Square divided by another Term, the Quote will be a Term of the Series as far distant from the Term squared on the one hand, as the Divisor is from the same Term on the other hand; and consequently the Quote will be as far from the Divisor as twice the Distance of the Term squared from the Divisor. Hence, *reversly*, The Term which is in the middle betwixt two Terms is the Square Root of their Product. So in this Series, $a:b:c:d:e:f:g:h$, if $d \times d$ is divided by b, the Quote is equal to f, as far distant from d as d is from b, and twice as far from b as d is. The *Reason* is because $b:d:f$ are $\div l$, and therefore $dd = bf$; whence, $dd \div b = f$, also $\overline{bf}|^{\frac{1}{2}} = d$.

3. Again, more universally; If any Term of a Series is raised to any Power, and that Power divided by such a Power of any other Term whose Index is 1 less than that of the Dividend, the Quote is a Term of the Series whose Distance from the Root of the Divisor is equal to the Product of the Index of the Power in the Dividend by the Distance of its Root from that of the Divisor. The *Reason* will be plain from this *Example*: Let any two Terms of a Series be called $a:b$; a Series continued from these is (by *Probl.* III.) $a:b:\frac{b^2}{a}:\frac{b^3}{a^2}:\frac{b^4}{a^3}:$ &c. But, by the two preceding *Corollaries*, each of these Terms is a Term of any Series to which a, b can belong, since $\frac{b^2}{a}$ is a 3d $\div l$ to $a:b$, and each of the rest a 4th to $a:b$, and the preceding Term. And here it's evident that each Term is twice or thrice, &c. as far from a as b is, according to the Index of b.

4. If any three or more Terms of a Geometrical Series are continually multiplied together, and the Product divided by such a Power of a Term less than any of them, or greater, whose Index is 1 less than the Number of Terms multiplied, the Quote will be that Term of the Series whose Distance from the Root of the Divisor is equal to the Sum of the Distances of these multiplied Terms from the same Root. The *Reason* of this is easily deduced from the *Theorem*, thus: Suppose any Term of a Series is a; and b, c, any two other Terms both greater or lesser than a; then $\frac{bc}{a}$ expresses a Term distant from a by the Sum of the Distances of b and c (by this *Theorem*). Take another Term d, and multiply into the last found, it is $\frac{bc}{a} \times d = \frac{bcd}{a}$, which divided by a is $\frac{bcd}{a^2}$, which is a Number formed according to the Proposition, and is also by this *Theorem* a Term of the Series as far distant from a as the Sum of the Distances of d and $\frac{bc}{a}$; but the Distance of $\frac{bc}{a}$ is the Sum of the Distances of b and c, therefore the Distance of $\frac{bcd}{a^2}$ is the Sum of the Distances of b, c, d; and it's obvious that the same Reasoning will be good in the next Case, *i. e.* where four Terms are multiplied, and so *in infinitum*.

Or the *Demonstration* of this Truth may be deduced without the *Theorem*, thus: Any Term of a Series may be called a, and if the common Ratio of the Series is r, then the several Terms after a will be ar, ar^2, ar^3, &c so that any Term after a may be represented ar^n or ar^m, or with any other Index which will express the Distance of that Term from a. Now if we take any two or more of these Terms after a, and multiply them together, the Product will be equal to the Product of these two Factors, *viz.* such a Power of a whose Index is the Number of Terms multiplied; and such a Power of r whose Index is the Sum of the Indexes of r in the several Terms multiplied (thus $ar^n \times ar^m = a^2 \times r^{m+n}$, and so of more Terms). Let us then suppose any Number of Terms after a multiplied

tiplied together, if that Number of Terms is n, and the Sum of the Indexes of r in the ſeveral Terms is m, then is the Product $a^n \times r^m$, which divide by a^{n-1}, the Quote is $a \times r^m$, a Term as far diſtant from a as m expreſſes, or as far as the Sum of the Diſtances of all the Terms multiplied.

Scholium. The immediate Uſe and Application of theſe *Corollaries* is in the Solution of the following *Problems.*

(1.) To find any Term of a Series by means of its Diſtance from the firſt Term, together with the firſt Term, and any two other, the Sum or Difference of whoſe Diſtance from the firſt is equal to the Diſtance of the Term ſought: The Solution of which is plainly contained in *Coroll.* 1. which need not to be repeated.

(2.) By the firſt and any three or more others, the Sum of whoſe Diſtances from the firſt is equal to the Diſtance of the Term ſought from the firſt, the Solution of which is in *Coroll.* 4.

(3.) By the firſt Term, and any one other whoſe Diſtance from it is an aliquot Part of the Diſtance of the Term ſought from the firſt; the Solution of which is contained in *Coroll.* 2. and 3.

(4.) When the Term ſought is in the very middle betwixt the given Terms, as in *Coroll.* 2.

Obſerve, If the Term ſought is betwixt the Terms given, but not in the middle, you have a Rule for ſolving this in *Probl.* 3. See Article 2. of the *Schol.* to *Coroll.* 8.

Theorem XXI.

In any Geometrical Series the Product of the Extremes is equal to the Product of any two middle Terms equally diſtant from the reſpective Extremes, (*i. e.* the leſſer Mean from the leſſer Extreme, and the greater from the greater) and to the Square of the middle Term, when the Number of Terms is odd. That is, theſe Products are all equal, *viz.* that of the Extremes, and theſe of every two Means, taken at equal Diſtance from the Extremes, and that of the middle Term by itſelf, *i. e.* its Square, when the Number of Terms is odd.

Example. 3 : 12 : 48 : 192. 168 : 3072 : 12288, being a Geometrical Progreſſion, theſe Products are equal, *viz.* $3 \times 12288 = 12 \times 3072 = 48 \times 168 = 192 \times 192$; each of them being $= 36864$.

Demonstr. This follows eaſily from the preceding, compared with *Theor.* I. for the *Extremes* of a Series, and any two Terms equally diſtant from them, are :: l by the preceding, and by *Theor.* I. the Products of the Extremes and Means of four Proportionals are equal; and the Product of the Extremes of three Numbers in continued Proportion is equal to the Square of the middle Term; hence the thing to be proved is manifeſt.

Example. $a : b : c : d : e : f : g$, being a continued Series, theſe are proportional, $a : b :: f : g$; hence $ag = bf$. Again; $a : c :: e : g$; hence $ag = ce$. Alſo $a : d : g$ are $\div l$; hence $ag = dd$; wherefore $ag = bf = ce = dd$. However long the Series be, the Truth of the *Theorem* is equally clear.

Which we may alſo ſhew by this other Repreſentation: let a be the leaſt Extreme, r the Ratio, and d the Diſtance of the Extremes, then the greater Extreme is ar^d, (*Cor.* 6. *Prob.* 3.) and the Series of increaſing Terms from a, and of the decreaſing from ar^d, are as in the Margin. From whence it's obvious, that the Products of the ſeveral correſponding Terms ſtanding againſt one another in the two Series are all equal to

$$\begin{array}{l}
a : ar : ar^2 : ar^3 : ar^4 : \&c.\ ar^d. \\
ar^d : ar^{d-1} : ar^{d-2} : ar^{d-3} : ar^{d-4} : \&c.\ a. \\
\hline
\qquad\qquad a^2 \times r^d
\end{array}$$

to $a^2 \times r^d$; for r being involved once more in every Term from a, and once less in every Term from ar^d, makes still the same Product $a^2 \times r^d$.

Or it will be as clear if we represent a Series in this manner; $a : ar : ar^2 :$ &c. $: \frac{l}{r^2} : \frac{l}{r} : l$, increasing from a for the one half of the Series, and decreasing from l for the other half: and if there is a middle Term let both Parts of the Series include it: And then the Truth proposed is obvious; each Product being $= a\,l$. For tho' one of the Factors is a multiplied by some Power of r; yet the other is l divided by the same Power of r, which Multiplier and Divisor destroy one another in the Product proposed.

SCHOLIUMS.

1. Where a Series has an even Number of Terms, there are two Terms which we may call the two middle Terms, and in this Case the *Theorem* may be expressed thus: The Product of the two middle Terms, and of every two equally distant from them are equal: And we may also see this Truth in a Representation different from any of the former, thus; If $m : n$ be the two middle Terms, the Series ascending from n will be the continual Products of n by the Ratio, and descending from m it will be the continual Quotes of m divided by the Ratio, which in this Case is taken for the Quote of the greater Term divided by the lesser, as $n \div m$: Thus; *&c.* $: \frac{m}{r^3} : \frac{m}{r^2} : \frac{m}{r} : m : n : nr : nr^2 : nr^3 :$ *&c.* In which the *Theorem* is manifest; for n being multiplied by any Power of r, and m being divided by the same, the Product of that Quote and Product is $m\,n$, the Divisor and Multiplier destroying one another: So $\frac{m}{r} \times nr = \frac{mnr}{r} = mn$; and universally, $\frac{m}{r^x} \times nr^x = \frac{mnr^x}{r^x} = mn$.

2. Where the Series has an odd Number of Terms, *i.e.* has a middle Term equally distant from both Extremes, then it's the same thing to say, The Products of the several Terms equally distant from the Extremes, or, The Products of Terms equally distant from the middle Term, are equal to another, and to the Square of the middle Term; and such a Series may be thus represented; *&c.* $\frac{m}{r^3} : \frac{m}{r^2} : \frac{m}{r} : m : mr : mr^2 : mr^3$, *&c.* which also does clearly shew the Truth of the *Theorem*.

3. *Observe*, For the same Reasons here explained, the Product of any two Terms in a Series is equal to the Product of any other equally distant from the former two; and whereof one is taken as far above one of these former as the other is below the other of them: Because such four Terms are proportional; and of the Terms multiplied, the one Couplet are the Extremes, and the other the Means.

Also, The Product of any two Terms is equal to the Square of that Term which is in the midst, equally distant from either of them, because these three Terms are $::l$; so in the preceding Series $af = be$, because $a : b :: e : f$; and $ae = cc$, because $a : c :: c : e$.

Again; When a Series has an even Number of Terms, tho' the two middle Terms are not in the continued Ratio of all the rest above and below, yet the Products of the Extremes and middle Terms equally distant from them, will still be equal, because these Factors are $::l$, at least disjunctly.

COROLL. The continual Product of any three or more Numbers in Geometrical Progression is a Power of some Order; particularly,

(1.) If the Number of Terms, n, is even (*i.e.* a Multiple of 2.) and the Extremes a, l, then suppose $n \div 2 = d$; I say, the continual Product of all the Terms of the Series is the d Power of the Product of the Extremes, or $\overline{a\,l}|^d$: For since, as has been shewn,

 the

the Product of the Extremes, and of every two mean Terms equally distant from the Extremes, are all equal; and there are as many of these equal Products as the half Number of Terms, or d: Therefore the continual Product of all these equal Products (which is manifestly the continual Product of all the Terms in the Series) is equal to such a Power of any one of them (as the Product of the Extremes) whose Index is d. *Example:* Of this Series, $2:4:8:16:32:64$, the continual Product is $2\times4\times8\times16\times32\times64 = 2097152$ $= \overline{2\times64}|^3 = \overline{128}|^3$.

(2.) If the Number of Terms is odd, *i.e.* If there is a middle Term equally distant from each Extreme; suppose that middle Term is m, and the Number of Terms n; I say, the continual Product of all the Terms is the n Power of the middle Term, m, or m^n: For the Products of every two Terms equally distant from the middle Term, being equal to m^2 or $m\times m$; therefore, as to the continual Product of the whole Terms, the Series is the same in effect as if it were a Series of Terms all equal to m, in which case it's evident that the Product of the whole is m^n. *Example:* Of this Series, $2:4:8:16:32$, the continual Product is $2\times4\times8\times16\times32 = 32768 = 8^5$.

Observe, The continual Product of the Series, *viz.* $\overline{al}|^d$, or m^n, may also be a Power of as many other Orders as are denominated by the integral Numbers, which divide the Index d or n, without a Remainder, as has been already explained in *Book* III. Again; In some Cases the Product may be a Power of other Orders than what are assigned either by this *Corollary* or last *Observation,* according to the Nature and Composition of the 1st Term of the Series, and of the Ratio; but these not flowing immediately from the Nature of Progressions, are not to be considered here. This only I shall further *observe,* That if the Index n (*viz.* of the Power m^n the Product of an odd Series) is an odd Number, the Product cannot be a Power of the Order d. Supposing in this Case also $d=\frac{n}{2}$; because n being an odd Number, is not divisible by 2 without a Remainder; therefore $\frac{n}{2}$ or d is not the Index of a simple Power. Also, If n is an even Number the Product cannot be a Power of the Order n; for that Product being $\overline{al}|^d$, if this is also a Power of the Order n, then d is divisible by n without a Remainder, which is impossible, because $d=\frac{n}{2}$.

Theorem XXII.

In any Geometrical Progression, which soever of the Extremes you call the 1st Term, the Difference of the 1st and 2d Term is, to the Difference of the Extremes, in the same Ratio as the 1st Term is to the Sum of the whole Series, except the last Term: Or also, as the 2d Term to the Sum of the whole Series, except the 1st.

Example: In this Series, $2:6:18:54:162$, it is as 4 ($=6-2$) to 160 ($=162-2$) so is 2 to 80 ($=2+6+18+54$); or take the Series decreasing, it is as 108 ($=162-54$) to 160 so is 162 to 240 ($=162+54+18+6$.) Again; it is $4:160::6:240$; and $108:160::80$.

Universally: If a Geometrical Series is $a:b:c:d:e$, *&c.* l, for the whole Sum put s, and let a be the 1st, and l the last Term; so that $s-a$, $s-l$, express the Sum of the whole Series, except the 1st or last Term; then is it $b-a$ (or $a-b$) $:l-a$ (or $a-l$) $::a:s-l::b:s-a$.

Demonstr. 1. Let us suppose the 1st Term to be the lesser Extreme, and the thing to be demonstrated is, that $b-a:l-a::a:s-l::b:s-a$. Thus; Of any Number of similar and equal Ratios, the Sum of all the Antecedents is to the Sum of all the Consequents as any one of the Antecedents to its Consequent (by *Theor.* IV. *Coroll.* 1.): But

But in caſe of a continued Progreſſion all the Terms except the Laſt are Antecedents, and all except the 1ſt are Conſequents; ſo that it is $s-l:s-a::a:b$; and *diviſively* $\overline{s-a}-\overline{s-l}:b-a::s-l:a::s-a:b$; but $\overline{s-a}-\overline{s-l}=s-a-s+l=l-a$; therefore $l-a:b-a::s-l:a::s-a:b$; or, *reverſly*, $b-a:l-a::a:s-l::B:s-a$.

2. If we ſuppoſe a to be the greater Extreme, the *Demonſtration* proceeds the ſame way; only inſtead of $b-a$ and $l-a$, it is $a-b:a-l::a:s-l::b:s-a$.

COROLLARIES.

1. Having the Extremes of a Series, and the ſecond Term, (or that next either of the Extremes) we can find the Sum of the whole Series, without knowing or finding any more of the Terms. Thus:

Multiply the Difference of the Extremes by the firſt Term, and divide the Product by the Difference of the firſt and ſecond Term, the Quote is the Sum of all the Series except the laſt Term; to which Quote add the laſt Term, and the Sum is the thing ſought. For ſince $b-a:l-a::a:s-l$, therefore $s-l=\frac{\overline{l-a}\times a}{b-a}$, and $s=\frac{\overline{l-a}\times a}{b-a}+l$. Or inſtead of multiplying by the firſt Term, multiply by the ſecond Term; and you'll find the Sum of all the Series, except the firſt Term. For $b-a:l-a::b:s-a$; therefore $s-a=\frac{\overline{l-a}\times b}{b-a}$, and $s=\frac{\overline{l-a}\times b}{b-a}+a$. If we ſuppoſe b leſs than a, yet the Rule is the ſame, and the Reaſon of it alſo; by putting $a-b$ and $a-l$ in place of $b-a$ and $l-a$.

Again: If we take either of theſe Expreſſions for the Sum, *viz.* $\frac{\overline{l-a}\times a}{b-a}+l$, or $\frac{\overline{l-a}\times b}{b-a}+a$; and reduce them, by the common Rules, to a more ſimple Expreſſion, we ſhall find it to be this, $s=\frac{bl-a^2}{b-a}$; *i. e.* multiply the laſt and ſecond Terms, and from the Product ſubtract the Square of the firſt Term, and divide the Remainder by the Difference of the firſt and ſecond Term, the Quote is the Sum. And the Truth of this we may alſo deduce otherwiſe. Thus, $a:b::s-l:s-a$, (as we ſaw above) and the Product of Extremes and Means are equal, *viz.* $as-a^2=bs-bl$. To each add bl, and it is $bl+as-a^2=bs$; and ſubtracting as, it is $bl-a^2=bs-as=\overline{b-a}\times s$; and dividing by $b-a$, it is $\frac{bl-a^2}{b-a}=s$.

2. From this Expreſſion of the Sum, *i. e.* from this Equality, $as-a^2=bs-bl$; we have alſo the Solution of theſe *Problems*, *viz.* (1.) Having b, s, l to find a, which is, $a=\frac{s}{2}+\overline{\frac{s^2}{4}-bs-bl}\Big|^{\frac{1}{2}}$ (*Prob.* 6. C. 2. B. III.) (2.) Having a, s, l to find b, which is, $b=\frac{as-a^2}{s-l}$. (3.) Having a, s, b to find l, which is $l=\frac{bs-as+a^2}{b}$. For ſince $bl-a^2=bs-as$. Add a^2 to each, it is $bl=bs-as+a^2$; and dividing by b, it is $l=\frac{bs-as+a^2}{b}$. (4.) Having $a, b, s-l$ to find $s-a$, which is, $s-a=\frac{\overline{s-l}\times b}{a}$; which flows immediately from $as-a^2=bs-bl$. (5.) Having $a, b, s-a$ to find $s-l$, which is, $s-l=\frac{\overline{s-a}\times a}{b}$.

In the ſame manner having any three of theſe four, *viz.* $a, \overline{b-a}, \overline{l-a}, \overline{s-l}$, we can find the fourth; thus, $s-l=\frac{\overline{l-a}\times a}{b-a}$, (which is above demonſtrated, and from which follow the reſt, *viz.*) $b-a=\frac{\overline{l-a}\times a}{s-l}$; $l-a=\frac{\overline{s-l}\times\overline{b-a}}{a}$; and $a=\frac{\overline{s-l}\times\overline{b-a}}{l-a}$. Again: Having any three of theſe four, *viz.* $b, \overline{b-a}, \overline{l-a}, \overline{s-a}$; we can find the fourth thus, $s-a=\frac{\overline{l-a}\times b}{b-a}$. Which is above demonſtrated; and from which follows, that $b-a=\frac{\overline{l-a}\times b}{s-a}$; $l-a=\frac{\overline{s-a}\times\overline{b-a}}{b}$; and $b=\frac{\overline{s-a}\times\overline{b-a}}{l-a}$.

Scholium. In every Geometrical Progreſſion theſe five things are conſiderable, *viz.* the two Extremes, the Ratio, the Number of Terms, and the Sum: From which a Variety of *Problems* ariſes, whereof theſe are the chief and moſt uſeful, in which are given any three of theſe things to find the other two. But there is one Caſe, *viz.* having the Sum, Number of Terms, and either Extreme, to find the other and the Ratio; the Solution of which is too much above the Method I am limited to: however, I ſhall not entirely paſs it over, but bring the Solution ſo far as my Method permits, and muſt leave the reſt to ſuperior Methods.

The Uſe of the Symbols *employed in the following* Problems.

a = the leſſer Extreme. l = the greater Extreme. s = Sum of the whole Series.	r = the Ratio, which is to be taken as the Quote of the greater Term divided by the leſſer, and conſequently the reciprocal Ratio taken the other way.	n = Numb. of Terms. $d=n-1$ = the Number of Terms leſs 1, or the Diſtance of the Extremes.

Problem IV.

Having the Extremes and Ratio A, l, r, *to find the Sum and Number of Terms* s, n.

Solutions.

1. For the Sum. It may be done various ways; thus,

1ſt *Method:* Having a, r, we have alſo b the ſecond Term, for $b=ar$; then by a, b, l we may find s, as in *Cor.* 1. *Theor.* 22. thus, $s=\frac{bl-a^2}{b-a}$.

But without finding b, we have various other ways.

2d *Method:* Divide the Difference of the Extremes by the Ratio leſs 1, the Quote is the Sum leſs the greater Extreme. Thus, $s-l=\frac{l-a}{r-1}$, and $s=\frac{l-a}{r-1}+l$.

Example: $a=3$, $l=48$, $r=2$; then is $s-l=45=\frac{48-3}{2-1}$, and $s=45+48$; as in this Series, $3:6:12:24:48$.

Demonstr. By *Theor.* 22. it is $b-a:l-a::a:s-l$, or $b-a:a::l-a:s-l$. But $b=ar$, therefore $b:a::r:1$; and diviſively, $b-a:a::r-1:1$. Hence $r-1:1::l-a:s-l$, and $s-l=\frac{l-a}{r-1}$; alſo $s=\frac{l-a}{r-1}+l$.

3d *Me-*

3d *Method.* Multiply the Difference of the Extremes by the Ratio, and divide the Product by the Ratio less 1, the Quote is the Sum wanting the lesser Extreme. Thus; $s-a=\frac{\overline{l-a}\times r}{r-1}$; and hence $s=\frac{\overline{l-a}\times r}{r-1}+a$.

Example: $a=3$, $l=48$, $r=2$; then is $s-a=90=\frac{\overline{48-3}\times 2}{2-1}$, and $s=93$.

DEMON. Since $b:a::r:1$, (as before); therefore $b-a:b::r-1:r$. But $b-a:b::l-a:s-a$, (*Theorem* 22.) Hence $r-1:r::l-a:s-a$, and $s-a=\frac{\overline{l-a}\times r}{r-1}$, and $s=\frac{\overline{l-a}\times r}{r-1}+a$.

4th *Method.* Take the Product of the lesser Extreme and the Ratio out of the greate Extreme, and divide the Difference by the Ratio less 1, the Quote is the Sum wanting both Extremes; thus, $s-a-l=\frac{l-ra}{r-1}$, and $s=\frac{l-ra}{r-1}+a+l$.

Exam. In the preceding Series, $s-a-l=42=\frac{48-6}{2-1}$, and $s=42+51=93$.

DEMONSTR. By the 2d *Method*, $s-l=\frac{l-a}{r-1}$; hence $s-l-a=\frac{l-a}{r-1}-a=\frac{l-a-ar+a}{r-1}=\frac{l-ar}{r-1}$. Or thus; If ar (which is the second Term, when a is the first) be made the first Term; then, of such a Series $s-l=\frac{l-ar}{r-1}$, by the second *Method*: But this is plainly $=s-l-a$ of that Series which begins with a. Therefore, &c.

5th *Method.* Multiply the greater Extreme by the Ratio, and from the Product take the lesser Extreme; divide the Difference by the Ratio less 1, the Quote is the Sum; thus, $s=\frac{rl-a}{r-1}$.

Exam. In the preceding Series, $s=\frac{2\times 48-3}{2-1}=93$.

DEMONSTR. This Rule is deduced from the 2d Method, thus, $s=\frac{l-a}{r-1}+l=\frac{l-a+rl-l}{r-1}=\frac{rl-a}{r-1}$; and it may be deduced the same way from the 3d or 4th Methods. Or we may deduce it from the 1st Method, by putting ar in the Place of b, thus, $\frac{arl-aa}{ar-a}=\frac{rl-a}{r-1}$, by dividing Numerator and Denominator by a. Or, again, from the 2d Method, Thus; If one Term more is joined to that Series whose greatest Term is l, that new Term is rl; and putting rl in place of l, in *Rule* 2. it is $s-rl=\frac{lr-a}{r-1}$: But $s-rl$ in this new Series is equal to s in the former, whose greater Extreme was l, wherefore its Sum is $\frac{rl-a}{r-1}$.

But we may also demonstrate this Rule independently of any of the preceding, thus: $a:b::s-l:s-a$ (as has been shewn in *Theor.* 22.), but $1:r::a:b$; hence $1:r::s-l:s-a$, and $s-a=rs-rl$, and $s-a+rl=rs$; also $rl-a=rs-s$; and, lastly, $\frac{rl-a}{r-1}=s$.

A:

$$\begin{array}{l} a : ar : ar^2 : \&c. : ar^{n-1} : ar^n \\ r - 1 \\ \hline ar + ar^2 + ar^3 : \&c. + ar^n + ar^{n+1} \\ -a - ar - ar^2 - ar^3 : \&c. - ar^n. \\ \hline -a + ar^{n+1} \text{ or } ar^{n+1} - a \end{array}$$

Or also thus, (independently of *Theorem* 22.) Let the Series be expressed thus, $A : Ar : Ar^2 : \&c. : Ar^n$, and let it be multiplied by $r - 1$, it's manifest from the Work in the Margin, that the Product is $= Ar^{n+1} - A$; and if this Product is divided by $r - 1$, the Quote is $\frac{ar^{n+1} - a}{r - 1} = \frac{rl - a}{r - 1}$; because $ar^n = l$, and $rl = ar^{n+1}$.

2. For the Number of Terms n, Raise the Series from a till you find a Term equal to l, and then you'll have n. The *Reason* is obvious.

Or thus: Divide the greater by the lesser Extreme, and raise the Ratio to a Power equal to that Quote, its Index is equal to $n - 1$, or d; because $l = ar^d$ (*Cor.* 6. *Probl.* III.) therefore $\frac{l}{a} = r^d$; so in the preceding Series, $48 = 3 \times 2^4 = 3 \times 16$; wherefore 5 is the Number sought.

Observe, There is another Method of finding n, *viz.* By the *Logarithms*: And tho' the Nature and Use of these Numbers is not yet explained, I shall here, nevertheless, deliver the Rule, that what belongs to Progressions may be found together; but the *Demonstration* and *Application* must both be referred till *Logarithms* are explained. See *Page* 507.

RULE. To find n by *Logarithms*, Subtract the *Log.* of a from that of l, and divide the Remainder by the *Log.* of r, the Quote is equal to d or $n - 1$; thus, $d = \frac{\text{Log.}\, l - \text{Log.}\, a}{\text{Log.}\, r}$. But *observe*, In this and all the following *Solutions* by *Logarithms*, that the Answer will not always be accurately true; and therefore to be sure of an exact Answer, the former Method is to be chosen.

PROBLEM V.

Having the Extremes a, l, and Number of Terms n, to find the Sum s and Ratio r.

1. For the Ratio. Divide the greater Extreme by the lesser; extract that Root of the Quote whose Index is $n - 1$, or d, it is the Ratio. Thus, $r = \overline{l \div a}|^{\frac{1}{d}}$.

Example. $a = 3$, $l = 48$, $n = 5$, then is $r = \overline{48 \div 3}|^{\frac{1}{4}} = \overline{16}|^{\frac{1}{4}} = 2$.

DEMONSTR. $l = Ar^d$ (*Coroll.* 6. *Probl.* III.) therefore $l \div a = r^d$, and $r = \overline{l \div a}|^{\frac{1}{d}}$ (*Ax.* 1. *Book* III.)

SCHOLIUM. As to the Extraction of this Root, see what is said in *Book* III.; and if the Root is *Surd*, the *Problem* is impossible in rational Numbers.

2. For the Sum. Find it by a, l, r, as above, *viz.* $s = \frac{rl - a}{r - 1}$.

Or instead of r put its Equal $\overline{l \div a}|^{\frac{1}{d}}$; and so the Rule will be $s = \overline{\overline{l \div a}|^{\frac{1}{d}} \times l - a} \div \overline{\overline{l \div a}|^{\frac{1}{d}} - 1}$. Where *observe*, That the Expression for r standing here distinctly by itself, this is no other thing than bidding you first find r, and then by a, l, r, find s. But this Expression may be reduced to another Form wherein the Expression of r is lost, thus; $\overline{\overline{l \div a}|^{\frac{1}{d}} \times l - a} \div \overline{\overline{l \div a}|^{\frac{1}{d}} - 1} = \overline{l \times l^{\frac{1}{d}} - a \times a^{\frac{1}{d}}} \div \overline{l^{\frac{1}{d}} - a^{\frac{1}{d}}}$. The Manner and Reason of which Reduction

duction is this, $\overline{l \div a}|^{\frac{1}{d}} = l^{\frac{1}{d}} \div a^{\frac{1}{d}}$ (*Theor.* 4. *Book* III.), and therefore $\overline{l \div a}|^{\frac{1}{d}} \times l = l \times l^{\frac{1}{d}} \div a^{\frac{1}{d}}$; from which subtract a and it is $\underline{l \times l^{\frac{1}{d}} - a \times a^{\frac{1}{d}}} \div a^{\frac{1}{d}}$. Again; $\overline{l \div a}|^{\frac{1}{d}} - 1 = l^{\frac{1}{d}} \div a^{\frac{1}{d}} - 1 = \underline{l^{\frac{1}{d}} - a^{\frac{1}{d}}} \div a^{\frac{1}{d}}$. And the former Remainder, divided by this, quotes $\underline{l \times l^{\frac{1}{d}} - a \times a^{\frac{1}{d}}} \div \underline{l^{\frac{1}{d}} - A^{\frac{1}{d}}}$. If you express all the Divisions here fraction-wise you will perceive the Method of the Reduction more easily. Yet here again *observe*, that tho' the *Problem* be possible in rational Numbers, it cannot be solved in this Form unless l and a are both Powers the Order d. And if it be here objected, That either this Solution is false, or the *Problem* impossible, since a true Solution must necessarily give the Answer to a possible Problem, this Difficulty is removed by considering that this new Expression is only the Effect of taking $l^{\frac{1}{d}} \div a^{\frac{1}{d}}$ for $\overline{l \div a}|^{\frac{1}{d}}$; which last may be rational (and must be so if the Problem is possible) tho' $l^{\frac{1}{d}}$ and $a^{\frac{1}{d}}$ are both Surd, (as has been shewn in *Book* III.) and the Consequence we are to draw from $l^{\frac{1}{d}}$ and $a^{\frac{1}{d}}$ being Surd, is not that the Problem is impossible in rational Numbers; but rather this, that making these Extractions further and further *in infinitum*, and applying them in this Rule, we shall approach infinitely nearer to the Answer of the Problem.

PROBLEM VI.

Having the Extremes a, l, and Sum s, to find the Ratio r, and Number of Terms n.

1. For the Ratio. Divide the Difference of the Extremes by the Difference of the Sum and greater Extreme, the Quote is the Ratio less 1: Or thus; Divide the Difference of the Sum and lesser Extreme, by the Difference of the Sum and greater Extreme, the Quote is the Ratio: thus, $\frac{l-a}{s-l} = r - 1$, or $r = \frac{s-a}{s-l}$.

Exam. $a = 3$, $l = 48$, $s = 93$, then is $r - 1 = \frac{48-3}{93-48} = \frac{45}{45} = 1$, or $r = \frac{93-3}{93-48} = 2$.

DEMONSTR. $s - l = \frac{l-a}{r-1}$ (*Probl.* IV.), hence $r - 1 = \frac{l-a}{s-l}$ (the 1st *Rule*); hence $r = \frac{l-a}{s-l} + 1 = \frac{l-a+s-l}{s-l} = \frac{s-a}{s-l}$ (the 2d *Rule*). Which is also demonstrated thus; $s - l : s - a :: a : b :: 1 : r$; hence $r = \frac{s-a}{s-l}$.

2. For n. Find it by a, l, r, as in *Probl.* IV.

Or by *Logarithms*, thus: Divide the Difference of the Logarithms of l and a by the Difference of the Logarithms of $s - a$ and $s - l$, the Quote is $n - 1$ or d: Thus $d = \frac{\text{Log. } L - \text{Log. } a}{\text{Log. } \overline{s-a} - \text{Log. } \overline{s-l}}$.

COROLL. Having r and the Difference of the Sum from either of the Extremes, we can find its Difference from the other Extreme; for since $r = \frac{s-a}{s-l}$, therefore $s - l = \frac{s-a}{r}$, and $s - a = \overline{s-l} \times r$.

PROB-

Problem VII.

Having either of the Extremes a or l, with the Ratio r, and Number of Terms n, to find the other Extreme l or a, and the Sum s.

1. If a is given, multiply it by the d Power of r, the Product is l: thus, $l = ar^d$. If l is given, then divide it by r^d, the Quote is a: Thus $a = l \div r^d$.

Example: $a = 2$, $r = 3$, $n = 5$; then is $l = 2 \times 3^4 = 2 \times 81 = 162$. And l being given, then is $a = 162 \div 81 = 2$.

Demonstr. By *Coroll. 6. Probl.* III. it is $l = ar^d$, and hence $a = l \div r^d$.

2. For the Sum. Apply a, l, r, as in *Probl.* IV. thus, $s = \frac{rl - a}{r - 1}$. Or it may be found otherwiſe in the given Terms, thus:

(1.) If a is given, then is $s = \frac{ar^n - a}{r - 1}$. For $l = ar^{n-1}$, therefore $rl = ar^{n-1} \times r = ar^n$, whence $s = \frac{rl - a}{r - 1} = \frac{ar^n - a}{r - 1}$.

(2.) If l is given, then is $s = \frac{lr^n - l}{r^n - r^{n-1}}$. For $a = \frac{l}{r^{n-1}}$, and $rl - a = rl - \frac{l}{r^{n-1}} = \frac{r^{n-1} \times rl - l}{r^{n-1}} = \frac{lr^n - l}{r^{n-1}}$; therefore $s = \frac{lr^n - l}{r^{n-1}} \div \overline{r - 1} = \frac{lr^n - l}{r^n - r^{n-1}}$.

Problem VIII.

Having the Sum s, Number of Terms n, and Ratio r, to find the Extremes a, l.

1. For a, Raiſe r to the n Power; multiply the Sum by $r - 1$, and divide the Product by $r^n - 1$, the Quote is a: Thus, $a = \frac{s \times \overline{r - 1}}{r^n - 1}$.

Example: $r = 3$, $n = 5$, $s = 242$; then is $a = \frac{242 \times 2}{3^5 - 1} = \frac{484}{242} = 2$; as in this Series, $2 : 6 : 18 : 54 : 162$.

Demonstr. $s = \frac{rl - a}{r - 1}$ (*Probl.* IV.), and $l = ar^{n-1}$ (*Probl.* III. *Coroll.* 6.); hence $rl = r \times ar^{n-1} = ar^n$, and $rl - a = ar^n - a = a \times \underline{r^n - 1}$; wherefore $s = \frac{a \times \underline{r^n - 1}}{r - 1}$, and $s \times \overline{r - 1} = a \times \underline{r^n - 1}$, and laſtly $a = \frac{s \times \overline{r - 1}}{r^n - 1}$.

2. For the greater Extreme, Find it by a, r, n, as in *Probl.* VII.
Or alſo in the given Terms, thus, find the n and $n - 1$ Powers of r, and multiply their Difference by the Sum; then divide the Product by $r^n - 1$, the Quote is the greater Extreme, thus, $l = \frac{s \times \underline{r^n - r^{n-1}}}{r^n - 1}$.

Example: In the laſt Series, $r^n = 243$, and $r^{n-1} = 81$, and $l = \frac{242 \times \overline{243 - 81}}{242} = 243 - 81 = 162$.

DEMONSTR. This proceeds from putting $\frac{s\times\overline{r-1}}{r^n-1}$ in the place of a in the other Rule, *viz.* wherein $l=ar^{n-1}$, for thus, $l=\frac{s\times\overline{r-1}}{r^n-1}\times r^{n-1}=\frac{s\times\overline{r-1}\times r^{n-1}}{r^n-1}$; but $r\times r^{n-1}=r^n$; hence $\overline{r-1}\times r^{n-1}=r^n-r^{n-1}$; wherefore, lastly, $l=\frac{s\times\overline{r^n-r^{n-1}}}{r^n-1}$.

PROBLEM IX.

Having either Extreme, a, or l, the Sum s, and Ratio r, to find the other Extreme l, or a, and Number of Terms n.

1. For the unknown Extreme. And

(1.) If l is given, to find a. Multiply the Difference of the Sum and greater Extreme, by the Ratin less 1; subtract the Product from the greater Extreme, the Remainder is the lesser Extreme. Thus; $a=l-\overline{s-l}\times\overline{r-1}$; which being reduced gives also this other Rule, *viz.* $a=rl+s-rs$.

Example: $s=242$, $l=162$, $r=3$, then is $A=162-80\times2=162-160=2$.

DEMONSTR. In *Probl.* IV. *Method* 2. it is shewn that $s-l=\frac{l-a}{r-1}$, hence $\overline{s-l}\times\overline{r-1}=l-a$, and $a=l-\overline{s-l}\times\overline{r-1}=rl+s-rs$.

(2.) If a is given, to find l. Multiply the Difference of the Sum and lesser Extreme by the Ratio less 1, and divide the Product by the Ratio, then to the Quote add a, the Sum is l; thus $l=\frac{\overline{s-a}\times\overline{r-1}}{r}+a$; which being reduced is also $=\frac{rs+a-s}{r}$.

Example: $s=242$. $a=2$, $r=3$, then is $l=\frac{240\times2}{3}+2=\frac{480}{3}+2=160+2=162$.

DEMONSTR. In *Probl.* IV. it's shewn that $s-a=\frac{\overline{l-a}\times r}{r-1}$ hence $\overline{s-a}\times\overline{r-1}=\overline{l-a}\times r$, and $l-a=\frac{\overline{s-a}\times\overline{r-1}}{r}$, and $l=\frac{\overline{s-a}\times\overline{r-1}}{r}+a=\frac{rs+a-s}{r}$.

2. For the Number of Terms, Apply a, l, r, by *Probl.* 4.

Or by Logarithms, thus: (1.) If a is given, then subtract the Logarithm of r from that of $rs+a-s$; and from this Remainder subtract again the Logarithm of a: This last Remainder divide by the Logarithms of r, the Quote is $n-1$: thus, $n-1=\frac{\text{Log.}\,\overline{rs+a-s}-\text{Log.}\,r-\text{Log.}\,a}{\text{Log.}\,r}$.

(2.) If l is given, From the Logarithm of l take the Logarithm of $rl+s-rs$, and divide the Remainder by the Logarithm of r, you have $n-1=\frac{\text{Log.}\,l-\text{Log.}\,\overline{rl+s-rs}}{\text{Log.}\,r}$.

PROBLEM X.

Having either Extreme, a, or l, the Sum s, and Number of Terms n, to find other Extreme, l or a, and the Ratio r.

1. For

1. For the unknown Extreme. Take the Difference of the Sum and given Extreme; raiſe it to the $n-1$ Power, which multiply by the given Extreme: This Product is equal to the Product of the $n-1$ Power of the Difference of the Sum and unknown Extreme, multiplied by the unknown Extreme; thus, $l\times\overline{s-l}^{\,n-1} = a\times\overline{s-a}^{\,n-1}$; wherefore, to finiſh the Solution, we muſt find a Number which being taken from s, and the Remainder raiſed to the $n-1$ Power, and this Power multiplied by that ſame Number, the Product ſhall be equal to $a\times\overline{s-a}^{\,n-1}$ if l is ſought; or equal to $l\times\overline{s-l}^{\,n-1}$ if a is ſought. But how to diſcover ſuch a Number is the greateſt Difficulty: Nor does my Method and Limits allow me to give any farther Direction about it. Therefore I ſhall only illuſtrate the Rule by an *Example*, and demonſtrate the Solution ſo far as it is deduced.

Example: $s=242$, $n=5$, $a=2$, then is $s-a=240$, $\overline{s-a}^{\,n-1}=240^4=3317760000$, which multiplied by $a=2$, is $6635520000=a\times\overline{s-a}^{\,n-1}$, and the Number ſought is $162=l$; for $s-l=242-162=80$, and $\overline{s-l}^{\,n-1}=80^4=40960000$, and $l\times\overline{s-l}^{\,n-1}=162\times40960000=6635520000$.

Demonstr. $l=ar^{n-1}$ (*Coroll.* 6. *Probl.* 3.), and $r=\frac{s-a}{s-l}$ (*Probl.* 6.) Hence $l=a\times\overline{\overline{s-a}\div\overline{s-l}}^{\,n-1}=a\times\overline{s-a}^{\,n-1}\div\overline{s-l}^{\,n-1}$; therefore $l\times\overline{s-l}^{\,n-1}=a\times\overline{s-a}^{\,n-1}$.

2. For the Ratio. (1.) If a is given. Find a Number, whoſe n Power ſubtracted from its Product by $\frac{s}{a}$ ſhall leave a Number equal to $\frac{s-a}{a}$, that Number is the Ratio; *i.e.* $\frac{sr}{a}-r^n=\frac{s-a}{a}$.

Demon. $l=ar^{n-1}$, (*Cor.* 6. *Prob.* 3.) and $l=\frac{rs+a-s}{r}$ (*Prob.* 9.); therefore $ar^{n-1}=\frac{rs+a-s}{r}$. Hence $ar^n=rs+a-s$, and $ar^n+s-a=rs$; alſo $s-a=rs-ar^n$; and laſtly, $\frac{s-a}{a}=\frac{rs-ar^n}{a}=\frac{rs}{a}-r^n$.

(2.) If l is given, find r ſo that $\frac{sr^{n-1}}{s-l}-r^n=\frac{l}{s-l}$.

Demon. By *Prob.* 7. $s=\frac{lr^n-l}{r^n-r^{n-1}}$; hence $sr^n-sr^{n-1}=lr^n-l$; and $sr^n-sr^{n-1}+l=lr^n$; alſo $l=lr^n-sr^n+sr^{n-1}=sr^{n-1}-sr^n+lr^n=sr^{n-1}-\overline{s-l}\times r^n$; and laſtly, $\frac{l}{s-l}=\frac{sr^{n-1}}{s-l}-r^n$.

TABLE *of the laſt Seven* PROBLEMS, *with their Solutions.*

Prob.	Given.	Sought.	SOLUTIONS.
4	a, l, r	s, n	$s = \frac{l-a}{r-1} + l = \frac{rl-ra}{r-1} + a = \frac{l-ar}{r-1} + a + l = \frac{rl-a}{r-1}$. $n-1 =$ the Index of that Power of r which is $= \frac{l}{a}$, or $n-1 = \frac{\log l - \log. a}{\log. r}$
5	a, l, n	r, s	$r = \overline{l \div a}^{\frac{1}{d}}$ s as above, or $s = \overline{l \div a}^{\frac{1}{d}} \times l - a \div \overline{\overline{l \div a}^{\frac{1}{d}} - 1}$.
6.	a, l, s	r, n	$r = \frac{l-a}{s-l} + 1 = \frac{s-a}{s-l}$, and $n - 1 = \frac{\log. l - \log. a}{\log. \overline{s-a} - \log \overline{s-l}}$.
7.	a or r, n l	l or s a	$l = ar^d$ $a = l \div r^d$ s, as above; or $s = \frac{ar^n - a}{r-1}$. $s = \frac{lr^n - l}{r^n - r^{n-1}}$.
8.	s, n, r	a, l	$a = \frac{s \times \overline{r-1}}{r^n - 1}$. $l = \frac{s \times \overline{r^n - r^{n-1}}}{r^n - 1}$.
9.	a or s, r l	l or n a	$l = \frac{\overline{s-a} \times \overline{r-1}}{r} + a = \frac{rs + a - s}{r}$. $n - 1 = \frac{\log. l - \log. \overline{rl + s - rs}}{\text{Log. } r}$ $a = l - \overline{s-l} \times \overline{r-1} = rl + s - rs$. $n - 1 = \frac{\log \overline{rs + a - s} - \log. r - \log. a}{\log. r}$
10	a or n, s l	l or r a	For the unknown Extreme, $l \times \overline{s-l}^{n-1} = a \times \overline{s-a}^{n-1}$. For the Ratio, $\frac{sr}{a} - r^n = \frac{s-a}{a}$, and $\frac{sr^{n-1}}{s-l} - r^n = \frac{l}{s-l}$

SCHOL. What has been before obſerved upon Arithmetical Progreſſions holds alſo in Geometrical, *viz.* that any three Numbers taken at random, will not make a poſſible *Problem* in every Caſe; and therefore we ſhall here alſo conſider and explain the Invention of theſe Numbers, and how by means of any two given things the other three may be found. And in the firſt place obſerve, That as we have ſome general Rules in Arithmeticals for inventing three things to make a *Problem* poſſible, ſo we have here alſo. But again; in theſe we found alſo a Variety of particular Rules for the Invention of any one of the three things ſeparately, without being obliged to find any of the other two, (though that Invention is owing to the Rules for finding theſe others); whereas here, the Solutions of the preceding *Problems* do not afford us Rules for this purpoſe in every Caſe; and therefore we muſt be content with the general Rules. And for this Reaſon the Solution of the *Problems*, wherein two things only are given, are comprehended in theſe general Rules; ſuch of them at leaſt as are ſolvable upon our Method. Now the firſt thing I ſhall do here, is to give you the general Limitations, which the ſeveral things belonging to theſe *Problems* muſt have with reſpect to one another; which are theſe:

a may be any Number leſs than s, and not greater than l.
l any Number not leſs than a, nor greater than s.
r any Number not leſs than 1.
n any Integer greater than 1.
s any Number not leſs than a or l.

The

The Reasons of these Limitations, and that there are no other, will be manifest upon a small Attention to the Nature of a Geometrical Progression.

Now for the Invention of any three of these things to make a possible *Problem*.

(1.) If the things to be invented are a, n, r, or l, n, r, or s, n, r, they may be taken at pleasure within the general Limitation; because from any Number a or l, we may raise a Series in any Ratio. Again; Chusing any Ratio and Number of Terms, the Sum may be any Number whatever, since this has no Dependance on r or n.

(2.) If these are to be invented, *viz.* a, l, r, or a, l, n, or a, n, s, or l, n, s, or r, l, s, or a, r, s. Then take such one of the preceding Cases, in which are any two of the things to be now invented; and with the three things of that Case find the remaining thing to be invented (by *Prob.* 7, or 8.)

(3.) If a, l, s are to be invented, you must find all the five, by taking, first, any one of the three Cases in *Art.* 1. (in any of which you have one of the things now required) and by these find the other two.

For the *Problems*; wherein two things only are given, they are all possible, except three, *viz.* when a, s, or l, s, or a, l are given. For any other two they are both found in one of the Cases of *Art.* 1. *viz.* a, n, r, or l, n, r, or s, n, r. And therefore, since the Numbers of any of these three Cases may be taken at pleasure within the general Limits, this shews how one of the three things now required (by means of the two things given) may be found; and then by *Prob.* 7, or 8. the other two are found. For *Example*: Suppose a, r given, then is n also given; for we may take any Integer greater than 1; and by a, n, r we can find l, s by *Prob.* 7. But if a, l, or a, s, or l, s are given, we cannot find the other three; because we have no Method of finding any one of them consistently with the two given things; since these two are not two of the things in any of these Cases wherein three things may be taken at pleasure within general Limits, as a, n, r; l, n, r; s, n, r; which are the Foundation of all these Rules.

CHAP. IV.

Of the Composition *of* Ratios *and* Proportions, *and things depending thereon.*

DEFIN. I. THE Ratio of one Number to another is said to be compounded of the Ratios of two or more other Numbers compared to as many; when the Antecedent of the first Ratio (called the Compound) is to its Consequent, in the Ratio of the Product of the Antecedents to the Product of the Consequents of all the other Ratios or Numbers compared. Thus: The Ratio of a to b is said to be compounded of the Ratios of c to d, and of e to f, providing that these Numbers be $::$, *viz.* $a : b :: ce : df$. Or thus: Take Ratios fraction-wise, placing the Antecedent as Numerator over the Consequent. Then if one Ratio is a Fraction (or Quote) equal to the Product of several other Ratios multiplied as Fractions (or Quotes), that one is said to be compounded of these others; whether it be in the same Terms with that Product, or only in equivalent Terms. *Example:* $8 : 15$ is a Ratio compounded of these Ratios $2 : 3$ and $4 : 5$; because $\frac{8}{15} = \frac{2}{3} \times \frac{4}{5}$. Also $1 : 2$ is compounded of $2 : 3$, and $3 : 4$; for $\frac{1}{2} = \frac{2}{3}$

$\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$. Now that this Definition, or Character of a Compound Ratio, is the ſame in effect as the preceding, is evident; becauſe if $\frac{1}{2} = \frac{6}{12}$, which is the laſt Character; then $1:2::6:12$, which is the firſt Character; univerſally, if $\frac{a}{b} = \frac{ce}{df}$, then $a:b::ce:df$.

2. Any Number, Integer or Fraction, is alſo ſaid to be compounded of others, or to be a Compoſite of theſe others, to whoſe Product it is equal: Thus if $a = cde$, then is a compoſite of c, d, and e, and the compounding Numbers are called the Factors of the Compoſition.

3. Two Numbers are ſaid to be *like* or *ſimilar Compoſites*, when having an equal Number of Factors, they are all in the ſame Ratio, comparing the leſſer Factor of the one Compoſite to the leſſer of the other, and ſo on in order to the greateſt. Thus; ab, cd are *like Compoſites*, if $a:c::b:d$, (or $a:b::c:d$); alſo abc, def are *ſimilar Compoſites*, if $a:d::b:e::c:f$.

Hence it is evident, that all ſimilar Powers are ſimilar Compoſites, the Roots being the compounding ſimilar Factors. Thus; aa and bb are ſimilar Compoſites, becauſe $a:b::a:b$.

Scholium. The peculiar Doctrine of the *Compoſition* and *Reſolution* of Numbers, you have in the following Book: But the laſt two Definitions were neceſſary here, becauſe of ſome relative Properties of Numbers ariſing immediately from the Conſideration of compound Ratios, and which are equally applicable to Integers and Fractions: whereas the Compoſition afterwards explained regards Integers only, becauſe it's conſidered in Oppoſition to another thing which belongs not to Fractions, as they are diſtinguiſh'd from Integers.

4. Two Numbers, $a:b$, are ſaid to be in the duplicate Ratio of other two $c:d$, when the former are as the Squares of the other; *i. e.* if $a:b::c^2:d^2$; and if $a:b$ are as the ſquare Roots of $c:d$, *i. e.* if $a:b::c^{\frac{1}{2}}:d^{\frac{1}{2}}$; then $a:b$ are ſaid to be in the ſubduplicate Ratio of cd. Again: If $a:b::c^3:d^3$, then $a:b$ are ſaid to be in the triplicate Ratio of $c:d$; or if $a:b::c^{\frac{1}{3}}:d^{\frac{1}{3}}$, then are $a:b$ ſaid to be in the ſubtriplicate Ratio of $c:d$. Univerſally, if $a:b::c^n:d^n$, or $c^{\frac{1}{n}}:d^{\frac{1}{n}}$; then $a:b$ is ſaid to be in ſuch a Ratio as is named from $c:d$, with a complex Denomination expreſſing ſuch a Power or Root of $c:d$ as n expreſſes.

This was the antient way, and is ſtill in Uſe; but it's plain, that it's as ſimple and convenient a way to name the Order of the Power or Root, and ſay, that $a:b$ are in the Ratio of the Squares or Cubes, or n Powers or Roots of $c:d$; which is yet eaſier expreſſed in Characters, thus, $a:b::c^n:d^n$, or $c^{\frac{1}{n}}:d^{\frac{1}{n}}$.

Theorem I.

In any Series of Numbers, whatever $::$ *l* or not, as A : B : C : D : E, *&c.* the firſt is to the laſt, in the Ratio compounded of the Ratios of all the intermediate Couplets, comparing always every Number as Antecedent to that which is immediately next it.

Demonstr. A is to E in the compound Ratio of all the intermediate Terms; that is, $\frac{A}{E} = \frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} \times \frac{D}{E} = \frac{ABCD}{EBCD}$; for BCD is common to both Numerator and Denominator, therefore both being divided by it, the Quotes A and E make an equal Fraction or Ratio $\frac{A}{E}$.

Now

Now however many Terms the Series has, the Product of the Numerators will be the Product of all the Series except the last, and the Product of the Denominators will be the Product of all the Series except the first; and therefore the Product of all the intermediate Terms will be an equal Fraction, and contain nothing in it but the first Term divided by the last.

THEOREM II.

ANY two Numbers whatever are to one another in a Ratio compounded of an indefinite Number of other Ratios, *i. e.* we can assign as many other Couplets as can be required, whose Ratios compounded shall be equal to the given Ratio.

DEMONSTR. *and* RULE. Consider the Difference betwixt the given Numbers, and multiply them, by such a Number that the Difference of the Products shall be at least equal to the Number of the Ratios to be found; then betwixt these Products you can take as many intermediate Numbers as will answer the Problem. For *Example:* To find 4 Ratios whose Compound is equal to $\frac{2}{3}$; multiply 2, 3, by 4, the Products are 8, 12; betwixt which there are these Numbers, 9, 10, 11, and the whole Series is 8, 9, 10, 11, 12. But $\frac{2}{3}$ is $= \frac{8}{12}$; and this, by the preceding *Theorem*, is $= \frac{8}{9} \times \frac{9}{10} \times \frac{10}{11} \times \frac{11}{12}$.

Now the general *Reason* of the Rule is this, The given Terms being equally multiplied, the Products are in the same Ratio, and their Difference contains the Difference of the given Numbers as oft as the Multiplier contains 1. Therefore 'tis plain, that betwixt the Products can be found as many intermediate Terms as their Difference hath Units less 1; so if you multiply by such a Number as makes the Difference of the Products as great as the Number of Ratios required, you can have as many intermediate Numbers as you need.

SCHOLIUM. If the Difference of the given Numbers is great enough for the purpose, there is no need to multiply, as if in the former *Example*, the Numbers given had been 8, 12, instead of 2, 3. But *observe*, That the greater you make the Difference (of Numbers in the same Ratio), the greater Variety of Choice you have in the intermediate Terms and Ratios: So if instead of 8, 12, I take 32, 48 (*i. e.* multiply by 16); then I can chuse any of these Series to solve the Question, *viz.* 32 : 33 : 35 : 36 : 48; or 32. 35. 36 . 40 . 48, *&c.*

COROLL. 'Tis plain, that one Ratio is not only equal to the Compound of an indefinite Number of others, but any Number required can be taken in an indefinite Variety.

PROBLEM I.

Any Number of Ratios (or Couplets of Numbers compared) being given, to continue them, *i. e.* to find a Series of Numbers, which shall be to one another (comparing them each to the next in order from the first to the last) in the given Ratios taken in any Order assigned.

RULE. Multiply the Terms of the first Ratio (*i.e.* that which ought to be betwixt the two first Terms of the Series) by the Antecedent of the second, *&c* the Consequent of this by the Consequent of the first, and thus you have three Numbers which continue the two first Ratios. Then multiply this Series by the Antecedent of the third Ratio, and the

the Consequent of this by the last of that Series, and you have four Numbers which continue the first three Ratios. Go on thus, multiplying the last Series by the Antecedent of the next Ratio, and the Consequent of this by the last Term of that last Series. The Manner of the Work will be clear by an *Example:* Suppose these Ratios are to be continued in the Order proposed, *viz.* $2:3\,..\,4:5\,..\,6:7$, they reduce to this Series $48:72:90:105$. See the Work of *Example* 1.

The Method is the same if a Series is required in the continued Ratio of any two given Terms, so to continue the Ratio $2:3$. See *Example* 2.

Exam. 1.

$2:3$
$\quad 4:5$
$8:12:15$
$\qquad 6:7$
$48:72:90:105$

Exam. 2.

$2:3$
$\quad 2:3$
$4:6:9$
$\qquad 2:3$
$8:12:18:27$

Demonstr. The *Reason* of this Rule will easily appear without many Words from the Operation; by which it's manifest that the Terms of each Ratio are equally multiplied, and consequently the Numbers produced continue in the same Ratio; for $2:3$ are both multiplied by 4; then $4:5$ are both multiplied by 3. Again; $8:12:15$ are each multiplied by 6; and $6:7$ both multiplied by 15; and so on.

Scholium 1. Here we have a new Method of raising a Series in a given Ratio, so as all the Terms be Integers: In which *observe*, That none of the Extremes can be given Numbers, for that is a Limitation which in many Cases will bring in Mixt Numbers into the Series, as we shall afterwards learn.

COROLLARIES.

1. Since the Series produced expresseth in Order the several given Ratios, The Extremes of it are in the compound Ratio of these given (by *Theor.* 1.) And for this Reason, in whatever different Order the same Ratios may be continued, tho' the Series produced will be different; yet the Extremes will not only be in the same Ratio, because they are the Compound of the given Ratios, but they will be the very same Numbers; because by the Rule, the first is always the Product of all the given Antecedents, and the last the Product of all the Consequents: So take the preceding Ratios in another Order, the Series is different, but its Extremes are the same, as in this Scheme,

$4:5$
$\quad 6:7$
$24:30:35$
$\qquad 2:3$
$48:60:70:105$

2. 'Tis plain, that the Series raised will always have as many Terms as the Number of Ratios, and one more.

3. 'Tis evident, that the Extremes of a continued Series, (*i. e.* where all the given Ratios are the same Numbers) raised after this manner, are like Powers (of the Terms of the given Ratio) whose Index is the Number of Terms less 1, as in *Example* 2. where the Extremes 4, 9 are the Squares of 2 and 3; and 8, 27 are the Cubes or third Powers (3 being 1 less than 4, the Number of Terms); and in the same manner it will proceed *in infinitum*, because the lesser Extreme is multiplied by 2, and the greater by 3.

4. If the Extremes of the Series are only required, *i. e.* two Numbers which shall be in the compound Ratio of the Ratios given, then 'tis plain we have nothing more to do but multiply together all the Antecedents, and then all their Consequents, the two Products are in the Ratio sought; and if the same Ratio is continued, then raise the Antecedent and Consequent to a Power whose Index is the Number of Ratios (which is the Number of Terms that would be in the Series less 1.).

This follows also simply from the Definition of Compound Ratio. Scho-

SCHOLIUM 2. This is to be remark'd, That tho' the Method of this *Problem* be applied to all the different Expressions or Variety of Terms in which any Ratio can appear, *i. e.* all the possible different Choices of two Terms in the same Ratio; yet we cannot hereby find all the Variety of different Series, which are the Continuation of the same Ratio; because some of the Series found this way being equi-multiplied or divided, will always make a new Series in the same Ratio, and which will often be different from any of the Series raised this way. For *Example*: 8 : 12 : 18 : 27 is a Series raised from 2 : 3; and if one is raised from 4 : 6, which is the same Ratio, it is 64 : 96 : 114 : 216; and if you go on, still taking higher Terms of the Ratio, the Series will consist of greater Numbers; yet betwixt these two Series raised from 2 : 3 and 4 : 6 we have several others: For by multiplying the first of them by these several Numbers, 2, 3, 4, 5, 6, 7, we have six new Series in the same Ratio different from these two, and consequently from all others that can be raised from any other Terms of the Ratio. And if it is required to find all the different Series that continue the same Ratio to a given Number of Terms, the *Rule* for this you'll find afterwards. (See *Schol.* 2. to *Probl.* 6. *Ch.* 1. *Book* V.)

PROBLEM II.

To reduce any Number of Ratios (given as in the last *Problem*) to a common Antecedent or Consequent, that is, to find a Series whose first Term compared to all the rest as a common Antecedent, or to whose last Term as a common Consequent all the rest being compared, their Ratios shall be equal to certain given Ratios, taken in a certain Order.

RULE. Take the Ratios fractionally, and reduce them (by the Rule of Fractions) to one common Denominator for a common Consequent, or to a common Numerator for a common Antecedent: Then place the Terms in a Series according to the Order assigned.

Exam. These Ratios, $\frac{2}{3} \cdot \frac{4}{5} \cdot \frac{7}{8}$, reduced to one Antecedent or Numerator are $\frac{56}{84} \cdot \frac{56}{70} \cdot \frac{56}{64}$. So the Series is 56 . 84 . 70 . 64; but reduced to one Consequent or Denominator they are $\frac{80}{120} \cdot \frac{96}{120} \cdot \frac{105}{120}$, and the Series is 80 . 96 . 105 . 120.

The Reason is manifest.

THEOREM III.

Part I. If one Rank of ::*ls* (whether continued or not) is multiplied by another Rank in the same or another Ratio, the Products are also ::*l*, in the compound Ratio of the Factors.

Example.

If $A : B :: C : D$
And $a : b :: c : d$

Then $Aa : Bb :: Cc : Dd$

DEMONSTR. $AD = BC$, and $ad = bc$, therefore $AD \times ad = Bc \times bc$. But $AD \times ad = Aa \times Dd$, and $BC \times bc = Bb + Cc$; hence $Aa \times Dd = Bb \times Cc$, wherefore $Aa . Bb :: Cc : Dd$: Or thus, $\frac{A}{B} = \frac{C}{D}$, and $\frac{a}{b} = \frac{c}{d}$ (by *Definitions*) therefore $\frac{A}{B} \times \frac{a}{b} = \frac{C}{D} \times \frac{c}{d}$ (*i. e.* equal Quantities multiplied by equal), that is $\frac{Aa}{Bb} = \frac{Cc}{Dd}$. Therefore $Aa : Bb :: Cc : Dd$. Or it may be shewn by another Representation, thus:

Example.

$A : Ar :: B : Br$
$C : Cn :: D : Dn$

$AC : ACrn :: BD : BDrn$

Wherein the common Ratio of the Products is plainly rn.

Exam. in Numbers.

3 : 4 :: 6 : 8
5 : 7 :: 15 : 21

15 : 28 :: 90 : 168

Part

Part II. If one Rank of ::*ls* is divided by another, The Quotes are ::*l* in a Ratio which is the Quote of the Ratio of the first Rank divided by the Ratio of the other. Thus in the preceding *Examples* it is $\frac{A}{a} : \frac{B}{b} :: \frac{C}{c} : \frac{D}{d}$.

DEMONSTR. $AD = BC$, and $ad = bc$, therefore $\frac{AD}{ad}\left(=\frac{A}{a}\times\frac{D}{d}\right) = \frac{BC}{bc}$. $\left(=\frac{B}{b}\times\frac{C}{c}\right)$, hence $\frac{A}{a} : \frac{B}{b} :: \frac{C}{c} : \frac{D}{d}$.

Or thus: Dividing each Antecedent by its Consequent (by the Rules of Fractions), the Ratio of the first Pair is $\frac{Ab}{Ba} = \frac{A}{B}\times\frac{b}{a}$, and the second is $\frac{Cd}{Dc} = \frac{C}{D}\times\frac{d}{c}$. But $\frac{A}{B} = \frac{C}{D}$, and $\frac{b}{a} = \frac{d}{c}$, by Supposition; therefore $\frac{A}{B}\times\frac{b}{a} = \frac{C}{D}\times\frac{d}{c}$; that is, the Ratios are equal and consequently the Terms are ::*l*; also the Ratio is the Quote of the given Ratios, for $\frac{Ab}{Ba} = \frac{A}{B} \div \frac{a}{b}$, and $\frac{Cd}{Dc} = \frac{C}{D} \div \frac{c}{d}$.

But the whole will appear yet more easily in the other Representation, thus, $\frac{A}{C} : \frac{Ar}{Cn} :: \frac{B}{D} : \frac{Br}{Dn}$; for $\frac{Ar}{Cn} = \frac{A}{C}\times\frac{r}{n}$, and $\frac{Br}{Dn} = \frac{B}{D}\times\frac{r}{n}$; hence the Ratios are equal, and is also the Quote of the former Ratios, *viz.* r and n.

Exam. in Numbers.

2 :	5 ::	6 :	15
7 :	9 ::	14 :	18
14 :	45 ::	84 :	270

For $14\times 270 = 45\times 84 = 3780$, also $\frac{14}{45} = \frac{2}{5}\times\frac{7}{9}$. If the Products are again divided by one of the former Series, it makes an *Example* of the second Part.

The *Reverse* of this *Theorem* is not universally true; for two Ranks may produce ::*ls* which are not themselves ::*l*, as an Example will shew. This Rank, 6, 2, 2, 3, by this 2, 9, 9, 9, produces this ::*l* Rank, 12 : 18 :: 18 : 27.

SCHOLIUM. There are many Propositions demonstrable by the means of this *Theorem* alone, especially the 1st *Part*; which needing but a very small Help to demonstrate, I chuse to bring them here as *Corollaries*, and shall express them only in Characters, leaving you to express them in Words at length, and apply Numbers at pleasure, except the two last, which are of more frequent Use.

If A : B :: C : D, then follow these

COROLLARIES,	which arise from multiplying
1. $A^2 : B^2 :: AC : BD$	A : B :: C : D by A : B :: A : B
2. $A^2 : C^2 :: AB : CD$	A : C :: B : D by A : C :: A : C
3. $A^2 : BC :: AC : CD$	A : B :: C : D by A : C :: A : C
4. $A^2 : AB :: CD : D^2$	A : B :: C : D by A : A :: D : D
5. $A^2 : BC :: BC : D^2$	A : B :: C : D by A : C :: B : D
6. AB : AC :: BD : CD	A : C :: B : D by B : A :: D : C
7. AB : BC :: BC : CD	A : B :: C : D by B : C :: B : C
8. AC : BC :: BC : BD	A : B :: C : D by C : C :: B : B
9. AB : AD :: AD : CD	B : A :: D : C by A : D :: A : D
10. AC : AD :: AD : BD	C : D :: A : B by A : A :: D : D

COROLL. 11. $A^n : B^n :: C^n : D^n$; that is, if any four Numbers are $::l$, so are their Like Powers, whatever the Index be; the Deduction of which from the *Theorem* is plain; for if $A : B :: C : D$ is multiplied by $A : B :: C : D$, the Products are $::l$, *viz.* $A^2 : B^2 :: C^2 : D^2$; and if this again is multiplied by the same Rank, $A : B :: C\,D$, and these Products also by the same, and so on, the last Product will still be a proportional Rank; and it's plain, they will be Like Powers of these Roots, $A : B :: C : D$, the Index still increasing by 1 at every Multiplication.

The Truth contained in this *Corollary* may also be proved thus: Since $A : B :: C : D$, then is $\frac{A}{B} = \frac{C}{D}$; therefore the Like Powers of these fractional Roots must be equal, which Powers are $\frac{A^n}{B^n} = \frac{C^n}{D^n}$; therefore $A^n ; B^n :: C^n : D^n$; Or also $A\,D = B\,C$, and $\overline{A\,D}^n$ $\overline{B\,C}^n$: But $\overline{A\,D}^n = A^n \times D^n$, and $\overline{B\,C}^n = B^n \times C^n$; whence $A^n : B^n :: C^n : D^n$.

Hence again, The *Reverse* of this *Corollary* will easily be proved, *viz.* That if four Like Powers are $::l$, so are their Roots; for since $A^n : B^n :: C^n : D^n$, then $\frac{A^n}{B^n} = \frac{C^n}{D^n}$; and the n Roots of these Fractions are $\frac{A}{B} = \frac{C}{D}$, therefore $A : B :: C : D$; or thus, $A^n\,D^n = B^n\,C^n$, but these are $\overline{A\,D}^n = \overline{B\,C}^n$ (by *Theor.* 1. *Ch.* 1. *Book* III.); and hence $A\,D = B\,C$, and $A : B :: C : D$.

Again; It follows, That of a Series $\div l$, the Like Powers are also $\div l$, and *reversly*.

COROLL. 12. $A^2 : A\,D :: A\,D : D^2$; that is, the Product of two Numbers is a mean Proportional betwixt their Squares; for $A : D :: A : D$, and $A : A :: D : D$; which two Ranks multiplied produce $A^2 : A\,D :: A\,D : D^2$; or thus, $A^2 \times D^2 = A\,D \times A\,D$.

SCHOLIUM. This is a *Corollary* of the *Theorem*, but independent of the Supposition of $A : B :: C : D$; and it has also a Demonstration independent of this *Theorem*; thus, $A : D :: A^2 : A\,D$, by Equi-multiplication. Again; $A : D :: A\,D : D^2$; and hence, $A^2 : A\,D :: A\,D : D^2$.

Example: Take the Numbers 5 and 8, their Squares are 25 and 64, their Products 40, and $25 : 40 :: 40 : 64$.

Or, All these *Corollaries* follow from the equal Products of Extremes and Means, founded all upon this, that $A\,D = B\,C$, and that $A\,D \times B\,C = \overline{A\,D}^2 = \overline{B\,C}^2$.

THEOREM IV.

Two composite Numbers of an equal Number of Factors, are to one another in the compound Ratio of these Factors compared one to one in any Order.

DEMONSTR. Take any two Numbers, A, B, and suppose $A = abc$, and $B = mno$, then is $\frac{A}{B} = \frac{abc}{mno}$; But $\frac{abc}{mno} = \frac{a}{m} \times \frac{b}{n} \times \frac{c}{o}$, therefore $\frac{A}{B} = \frac{a}{m} \times \frac{b}{n} \times \frac{c}{o}$.

SCHOLIUM. This Proposition is the same in effect as this, *viz.* If the several Factors of the one are divided by those of the other, and the Quotes multiplied together, the Product is equal to the Quote of one whole Dividend by the other.

LEMMA.

When several Fractions (or Ratios) are continually multiplied together, the Product is equal to the Product of as many equivalent Fractions (or Ratios) expressed in any other different Terms.

Exam.

Example: If $\frac{2}{3} = \frac{4}{6}$, and $\frac{3}{5} = \frac{12}{20}$, then $\frac{2}{3} \times \frac{3}{5} = \frac{4}{6} \times \frac{12}{20}$, *viz.* $\frac{6}{15} = \frac{48}{120}$.

DEMONSTR. The *Reason* is manifest from this *Axiom*. That equal Fractions multiplied by equal, produce equal; *i. e.* equal Fractions of equal Fractions are equal.

COROLL. The continual Product of any Number of equal Fractions (or Ratios) expressed in different Terms, is equal to such a Power of any one of them as has for its Index the Number of Factors (or Fractions multiplied), that is, the Square if they are 2, the Cube, if 3, &c.

Exam. If $\frac{2}{3} = \frac{4}{6}$, then $\frac{2}{3} \times \frac{4}{6} = \frac{2}{3} \times \frac{2}{3}$, or the Square of $\frac{2}{3}$.

Again: If $\frac{2}{3} = \frac{4}{6} = \frac{6}{9}$, then is $\frac{2}{3} \times \frac{4}{6} \times \frac{6}{9} = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}$, or the Cube of $\frac{2}{3}$.

Universally: If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h} =$, &c. then is $\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} \times \frac{g}{h} \times$, &c. $= \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \times$ &c. making as many Factors in this Part as the other; so that if the Number of Factors (or equivalent Fractions) is n, and any one of them is $\frac{a}{b}$, the Product of the whole is $= \frac{a^n}{b^n}$ (that is, $\frac{a}{b}$ raised to the n Power.)

THEOREM V.

TWO like composite Numbers are in the Ratio of the like Powers of any two of their similar Factors, the Index of the Power being equal to the Index of the Composition (or Number of Factors.) Thus:

If ab and AB are like Composites, *i. e.* $a : A :: b : B$; then $ab : AB :: a^2 : A^2$: (i. e. ab to AB is in the duplicate Ratio of a to b); and if abc, ABC, are like Composites, they are as $a^3 : A^3$ (or in the triplicate Ratio of a to A.) Generally; If $abcd$, &c. : ABCD, &c. are similar, let the Number of Factors be n; then it will be, $abcd$, &c. : ABCD, &c. $:: a^n : A^n$.

DEMON. The Reason of this is plain from the last *Theorem*, and *Corollary* to the *Lemma*. For $\frac{abcd, \&c.}{ABCD, \&c.} = \frac{a}{A} \times \frac{b}{B} \times \frac{c}{C} \times \frac{d}{D}$, &c. (by *Theor.* 5.) and these Factors being, by Supposition, equal Fractions or Ratios, their Product is equal to $\frac{a^n}{A^n}$ by *Corollary* to the *Lemma*; that is, $abcd$, &c. : ABCD, &c. $:: a^n : A^n$.

THEOREM VI.

IN every Progression, the first Term is to the last as such a Power of the first, whose Index is the Distance of the Extremes, or Number of Terms less 1, is to the like Power of the second Term; or as those like Powers of any two Terms of the Series next other. Thus: If the Number of Terms is 3 or 4, the Extremes are in the duplicate or triplicate Ratio (or as the Squares or Cubes) of the first and second Terms; and so of others. As in this Series, A : B : C : D : E : F : G, whose Number of Terms is 7, I say $A : G :: A^6 : B^6$. And *generally*, if we make A the first Term, B the second, and L the last of any such Series, and n the Number of Terms less one; then I say, $A : L :: A^n : B^n$.

DEMON. By *Theorem* 2. A is to L in the Ratio composed of all the intermediate Ratios, *i. e.* $\frac{A}{L} = \frac{A}{B} \times \frac{B}{C} \times \frac{C}{D}$, &c. which are in Number equal to n, and all equal to one another; wherefore their Product or Compound is equal to the n Power of any one of

 them,

them, *viz.* of the first, or $\frac{A}{B}$ (by *Corol.* to *Lem.* preceding); that is, $\frac{A}{L} = \frac{A^n}{B^n}$; whence $A : L :: A^n : B^n$.

This is demonstrable also without the *Lemma*. Thus: If A is the first Term, the Ratio r, the Distance of the Extremes n; then is the second Term Ar, and the last Ar^n; and $A : Ar^n :: A^n : \overline{Ar}^n$ ($= A^n \times r^n$). For the Product of Extremes and Means are equal; *viz.* $A \times A^n \times r^n = A^n \times Ar^n$.

The *Reverse* is not true, *viz.* that if the first is to the last of a Series in this Ratio, the Series is $\div\!\!\div l$; for the changing any one of the Terms will shew this to be false.

Corol. If the Extremes and Number of Terms of a Geometrical Series are known, we learn from this *Theorem* a Rule, how by these to find the second Term. Thus: Multiply the last Term by such a Power of the first, whose Index is the Number of Terms less one; and divide the Product by the first Term, the Quote is the like Power of the second Term, whose Root being found is the thing sought. The Reason of this Rule is obvious; for since $A : L :: A^n : B^n$, therefore $A \times B^n = L \times A^n$, and $B^n = \frac{L \times A^2}{A}$ according to the Rule. But observe, Since $\frac{A^n}{A} = A^{n-1}$, therefore $B^n = L \times A^{n-1}$; which saves the Trouble of one Multiplication in raising A, and also the Division by A; and since n is here supposed to be the Number of Terms less one, therefore $n - 1$ is that Number less 2; *i. e.* it is the Number of Means betwixt A and L; and if we call $n - 1 = m$, then is $n = m + 1$. And to find the first of any Number of Means betwixt A and L, (or the Mean next that Extreme which is called the first, as A) which Mean call B; The Rule is this: Multiply the last Term by the m Power of the first, and from the Product extract the $m + 1$ Root; which is thus, $B = \overline{A^m \times L}|^{\frac{1}{m+1}}$

Theorem VII.

If four Numbers are proportional, *viz* $A : B :: C : D$; then as many Means as fall betwixt $A : B$ (in any Ratio as r) so many also fall betwixt $C : D$ in the same Ratio.

Demon. From A raise a Series in the Ratio r, as $A : Ar : Ar^2$, *&c.* and 'tis certain you must come at last to a Term equal to B; because 'tis supposed that B is the last Term of a Series raised from A in the Ratio r. Now n being the Number of Terms $- 1$, 'tis plain that the last Term of the Series is $ar^n = B$. Raise also a like Series from C; thus, $C : Cr : Cr^2 : Cr^3 :$ *&c.* C^n. These two Series being in the same Ratio, it will be $A : Ar^n :: C : Cr^n$; *that is,* $A : B :: C : Cr^n$: but $A : B :: C : D$; therefore $Cr^n = D$. Again; $C : Cr^n$ have as many Means as $A : Ar^n$ (*i. e.* as $A : B$) by the nature of the Series raised. Therefore, *lastly*, $C : D$ ($= C : Cr^n$) admit as many Means as $A : B$ ($A : Ar^n$).

The *Reverse* is also true; for if $A : B$ and $C : D$ admit an equal Number of Means in the same Ratio, they are in the same compound Ratio; *i. e.* $A : B :: C : D$.

Scholium. Here we do not distinguish betwixt Integers and Fractions, but take Numbers in general; yet when we speak of Means, we always understand rational Terms. In the next Book, you'll learn what Qualifications of the Extremes make the Means Integers or Fractions, and what do or do not admit of rational Means.

Theorem VIII.

Any two Numbers which are like Powers, as $A^n : B^n$:, admit as many Means $::$ *ls*, as the Index less 1, (or $\overline{n - 1}$) in the Ratio of the Roots $A : B$; that is, 1 if they are square Numbers, 2 if they are Cubes, *&c.*

Demon. If a Series is raised (by *Problem* 1.) in the Ratio of $A : B$: to a Number of Terms equal to $\overline{n + 1}$, the Extremes will be $A^n : B^n$: (by *Corol.* 2. and 3. to that *Problem*); and when the Number of Terms is $\overline{n + 1}$, the Number of Means will be $\overline{n - 1}$; therefore, *&c.*

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The *Reverse* of this *Theorem* does not hold; as an Example will shew, *viz.* 16 : 24 : 36 : 54; wherein tho' there are two Means, yet the Extremes are not Cube Numbers. But if the lesser Extreme is 1, the Reverse must always hold; because the Terms of that Series are the several Powers of the second Term, (*Corol* 2. *Prob.* 3. *Chap.* 3.) Afterwards you'll find it demonstrated, that they are either such Powers, or Equi-multiples of such Powers.

SCHOLIUM. If it be required to fill up the Means betwixt A^n and B^n, knowing their Roots $A : B$; the whole Series may be rais'd by *Prob.* 1.

But I shall here add another Demonstration of this *Theorem*, from which we shall see more particularly the Composition of every Term in the Series, and a new Method of filling up the mean Terms required. Thus: Suppose a Series of the Powers of A^n : and B^n : beginning with 1; these are Geometrical Progressions in the Ratios of $1 : A$, and $1 : B$, and have an equal Number of Terms (by what has been already explained.) Place these two Series the one above the other in a reverse Order, as in the following Example: Multiply the one Series into the other, *i. e.* every Term of the one into the correspondent Term of the other, and the Products make a Series of $n+1$ Terms, whose Extremes are $A^n : B^n$: in the continued Ratio of $A : B$, (*viz.* the Compound of the Ratios of the two Series multiplied) as has been already demonstrated in *Theor.* 3. and is indeed obvious in this first Case by Inspection into the Series itself. And here we see how every middle Term is composed of the Powers of the Roots; and so we have a new Rule for solving *Problem* 1. when the given Ratios are all equal; being, for Example, $A : B$. Or if the Extremes are given, and also the Roots, we can fill up the middle Terms more easily than by *Prob.* 1. by making up the Series of the Powers of A and B, and then multiply them together in this manner.

$$
\begin{array}{llllll}
A^n : A^{n-1} & : A^{n-2} & : A^{n-3}, & \&c.\ A & : 1. \\
1 : B & : B^2 & : B^3, & \&c.\ B^{n-1} & : B^n. \\
\hline
A^n : BA^{n-1} & : B^2A^{n-2} & : B^3A^{n-3}, & \&c.\ AB^{n-1} & : B^n.
\end{array}
$$

But the same Truth, as to the component Parts of the middle Terms, will also appear from the Method of *Prob.* 1. as here: Suppose the Ratio $A : B$ to be continued, 'tis done thus by *Prob.* 1. And 'tis plain that continuing it on, the Root A will be once more compounded in every Term at every Multiplication, and one Term more will be added, which will be the next higher Power of B.

$$
\begin{array}{llll}
A : B & & & \\
\quad A & : B. & & \\
\hline
A^2 : AB & : B^2. & & \\
 & \quad A & : B. & \\
\hline
A^3 : A^2B & : AB^2 & : B^3, \&c. &
\end{array}
$$

THEOREM IX.

ANY two like composite Numbers admit as many Means as the Index (or Number of component Factors) $-$ 1, and in the Ratio of their similar Factors. Thus:

If A and B represent two like Composites of two Factors, they admit 1 Mean; if of three Factors, they admit 2 Means, and so on: Generally, if they are composite of an n Number of Factors, they admit $n-1$ Means in the Ratio of any two of their similar Factors; suppose $a : b$. (*i. e.* a in the composite A, and b in B.)

DEMON. By *Theorem* 5. A and B are in the Ratio of the n Powers of any two of their similar Factors, *i. e.* $A : B :: a^n : b^n$. But (by *Theor.* 8.) $a^n : b^n$: admit $n-1$ Means in the constant Ratio of $a : b$, and (by *Theor.* 7.) $A : B$: admit as many in the same Ratio.

The Reverse of this *Theorem* is also true, but the Proof of it depends upon some things not yet explained; therefore it must be referred to another place. (See *Theor.* 34. in §. 11. *Chap.* 1. *Book* V.)

SCHOLIUM. If the component Factors are known, the mean Terms may easily be fill'd up, and their particular Compositions discover'd, after the Method of this Example. Suppose $\overline{abcde}$, and $\overline{ABCDE}$, the two like Composites, the Series will be this, $abcde$

$: bc$

: $bcdeA : cdeAB : deABC : eABCD : ABCDE$. In which the Means are filled up, by taking out the first Term of the lesser Extreme, and putting in the first of the greater; and so on in Order, taking out the next Term of the first Extreme, and putting in the next Term of the last Extreme, till all the Terms of the first Extreme are taken out; and that the Series thus made is .·. *l* in the Ratio of the similar Factors, is manifest by Inspection.

But if we only know the Ratio, (*i. e.* any one Couplet of the similar Factors) the Means must be filled up by the common Rule of *Prob.* 3. *Chap.* 3.

COROLLARIES.

1. The Product of two Numbers which are like Composites of two Factors, is a Square Number; so if N, M are similarly composed of two Factors, they admit one Mean, as X, and then 'tis plain, $NM = X^2$: Or, independently of this *Theorem*, it may be demonstrated thus: Let $N = ab$, and $M = AB$, and because they are similar, $a : b :: A : B$, and $aB = bA$; wherefore $aB \times bA$ ($= ab \times AB$) is a Square Number, whose Root is $aB = bA$.

The *Reverse* of this *Corollary* is also true, *viz.* That if two Numbers, A, B, produce a Square, they are Like Composites of two Factors; but the *Demonstration* of this belongs to another Place. (See *Coroll.* 1. *Theor.* 34. in §. 11. *Ch.* 1. *Book* V.)

2. Any two Like Composites of an even Number of Factors produce a Square Number; for any two of the Factors of the one, and the similar two of the other, produce a Square by the 1st *Coroll.* The like will any other two do; and these two Squares multiplied together will produce a Square (by *Theorem* 1. *Book* III.) and so on, taking in the two next Factors. *Example:* If there are four Factors, $abcd : ABCD$, because 'tis $a : b : A : B$, and $c : d :: C : D$; hence $ab\,AB = \overline{aB}^2$, also $cd\,CD = \overline{cD}^2$: And, lastly, $abcd \times ABCD = \overline{aB}^2 \times \overline{cD}^2 = \overline{aBcD}^2$. Or thus; since $aB = bA$, and $cD = dC$, therefore $aBcD = bAdC$; and these multiplied produce $abcd\,ABCD = \overline{aBcD}^2$, or $\overline{bAdC}^2$.

3. If the Product of two Like composite Numbers of an odd Number of Factors is divided by the Product of any two of the similar Factors (one in each Composite) the Quotient is a Square Number. *Example:* If A, B, are two such Numbers, the least Number in A being a, and the least in B being b, then I say, $\frac{AB}{ab}$ is a Square Number; for when one Factor is taken out of each there remains an even Number of component Factors in each, whose Product makes a Square Number: But by dividing the total Product of all the Factors by the Product of any two similar ones, we effectually take them away, and reduce the Case to an even Number of Factors.

THEOREM X.

IF betwixt any Number A, and each of two others B and C, there falls an equal Number of Means, there fall as many betwixt B and C, which are in the Ratio of the 2d Terms of the two Series from A to B, and C.

DEMONSTR. Suppose the first Mean betwixt A and B is b, and the first betwixt A and C is c, then, by *Theorem* VI. these Proportions mark'd in the Margin are true (n being the Number of Terms — 1); and hence this is also true, $B : C :: b^n : c^n$; but (by *Theorem* 8.) betwixt $b^n : c^n$ there fall $n-1$ Means in the Ratio $b : c$, (*viz.* as many as fall by Supposition betwixt A : B and A : C; for n being the Number of Terms — 1, the Number of Means is $n-1$); therefore, (by *Theorem* 7.) there fall as many betwixt B : C, in the same Ratio of $b : c$.

$$\text{Series,}\begin{cases} A : b \text{ - - - - - } B \\ A : c \text{ - - - - - } C \end{cases}$$

$$\text{hence}\begin{cases} A : B :: A^n : b^n \\ A : C :: A^n : c^n \end{cases}$$

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SCHOLIUMS.

1. The Means that fall betwixt A : B and A : C must manifestly be in different Ratios; for they must be different Numbers, since there are an equal Number of them, and tend to different Extremes B and C; but the Means betwixt B and C may be in the same Ratio with one of the former, or different from both; for if we suppose A : B : : B : C, then the Ratio in the Series from B to C will be the same as that from A to B.

2. The *Reverse* of this *Theorem* is this; If any Number of Means fall betwixt two Numbers, C, B, there is another Number, A, betwixt which, and each of these, C, B, there fall as many Means. But the *Demonstration* of this depends upon something not yet explained, and must be referred to another Place. (See *Theor.* 27. and 28. in §. 11. *Ch.* 1. *B.* V.)

THEOREM XI.

If one Extreme of a Series is a Power whose Index n is the Number of Terms — 1, the other Extreme is also a like Power. Or thus; If two Numbers, whereof one is an n Power, admit $\overline{n-1}$ Means, the other is also an n Power. If the one Extreme is A^n and the other B, I say, B has an n Root; *i. e.* If n is 2 then B is a Square Number. If $n=3$ then B is a Cube Number, *&c.*

DEMON. Betwixt 1 and A^n there fall $\overline{n-1}$ Means, (*viz.* all the inferior Powers of A) and since there fall as many by Supposition betwixt A^n and B, therefore (by the last) there fall as many betwixt 1 and B, and hence, lastly, B is an n Power, *viz.* the n Power of the 2d Term of the Series; for all the Terms of a Series from 1 are the Powers of the 2d. (*Coroll.* 2. *Probl.* 3. *Ch.* 3.)

THEOREM XII.

Of four Numbers : : *l*, if any three of them are Like Powers, so is the 4th, and its Root is a 4th : : *l* to the Roots of the other three compared in the same Order as these Powers: Thus, if $a^n : b^n : : c^n : d$, I say, d is an n Power, and $a : b : : c : d^{\frac{1}{n}}$

DEMONSTR. Betwixt $a^n : b^n$, there fall $\overline{n-1}$ Means (*Theor.* 8.) and as many betwixt $c^n : d$ (*Theor.* 7.), therefore d is an n Power (*Theor.* 11.), which is the first Part. Again; Suppose $d=r^n$, so that $d^{\frac{1}{n}}=r$; then because $a^n : b^n : : c^n : r^n$, therefore $a : b : : c : r$ (*Theorem* 3. *Coroll.* 11.), that is, $a : b : : c : d^{\frac{1}{n}}$, because $r=d^{\frac{1}{n}}$, which is the last thing.

Or thus; Since $a^n : b^n : : c^n : d$, therefore $\frac{b^n c^n}{a^n}=d$; but $\frac{b^n c^n}{a^n}=\overline{\frac{bc}{a}}\Big|^n$, therefore $\frac{bc}{a}=d^{\frac{1}{n}}$, or $a : b : : c : d^{\frac{1}{n}}$.

CHAP. V.

Containing the Comparison *of unequal Ratios (wherein the Ratio is always to be understood as the Quote of the Antecedent by the Consequent): With the Comparisons of* Arithmetical *and* Geometrical PROPORTIONS.

Observe, *For the Words* greater *and* lesser, *I use these Signs,* 7 *and* ∠. *Thus,* A 7 B, *or* A ∠ B, *signifies* A *greater or lesser than* B; *and* A : B 7, *or* ∠ C : D, *signifies that the Ratio of* A *to* B *is greater or lesser than that of* C *to* D.

§. I. *The*

§. I. *The Comparison of* Unequal *Ratios.*

Theorem I.

If to each of two unequal Numbers, a, b, be added or subtracted an equal Number, the Ratio of the Sums is lesser, but the Ratio of the Differences greater, than that of the given unequal Numbers, comparing the greater Term to the lesser; or, contrarily, comparing the lesser to the greater: Thus, if $a \urcorner b$, then $a+c:b+c$, $\angle\, a:b$, and $a-c:b-c \urcorner a:b$. *Example*: $3 \urcorner 2$ and $3+4:2+4 \angle\, 3:2$, and $3-2:4-2 \urcorner 3:4$

Demonstr. If $a:b::c:d$. then $a+c:b+d::a:b$ (*Theor.* 4. *Ch.* 3. *B.* IV.) but because $a \urcorner b$, therefore $c \urcorner d$ and $b+c \urcorner b+d$; wherefore $a+c:b+c \angle\, a+c:b+d$ (*Ax.* 11.), and consequently $a+c:b+c \angle\, a:b$.

But comparing the lesser to the greater, proceed thus; $b+d:a+c::b:a$; but $b+c \urcorner b+d$, therefore $b+c:a+c \urcorner b+d:a+c$; hence $b+c:a+c \urcorner b:a$.

Again; For the Differences, $a-c:b-d::a:b$, but $a \urcorner b$, hence $c \urcorner d$, and $b-c \angle\, b-d$, therefore $a-c:b-c \urcorner a-c:b-d$, and consequently, $a-c:b-c \urcorner a:b$: But comparing the lesser to the greater, then $b-d:a-c::b:a$; and since $d \angle\, c$, therefore $b-d \urcorner b-c$, and hence $b-c:a-c-b \angle\, b-d:a-c$, and consequently, $b-c:a-c \angle\, b:a$.

The *Reverse* is also true, *viz.* If $a+c:b+c \angle\, a:b$, therefore $a \urcorner b$; and so of the rest of the Parts, which is easily demonstrated.

Theorem II.

If four Numbers are $::l$, $a:b::c:d$, the Antecedents a, c, being less than their Consequents, and $a\ b$, the two least of the four, then, by equally increasing or decreasing the two Antecedents, or the two Consequents, or both, the Proportion is destroyed, and the Ratio of the first Couplet so alter'd will be lesser or greater than that of the second in these Circumstances, *viz.*

1. By equal Addition to the Antecedents, the 1st Couplet has the greater Ratio; thus, $a+n:b \urcorner c+n:d$.

2. By equal Diminution of the Consequents, the 1st Couplet has the greater Ratio; thus, $a:b-n \urcorner c:d-n$.

3. By equal Addition to the Consequents, the Ratio of the 1st Couplet is least; thus, $a:b+n \angle\, c:d+n$.

4. By equal Diminution of the Antecedents, the Ratio of the 1st Couplet is least; thus, $a-n:b \angle\, c-n:d$.

5. By equal Addition to all the four, the Ratio of, the 1st Couplet is the greatest; thus, $a+n:b+n \urcorner c+n:d+n$.

6. By equal Diminution of all the four, the Ratio of the first Couplet is greatest; thus, $a-n:b-n \urcorner c-n:d-n$.

7. By equally increasing the Antecedents and diminishing the Consequents, the Ratio of the 1st Couplet is the greater; thus, $a+n:b-n \urcorner c+n:d-n$.

8. By equally diminishing the Antecedents and increasing the Consequents, the Ratio of the 1st Couplet is the least; thus, $a-n:b+n \angle\, c-n:d+n$.

Demonstration.

Case 1. $a+n:b \urcorner c+n:d$. For $a+n:c+n \urcorner a:c$ (*Theor.* 1.) but $a:c::b:d$, therefore $a+n:c+n \urcorner b:d$; and hence (by *Theor.* 4th, below; which is demonstrated independently of this) $a+n:b \urcorner c+n:d$.

Case

Case 2. $a:b-n \urcorner c:d-n$. For $b-n:d-n \angle b:d$ (*Theor.* 1.); but $b:d::a:c$; hence $b-n:d-n \angle a:c$; or $a:c \urcorner b-n:d-n$; therefore $a:b-n \urcorner c:d-n$ (*Theor.* 7.)

Case 3. $a:b+n \angle c:d+n$. For $b:d::a:c$, and $b+n:d+n \urcorner b:d$ (*Theor.* 1.), consequently $b+n:d+n \urcorner a:c$, and (by *Theor.* 7.) $b+n:a \urcorner d+n:c$; and, *reversly*, $a:b+n \angle c:d+n$ (*Theor.* 6.)

Case 4. $a-n:b \angle c-n:d$. For $a:c::b:d$; but $a-n:c-n \angle a:c$, or $b:d$, i.e. $b:d \urcorner a-n:c-n$, and (by *Theor.* 7.) $b:a-n \urcorner d:c-n$; and, *reversly*, $a-n:b \angle c-n:d$.

Case 5. $a+n:b+n \urcorner c+n:d+n$. For multiplying the Antecedent of the one into the Consequent of the other, the Products which (are the Antecedents when the two are reduced to a common Consequent) are $ad+an+dn+nn$, and $bc+bn+cn+nn$; the 1st of which is the greater, because taking equal Parts out of both, *viz.* $ad=bc$, and $nn=nn$, there remains in the first $an+dn=n\times\overline{a+d}$, and in the second $bn+cn=n\times\overline{b+c}$; but $\overline{a+d} \urcorner b+c$ (*Theor.* 14), therefore the 1st is greatest, *i.e.* $a+n:b+n \urcorner c+n:d+n$.

Case 6. $a-n:b-n \urcorner c-n:d-n$. For multiplying the Antecedent of each into the Consequent of the other, the Products are $ad-an-dn+nn$, and $bc-bn-cn+nn$; out of each take nn, and compare the Remainders, which are $ad-\overline{an+dn}$, and $bc-\overline{bn+cn}$; but $ad=bc$, and $a+d \urcorner b+c$, therefore the 1st is greatest, *i.e.* $a-n:b-n \urcorner c-n:d-n$.

Case 7. $a+n:b-n \urcorner c+n:d-n$. For $a+n:c+n \urcorner a:c$, or $b:d$ (*Theor.* 1.), and $b-n:d-n \angle b:d$ (*Theor.* 1.), therefore $a+n:c+n \urcorner b-n:d-n$, and $a+n:b-n \urcorner c+n:d-n$ (*Theor.* 7.).

Case 8. $a-n:b+n \angle c-n:d+n$. For $a-n:c-n \angle a:c$, or $b:d$ (*Theor.* 1.), and $b+n:d+n \urcorner b:d$; hence $a-n:c-n \angle b+n:d+n$; and lastly, $a-n:b+n \angle c-n:d+n$ (*Theor.* 7.).

COROLL. All these *Cases* are applicable to Numbers in Geometrical Progression. I shall only mention the Application of *Case* 5. and 6. thus: By Addition or Subtraction of the same Numbers, to or from each Term of a Series $\div l$, the Sums or Differences are not $\div l$; but (comparing them from the least to the greatest) does continually decrease; thus, if $a:b:c:d$, *&c.* are $\div l$, then $a+n:b+n \urcorner b+n:c+n \urcorner c+n:d+n$, *&c.* But from the greatest to the least, it does continually increase.

THEOREM III.

IF $a:b \urcorner c:d$, and $a \angle c$, or $a=c$, then is $b \angle d$.

Example: $6:7 \urcorner 8:13$, or $6:7 \urcorner 7:9$; also $6 \angle 8$, $7=7$, then $7 \angle 13$ or 9.

DEMONSTR. It cann't be that $b=d$; for since $a \angle$ or $=c$, then $a:b$ would be $\angle$ or $=c:d$, contrary to Supposition; and it would be yet more so if we suppose $b \urcorner d$.

The *Reverse* is also true, as you'll easily prove, *viz.* If $b \angle d$, and $a \angle$ or $=c$, then $a:b \urcorner c:d$.

THEOREM IV.

IF $a:b \urcorner c:d$, also $a+b=c+d$, then is $a \urcorner c$.

Example: $6:2 \urcorner 5:3$, and $6+2=5+3$, then $6 \urcorner 5$.

DEMONSTR. Suppose $a=c$, then must $b \angle d$ to make $a:b \urcorner c:d$; but if $a=c$, and $b \angle d$, then $a+b \angle c+d$, contrary to Supposition. And if we suppose $a \angle c$, it will yet more strongly follow that $a+b \angle c+d$.

The *Reverse* is also true, *viz.* If $a+b=c+d$, and $a \urcorner c$, then is $a:b \urcorner c:d$. The *Reason* is easy from the preceding Method.

THEOREM V.

IF $a:b \urcorner c:d$, then is $ad \urcorner bc$.

DEMONSTR. $c+n:d \urcorner c:d$; let n be taken so as $a:b::c+n:d$, and then $ad = bc+bn$, hence $ad \urcorner bc$.

Or this Truth is in effect already demonstrated in Fractions, where it's shewn, that if $\frac{a}{b} \urcorner \frac{c}{d}$, then $ad \urcorner bc$.

The *Converse* of this is also true, *viz.* If $ad \urcorner bc$, then $a:b \urcorner c:d$, for $\frac{ad}{b} \urcorner c$, and $\frac{a}{b} \urcorner \frac{c}{d}$, by equal Division.

THEOREM VI.

IF $a:b \urcorner c:d$, then *reversly*, $d:c \urcorner b:a$.

DEMONSTR. $\frac{a}{b} : \frac{c}{d} :: \frac{d}{c} : \frac{b}{a}$ (*Ch.* 1. *gen. Cor.* 13.) and if $\frac{a}{b} \urcorner \frac{c}{d}$, therefore $\frac{d}{c} \urcorner \frac{b}{a}$, which Consequence is also proved already from the Nature of Fractions.

Or thus, Take $a:b::c+n:d$, whence $d:c+n::b:a$; but $d:c \urcorner d:c+n$, therefore $d:c \urcorner b:a$.

THEOREM VII.

IF $a:b \angle c:d$, then, *alternately*, $a:c \urcorner b:d$.

DEMONSTR. $\frac{a}{b} : \frac{c}{d} :: \frac{a}{c} : \frac{b}{d}$ (*Ch.* 1. *gen. Cor.* 12.) and if $\frac{a}{b} \urcorner \frac{c}{d}$, therefore $\frac{a}{c} \urcorner \frac{b}{d}$, which is also already proved in the Doctrine of Fractions.

Or thus, Take $a::b:c+n:d$, whence $a:c+n::b:d$; but $a:c \urcorner a:c+n$, consequently $a:c \urcorner b:d$.

THEOREM VIII.

IF $a:b \urcorner c:d$, then, *compoundly*, $a+c:b+d \angle a:b$, but $\urcorner c:d$; also $a+b:c+d \angle a:c$, and $\urcorner b:d$.

DEMONSTR. Take $a:b::c+n:d$, then $a+\overline{c+n}:b+d::a:b$; but $a+c:b+d \angle a+\overline{c+n}:b+d$, therefore $a+c:b+d \angle a:b$. Again; Take $c:d::a:b+n$, then $c+a:d+\overline{b+n}::c:d$; but $a+c:b+d \urcorner a+c:d+\overline{b+n}$, therefore $a+c:b+d \urcorner c:d$.

For the second Part: Since $a:b \urcorner c:d$, therefore, *alternately*, $a:c \urcorner b:d$ (*Theor.* 7.) and then by the 1st Part it is $a+b:c+d \angle a:c$, and $\urcorner b:d$.

The *Alternations* of all these Conclusions is true by vertue of the preceding *Theorem*.

The *Converse* of this *Theorem* is also true, *viz.* if $a+c:b+d \urcorner c:d$, then $a:b \urcorner c:d$; for, (by the following *Theorem*) $a+c-c\ (=a):b+d-d\ (=b) \urcorner c:d$ (taking $a+c$, and $b+d$ instead of a, b, in the following *Theorem*), that is, $a:b \urcorner c:d$. The *Converse* of the other Conclusions will be found the same way.

THEOREM IX.

IF $a:b \urcorner c:d$, then, *divisively*, $a-c:b-d \urcorner a:b$, and also $\urcorner c:d$. Again, $a-b:c-d \urcorner a:c$, and also $\urcorner b:d$.

DEMONSTR. Take $a:b::c+n:d$, then $a-c-n:b-d::a:b$; but $a-c:b-d \urcorner a-c-n:b-d$, therefore $a-c:b-d \urcorner a:b$, and consequently also $\urcorner c:d$, which is $\angle a:b$. The other Part, or $a-b:c-d \urcorner a:c$, or $b:d$, is proved the same way, by taking $a:c \urcorner b:d$.

The

The *Converse* is also true, and depends upon the 7th in the same manner as the Converse of that upon this.

SCHOLIUM. As we have argued from the greater Ratio to the lesser in the preceding five *Theorems*, so we may argue the same way from the lesser to the greater; for this is all contained in the other; because if $a:b \urcorner c:d$, then $c:d \angle a:b$, and it's no matter which of the two Ratios we suppose to be greatest.

THEOREM X.

IF there are two Ranks of four Numbers each, whereof the Antecedents of the one are the Consequents of the other Rank, as a, b, c, d, and b, e, d, f; and if $a:b \urcorner c:d$, also $b:e \urcorner d:f$, then is, $a:e \urcorner c:f$.

If $a:b \urcorner c:d$
and $b:e \urcorner d:f$
then $a:e \urcorner c:f$

DEMONSTR. $a:c \urcorner b:d$, and $b:d \urcorner e:f$ (*Theor.* 7.) hence $a:c \urcorner e:f$, and $a:e \urcorner c:f$ (*Theor.* 7.)

THEOREM XI.

OF two Ranks of four Numbers each, where the Antecedents or Consequents of both are the same Numbers, as in the Margin, and the Ratio of the first two greater than that of the other two, the Ratio of the Sums of the Antecedents to that of the Consequents, is greater than the Ratio of the two common Terms.

If $a:b \urcorner c:d$
and $e:b \urcorner f:d$
then $a+e:b \urcorner c+f:d$
or
$a+e:c+f \urcorner b:d$

DEMONSTR. Take $a-n:b::c:d$, and $e-m:b::f:d$, then $\overline{a-n}+\overline{e-m}:c+f::b:d$ (*Theor.* 8. *Chap.* 3.), but $\overline{a+e} \urcorner \overline{a-n}+\overline{e-m}$, hence $a+e:c+f \urcorner \overline{a-n}+\overline{e-m}:c+f$, or $b:d$.

THEOREM XII.

OF two Ranks, whereof the Extremes of the one are the Means of the other, as in the Margin; if $a:b \urcorner c:d$, and $b:e \urcorner f:c$, then $a:e \urcorner f:d$.

$a:b \urcorner c:d$
$b:e \urcorner f:c$
$a:e \urcorner f:d$

DEMONSTR. $ad \urcorner bc$, and $bc \urcorner ef$ (*Theor.* 5.) hence $ad \urcorner ef$, and $a:e \urcorner f:d$, by Converse of the 5th.

SCHOLIUM. If the Ranks are so disposed that the Extremes or Means of the one are also the Extremes and Means of the other, as in the Margin; yet we can draw no Conclusion, because all that immediately follows, is only that $ad \urcorner bc$, and $ef \urcorner bc$; but leaves it undetermined, whether ad is $= ef$, or not. for it may be either way; as these *Examples* shew, *viz.* (1.) $3:6 \urcorner 4:15$, and $9:6 \urcorner 4:5$, and $3 \times 15 = 5 \times 9$. (2.) $3:6 \urcorner 4:15$, and $8:6 \urcorner 4:5$, and $3 \times 15 \urcorner 8 \times 5$.

$a:b \urcorner c:d$
$e:b \urcorner c:f$

§. II. Arithmetical *and* Geometrical *Proportions compared.*

LEMMA.

IF four Numbers are proportional, either Arithmetically or Geometrically, the *least* and *greatest* of the four are the two Means or the two Extremes.

a, b, c, d

DEMONSTR: The least of the four is either one of the Extremes or one of the Means. Suppose

1. That a is the least, then because $a \angle b$, so is $c \angle d$; but c is also greater than a (which is the least of the four), and hence $d \urcorner b$; consequently d is the greatest.

2. If b is the least, then, taking them *reversly*, b, a, d, c, because the one Extreme is the least, the other is the greatest (by the 1st *Case*), *i. e.* in the other Position, the two Means are the least and greatest.

Coroll. Of four Numbers, $: l$, or $:: l$, the two greatest or the two least, are one of them an Extreme, the other a Mean.

Theorem XIII.

If four Numbers are $:: l$, the Difference of that Couplet (*i. e.* of that Extreme and Mean) which consists of the greatest Numbers is the greatest; so if $a : b$ are greatest, $b - a \urcorner d - c$.

Example:

a : b :: c : d
4 : 6 :: 2 : 3
4 : 2 :: 6 : 3

Demonstr. $b - a : d - c :: a : c$; but by Supposition $a \urcorner c$, therefore $b - a \urcorner d - c$; or if a, c are greatest, then, because $c - a : d - b :: a : b$, and $a \urcorner b$, therefore $c - a \urcorner d - b$.

Coroll. Of Numbers in $\div l$, $a : b : c : d$, &c. the Differences taken from the lesser Extreme a, do continually increase; thus, $b - a \angle c - b \angle d - c$, &c. for $a : b :: b : c$, and b, c the two greatest, hence $c - b \urcorner b - a$, and so on, through the Series.

Scholium. In the last *Chapter* (*Theor.* 18.) it is demonstrated, that these Differeences make a Series $\div l$.

Theorem XIV.

Of four Numbers $:: l$ the Sum of the Extremes and Means are unequal; and that Sum whose Parts are the least and greatest of the four, is the greatest.

a : b :: c : d
3 : 2 :: 6 : 4
2 : 3 :: 4 : 6

Demonstr. If a is the least, then is d the greatest (*Lemma*), and I say, $a + d \urcorner b + c$; for c, d or d, b are the two greatest; suppose c, d, then $d - c \urcorner b - a$ (*Theor.* 13.); add $a + c$ to both, and $d - c + a + c$ $(= d + a)$ is $\urcorner b - a + a + c$ $(= b + c)$. Or if d, b are the two greatest, then also $d - b \urcorner c - a$; add $a + b$ to each, and $d + a \urcorner b + c$.

If b is the least, then is c the greatest, and $b + c \urcorner a + d$; which is plain from the Alternation of the Terms, *viz.* $b : a :: d : c$, whereby the Extremes are the least and greatest, as before.

Coroll. If three Numbers are $\div l$, the Sum of the Extremes exceeds double of the Means; thus, if $a : b :: b : c$, then $a + c \urcorner 2b$. And, particularly, it exceeds by the Product of the lesser Extreme into the Square of the Difference betwixt the Ratio and 1; for if we take $a : ar : ar^2$, then is $a + ar^2 = a \times \overline{r^2 + 1}$, and $2ar = a \times 2r$, also $a \times \overline{r^2 + 1} - a \times 2r = a \times \overline{r^2 + 1 - 2r} = a \times \overline{r - 1}^2$

Theorem XV.

If three Numbers, a, b, c, are $\div l$, and other three, d, e, f, also $\div l$, and in the same Ratio, *viz.* $a : b :: d : e$, then the Ratio of the Sum of the Extremes to double the Means, is the same in both Ranks.

$a,\ b,\ c$
4, 6, 9
$d,\ e,\ f$
8, 13, 18

$a + c : 2b :: d + f : 2e$
$4 + 9 : 2 \times 6 :: 8 + 18 : 2 \times 12$
13 : 12 :: 26 : 24

Demonstr. $a : c :: d : f$ (*Theor.* 19. *Chap.* 3.); hence $a + c : d + f :: a : d$: But since $a : b :: d : e$, therefore $a : d :: b : e$; consequently $a + c : d + f :: b : e :: 2b : 2e$.

Or, *alternately*, $a + c : 2b :: d + f : 2e$.

Theo-

THEOREM XVI.

IF three Numbers, $a, b, c,$ are $\div l$ (a the greateſt), and other three, $d, e, f,$ alſo $\div l$ (d the greateſt), but in a leſſer Ratio than the other; then the Ratio of the Sum of the Extremes to double the Mean in the 1ſt Rank is greater than that in the other; thus, $a+c : 2b \urcorner d+f : 2e$.

a, b, c
d, e, f
l, b, m

DEMONSTR. Take three Numbers, $l, b, m, \div l$, whoſe Mean is the ſame as that in the 1ſt Rank, and the Ratio equal to that in the 2d Rank, *viz* $d : e :: l : b$; then, becauſe $a : b \urcorner d : e$, therefore $a : b \urcorner l : b$, conſequently, $a \urcorner l$; and becauſe alſo $b : c \urcorner b : m$, therefore $c \angle m$; but $ac = bb = lm$, hence $a : l :: m : c$; and becauſe a is the greateſt, and c the leaſt of the four, therefore $a+c \urcorner l+m$ (*Theor.* 14.), and conſequently, $a+c : 2b \urcorner l+m : 2b$, which proves the *Theorem* in this Caſe. But, again (by laſt *Theor.*), ſince $d, e, f,$ and $l, b, m,$ are $\div l$ in the ſame Ratio, therefore $d+f : 2e :: l+m : 2b$; and it's now ſhewn, that $a+c : 2b \urcorner l+m : 2b$, therefore $a+c : 2b \urcorner d+f : 2e$.

THEOREM XVII.

OF four Numbers $: l$, the Ratio of that Couplet which conſiſts of the greater Numbers is greateſt, comparing the leſſer Term to the greater, and, contrarily, comparing the greater to the leſſer.

$a, b : c, d$
$5, 7 : 8, 10$

DEMONSTR. If a, b, c, d are $: l$, and c, d the two greater, alſo $d \urcorner c$, then $\frac{c}{d} \urcorner \frac{a}{b}$; for let $d = c+n$, and $b = a+n$, then $a. a+n : c. c+n$. Take their Ratios, *viz.* $\frac{a}{a+n}$ and $\frac{c}{c+n}$, and compare them by reducing them to one common Denominator, the new Numerators are $ac+an$, and $ac+nc$, which is greater than the former, becauſe ac is common to both, and c being $\urcorner a$, $nc \urcorner na$, conſequently, $\frac{c}{c+n} \left(=\frac{c}{d}\right)$ is $\urcorner \frac{a}{a+n} \left(=\frac{a}{b}\right)$.

COROLL. In a Progreſſion $\div l$, the Ratios of every Term to the next from the leſſer Extreme do continually increaſe; thus, if $a, b, c, d,$ &c. are $\div l$, and a the leſſer Extreme, then is $\frac{a}{b} \angle \frac{b}{c} \angle \frac{c}{d}$ &c. *Example:* Of this Progreſſion, 1, 2, 3, 4, 5, &c. the Ratios increaſe $\frac{1}{2} \angle \frac{2}{3} \angle \frac{3}{4} \angle \frac{4}{5}$, &c.

THEOREM XVIII.

OF four Numbers $: l$, the Product of the Extremes and Means are unequal, and that of the leaſt and greateſt Terms is the leaſt Product.

$a, b : c, d$
$3. 7 : 5. 9$

DEMONSTR. Suppoſe a the leaſt and d the greateſt, then either $c \urcorner$ or $\angle$ or $= b$. Suppoſe $c \urcorner$ or $= b$, then are $c, d,$ the two greater Terms, and, by the laſt, $\frac{c}{d} \urcorner \frac{a}{b}$, and, conſequently $cb \urcorner ad$. Again; Suppoſe $c \angle b$, then are b, d the two greateſt, and if we alternate the Terms thus, $a. c : b. d$, then, by the laſt, $\frac{b}{d} \urcorner \frac{a}{c}$, and hence $bc \urcorner ad$.

If the leaſt and greateſt are the two Means, then, by reverſing the Terms, they become the Extremes, and then it falls within the preceding *Demonſtration.* COROLL.

Coroll. If three Numbers are $\div l$, $a . b . c$, the Product of the Extremes, a, c, is less than the Square of the Mean bb, and that by the Square of the Difference; thus, Take $a . a+d$, $a+2d$, and $a \times \overline{a+2d} = a^2 + 2ad$; also, $\overline{a+d}^2 = a^2 + 2ad + d^2$.

Theorem XIX.

If there are three Numbers, $a, b, c, \div l$, and other three, d, e, f, also $\div l$, and with the same Difference, the Product of the Extremes, and Square of the Means have the same Difference in both Ranks; thus, $ac - bb = df - ee$.

a, b, c
d, e, f

Demonstr. Let the common Difference in the two Ranks be n, then (by *Theor.* 18.) $ac - bb = nn$; also $df - ee = nn$, therefore $ac - bb = df - ee$.

Theorem XX.

If three Numbers, a, b, c, are $\div l$, and other three, d, e, f, also $\div l$, but with a lesser Difference, the Difference of the Product of the Extremes, and the Square of the Mean in the 1st Rank, is greater than that in the 2d Rank.

a, b, c
d, e, f

Demonstr. Let $b - a = m$, and $d - e = n$, then (by *Theor.* 18.) $ac - bb = m^2$, and $df - ee = n^2$; but, by Supposition, $m \urcorner n$, hence $m^2 \urcorner n^2$, and consequently $ac - bb \urcorner df - ee$.

Theorem XXI.

The Mean Arithmetical betwixt two Numbers is a greater Number than the Mean Geometrical.

Demonstr. The Product of the Extremes is equal to the Square of the Mean Geometrical, but it's less than the Square of the Mean Arithmetical (by *Cor.* to *Theor.* 18.) consequently this Mean is greater than the other: Thus, betwixt A, B, let the Mean $\div l$ be C, and the Mean $\div\div l$ D; then is AB = DD, but AB $\angle$ CC, therefore DD $\angle$ CC, or CC $\urcorner$ DD. *Example*: Betwixt 2 and 8 the Mean $: l$ is 5, and the Mean $:: l$ is 4.

Scholium. If we express these Means according to their proper Rules, we find the Mean $: l$ exceeds the Mean $:: l$, by half of the Difference betwixt the Sum of the Extremes, and double of the Geometrical Mean; for these are $\div\div l$, $a : ar : arr$, and the Arithmetical Mean betwixt a and arr is $\frac{a + arr}{2}$: And, lastly, $\frac{a + arr}{2} - ar = \frac{a + arr - 2ar}{2}$.

Theorem XXII.

To the same three Numbers, a, b, c, take a fourth $: l$, n, and a 4th $:: l$, d, and order them so that the two first Terms, a, b, are either the two least or the two greatest of the four Terms, then is $n \urcorner$ or $\angle d$ in these different Circumstances, *viz.* (1.) If a, b are the two least, and $a \urcorner b$, then is $n \urcorner d$. (2.) If a, b are the two least, and $a \angle b$, then is $n \angle d$. (3.) If a, b are the two greater, and $a \urcorner b$, then is $n \angle d$. (4.) If a, b are the two greater, and $a \angle b$, then is $n \urcorner d$.

Demon-

$a\,.\,b : c\,.\,n$
$a : b :: c : d$

DEMONSTR. 1. If a, b are the two leſſer, and $a \urcorner b$, then is $\frac{a}{b} \urcorner \frac{c}{n}$ (*Theor.* 17.) But $\frac{a}{b} = \frac{c}{d}$, hence $\frac{c}{d} \urcorner \frac{c}{n}$, and conſequently $n \urcorner d$.

(2.) a, b, the two leſſer, and $a \angle b$, then is $\frac{a}{b} \angle \frac{c}{n}$ (*Theor.* 17.); but $\frac{a}{b} = \frac{c}{d}$, hence $\frac{c}{d} \angle \frac{c}{n}$, and $n \angle d$. (3.) a, b the two greateſt, and $a \urcorner b$, $\frac{a}{b} \angle \frac{c}{n}$ (*Theorem* 17.); but $\frac{a}{b} = \frac{c}{d}$, hence $\frac{c}{d} \angle \frac{c}{n}$, and $n \angle a$. (4.) a, b the two greater, and $a \angle b$, then is $\frac{a}{b} \urcorner \frac{c}{v}$; hence $\frac{c}{d} \urcorner \frac{c}{n}$, and $n \urcorner a$.

COROLL. To two Numbers a third $\div l$ increaſing, is leſs than a third $::l$; but, decreaſing it is greater.

THEOREM XXIII.

OF four Numbers $:l$, take their Squares: They cannot poſſibly be $:l$, but the Difference of the 1ſt and 2d Square will be to the Difference of the 3d and 4th in the Ratio of the Sums of the common Difference, and double the leſſer Term in the reſpective Couplets. Thus, if $a\,.\,b : c\,.\,d$, then $b^2 - a^2 : d^2 - c^2 :: 2\,a + \overline{b - a} : 2\,c + \overline{d - c}$.

DEMONSTR. Take four Terms, thus, $a\,.\,a + d : b\,.\,b + d$, their Squares are a^2, $a^2 + 2\,a\,d + d^2$, b^2, $b^2 + 2\,b\,d + d^2$, and the Differences are $2\,a\,d + d^2$, $2\,b\,d + d^2$, which, by equal Diviſion, are as $2\,a + d : 2\,b + d$.

COROLL. The Squares of a Series $:l$ cannot poſſibly be $:l$.

THEOREM XXIV.

OF four Numbers $:l$, the Difference betwixt the Sum of the Squares of the Extremes and the Sum of the Squares of the Means, is equal to the Product of twice the common Difference, multiplied into the Difference of the 1ſt and 3d Terms, the Terms being ſo ordered that the 1ſt and 3d are leſs than the 2d and 4th. Thus, take $a\,.\,b : c\,.\,d$, where $a \angle b$, and $c \angle d$, then $a^2 + d^2 - b^2 + c^2 = \overline{c - a} \times 2 \times \overline{b - a}$.

DEMONSTR. Take $a\,.\,a + d : b\,.\,b + d$, their Squares are a^2, $a^2 + 2\,a\,d + d^2$, b^2, $b^2 + 2\,bd + d^2$; the Sum of the Squares of the Extremes is $a^2 + b^2 + 2\,b\,d + d^2$, and of the Means it is $b^2 + a^2 + 2\,a\,d + d^2$; and the Difference of theſe Sums is plainly the Difference of $2\,bd$ and $2\,a\,d$ (the other Parts being equal), *i.e.* $2\,b\,a - 2\,ad = 2\,d \times \overline{b - a}$, or $2\,a\,d - 2\,bd = 2\,d \times \overline{a - b}$.

COROLL. If three Numbers, a, b, d, are $\div l$, or $2\,b = a + d$, then the Difference of the Sum of the Squares of the Extremes and double the Square of the Mean, is double the Square of the common Difference.

THEO-

THEOREM XXV.

IF three Numbers are $\div l$, the Ratio of the Extremes cannot poſſibly be the ſame as that of the Mean and common Difference.

DEMONSTR. Take $a\,.\,a+d\,.\,a+2d$; I ſay, it's impoſſible that it can be $a:a+2d::d:a+d$; for if it is, then $a\times\overline{a+d}=d\times\overline{a+2d}$, i. e. $a^2+ad=ad+2d^2$; whence, taking ad from both, it is $a^2=2d^2$, which is impoſſible ſince 2 is not a Square; for then a Square d^2, and a Number not a Square would produce a Square a^2, contrary to what is demonſtrated in *Ch.* 1. *Book* III. See *Coroll.* after *Theor.* 2.

SCHOLIUM. If the Extremes are in the Ratio of 1 : 2, the Difference and Mean will be as 1 : 3, as this *Example* ſhews; $a\,.\,a+d\,.\,a+2d=2a$, whence $a=2d$, and $a+d=3d$; and, reciprocally, if the Extremes are in the Ratio of 1 : 3, the Difference and Mean are in the Ratio of 1 : 2, as here, a, $2a$, $3a$. In *Theorem* 26. you'll find this Reciprocality demonſtrated univerſally, whatever the Ratios are.

THEOREM XXVI.

IF there are two Ranks of three Numbers each, $\div l$, and if the Ratio of the Extremes in the one Rank is equal to the Ratio of the Mean and Difference of the other Rank, then, reciprocally, the Ratio of the Extremes in the laſt Rank is equal to the Ratio of the Mean and Difference in the former Rank.

a .	$a+d$.	$a+2d$
6 .	8.	10
d	$\frac{a+2d}{2}$	$a+d$
2	5	8

DEMONSTR. In the annex'd *Example:* The Extremes of the 2d Series are the Difference and Mean of the 1ſt Series, and therefore in this Caſe it remains only to be ſhewn that the Ratio of the Extremes of the 1ſt Series is that of the Mean and Difference of the 2d; which is plain, for $a+2d:a::\frac{a+2d}{2}:\frac{a+2d}{2}-d=\frac{a+2d-2d}{2}=\frac{a}{2}$; but $\frac{a+2d}{2}:\frac{a}{2}::a+2d:a$, therefore.

Again, Whatever three Numbers we take that are $\div l$, and whoſe Extremes are as $d:a+d$; for *Example*, D, E, F, where we ſuppoſe $D:F::d:a+d$, then (by *Theor.* 31. below) $D:E::d:\frac{a+2d}{2}$, and $E-D:E::\frac{a+2d}{2}-d\left(=\frac{a}{2}\right):\frac{a+2d}{2}::a:a+2d$.

THEOREM XXVII.

IF three Numbers $\div l$ admit betwixt each of them a Geometrical Mean, the Square of the Mean $\div l$ is a Mean $\div l$ betwixt the Squares of the two Geometrical Means.

$A : B : C$

$\overline{AB}|^{\frac{1}{2}} \quad \overline{BC}|^{\frac{1}{2}}$

Demonstr. Let $A:B:C$ be $\div l$, and the two Geometrical Means as in the Margin, *viz.* $\overline{AB}|^{\frac{1}{2}}$ betwixt A and B, $\overline{BC}|^{\frac{1}{2}}$ betwixt B and C; then the Squares of these two, with the Square of the Mean $\div l$, B, are $\div l$; for these Squares are $AB:BB:BC$, which are the Products of the Series $A.B.C$ by B, and consequently they are $\div l$ (*Th.* 3 *Ch.* 2)

Theorem XXVIII.

It's possible to find three Numbers $\div l$, such, that the greater shall be equal to the Sum of the two lesser; but to find such three Numbers $\div l$ is impossible.

Demonstr. For the 1st Part, it's evident, and the Rule is this; Let the lesser Extreme be equal to the Difference, thus, $a.2a.3a$, where $3a=2a+a$. For the 2d Part, it's demonstrated thus; Let a, b, c be any three Numbers $\div l$; then a being the greater Extreme, and if $a=b+c$, then $b+c:b:c$, are $\div l$, whence $bc+cc=bb$; and if this is possible, then (by *Problem* 6. *Chap.* 2. *Book* III.) $c=\sqrt{bb+\frac{bb}{4}}-\frac{c}{2}$, or $\sqrt{\frac{5bb}{4}}-\frac{c}{2}$; but this is impossible, $\frac{5bb}{4}$ not being a Square Number; for tho' bb is a Square, yet 5 not being a Square, $5bb$ cannot be a Square (*Theor.* 2. *Cor.* 4. *Ch.* 1. *B.* III.) wherefore it's impossible that $b+c:b:c$ should be $\div l$.

Coroll. If three Numbers are $\div l$, it's impossible that the lesser Extreme should be equal to the Difference of the greater and Mean: For, in the preceding *Example* let $a-b=c$, then is $a=b+c$, which is inconsistent with $a:b:c$ being $\div l$.

Theorem XXIX.

If the Extremes of two Series $\div l$ having an equal Number of Terms, are $::l$ (or in the same Ratio), any two corresponding Terms in the one and other, are also $::l$.

$a \,.\, b \,.\, c \,.\, d \,.\, e$

$A \,.\, B \,.\, C \,.\, D \,.\, E$

Demonstr. If $a:e::A:E$, then $a:b::A:B$; for the two Differences are $b-a=\frac{e-a}{n-1}$, and $B-A=\frac{E-A}{n-1}$ (n being the Number of Terms, *Problem* 5. *Chap.* 2.); hence $b=a+\frac{e-a}{n-1}$, and $B=A+\frac{E-A}{n-1}$; but since $a:A::e:E$, therefore $a.A::e-a:E-A::\frac{e-a}{n-1}:\frac{E-A}{n-1}$, and hence again, $a:A::a+\frac{e-a}{n-1}:A+\frac{E-A}{n-1}$, or $b:B$; wherefore, lastly, $a:b::A:B$.

Again; Since $a:A::e:E$: and $a:A::b\,B$, therefore $b:B::e:E$, or $b:e::B:E$; and because we may now consider b, e and B, E as Extremes, therefore by the same Reasoning as before, it will follow that $b:c::B:C$, and so on, thro' the whole Series. And again; Taking any two corresponding Terms in each, at whatever Distance, they will be $::l$; because they are the Compound of the same simple Ratios; so $a:d::A:D$, also $b:d::B:D$, and so of others.

THEOREM XXX.

BETWIXT each Term of a Series $\div l$, take a Mean $: l$, and these Means are also $\div l$.

DEMONSTR. If $a . b . c . d . e.$ &c. are $\div l$, then the $-l$ Means are $\frac{a+b}{2} : \frac{b+c}{2} : \frac{c+d}{2}$, &c. But (by *Theor.* 18. *Chap.* 3.) $a+b : b+c : c+d$, &c. are $\div l$, therefore their Halfs are so (*Gen. Coroll.* 15. *Chap.* 1.).

THEOREM XXXI.

IF three Numbers are $\div l$, and if to any one of the Extremes be added that Number which is a 3d $\div l$ to the other Extreme and the Difference, the Sum is a 3d $\div l$ to the preceding two Terms.

Example: $4 . 6 . 8$ are $\div l$, and $4 . 6 . 9$ are $\div l$, $9 = 8 + 1$, and 1 the 3d $\div l$ to 4, 2, and 2 the common Difference in the Series $4 . 6 . 8$. Again; $8 . 6 . 4\frac{1}{2}$ are $\div l$, and $\frac{1}{2}$ the 3d $\div l$ to 8, 2.

DEMONSTR. $a . a+b, a+2b$ are $\div l$. Let a, b, c be $\div l$, then are $a : a+b : a+2b+c$, $\div l$: For $a \times \overline{a+2b+c} = a^2 + 2ab + ac, = a^2 + 2ab + b^2$ (because $ac = b^2$, since $a : b : c$ are $\div l$) $= \overline{a+b}^2$, therefore $a : a+b : a+2b+c$ are $\div l$, the Product of the Extremes being equal to the Square of the Mean.

Or, if we take these $-l$, a, $a-b$, $a-2b$, and these, $a : b : c$, $\div l$, then $a : a-b : a-2b+c$ are $\div l$, for $a \times \overline{a-2b+c} = a^2 - 2ab + ac = a^2 - 2ab + b^2 = \overline{a-b}^2$.

COROLL. If three Numbers are $\div l$, $a : b : c$, then any one of the Extremes, a, the Sum of that Extreme and the middle Term, $a+b$, and the Sum of both Extremes with double of the middle Term, $a+2b+c$, make a Series $\div l$.

PROBLEM I.

To find three Numbers $\div l$ such that the Quote (or Ratio) of the Ratio of the Extremes, and the Ratio of the Mean and common Difference, shall be greater than any assigned Ratio (taking the Ratios of the greater to the lesser).

RULE. Suppose the given Ratio is $d : a$, where $d > a$, then the three Numbers sought are $a : a+d : a+2d$; for the Ratio of the Extremes is $\frac{a+2d}{a}$, that of the Mean and Difference is $\frac{a+d}{d}$; then $\frac{a+2d}{a} \div \frac{a+d}{d} = \frac{da+2d^2}{a^2+da} = \frac{a+2d}{a+d} \times \frac{d}{a}$, which is greater than $\frac{d}{a}$.

PROBLEM II.

To find a given Number of Terms $\div l$, whose Extremes are in the Ratio of two given Numbers, and the common Difference is the Difference of the same two Numbers.

RULE.

RULE. Suppose the given Numbers are a, b, ($a \angle b$) and n the Number of Terms; these Products, *viz.* $\overline{n-1} \times a$, and $\overline{n-1} \times b$, are the Extremes of the Series sought; by which, with the given Difference, $b-a$, the Series may be filled up.

Example: The given Ratio 3 : 5, and Number of Terms 7, the Series is 18 . 20 . 22 . 24 . 26 . 28 . 30.

DEMONSTR. $a : b :: a \times \overline{n-1} : b \times \overline{n-1}$, and these, $a \times \overline{n-1}$, and $b \times \overline{n-1}$, being made the Extremes of a Series $\div l$, whose Number of Terms is n, the common Difference is $\frac{b \times \overline{n-1} - a \times \overline{n-1}}{n-1}$ (by *Probl.* 5. *Chap.* 2) $= b - a$, dividing the Numerator and Denominator equally by $n-1$.

PROBLEM III.

To find two Series $\div l$ in which the Extremes are in two given Ratios, and the common Difference equal in both.

RULE. Suppose the given Ratios are $a : b$ and $c : d$, find (by the last) two Series whose Extremes are (the one) in the Ratio of $a : b$, and (the other) of $c : d$, with any proposed Number of Terms in each; then multiply each of these Series by the common Difference of the other.

Example: The Ratios 2 : 3, and 4 : 7. I first find these two Series, 6 . 7 . 8 . 9, whose Extremes are 6 : 9 (:: 2 : 3) and 12 . 15 . 18 . 21, whose Extremes are 12 : 21 (:: 4 : 7); then multiplying the 1st Series by 3, and the other by 1, the Products are 18 . 21 . 24 . 27, and 12 . 15 . 18 . 21 the Series sought.

DEMONSTR. All that needs Demonstration here is, that the two Series first found being multiplied by one another's Differences, the Products are two Series having the same Differences; which is plain (from *Theor.* 3. *Chap* 2.); for let the Differences of the first two Series be d, c, then the Series whose Difference is d being multiplied by c, the Products are in the Difference $d \times c$; and the Series whose Difference is c being multiplied by d, the Products are in the Difference $c \times d$; but $d \times c = c \times d$, therefore the *Rule* is true.

PROBLEM IV.

To find a Series of Numbers $\div l$, whose Extremes are in a Ratio not less than a given one, and the Difference of the lesser Extreme and the Term next it not less than a given Number, the Number of Terms being also given.

RULE. Suppose the given Ratio $a : b$, and the given Number n: Then I find (by *Probl.* 2.) a Series $\div l$ of the Number of Terms proposed, whose Extremes are in the Ratio $a : b$, and whose Difference is $b - a$; if $b - a \angle n$, then I multiply the Series found by such a Number as shall make the Difference of the lesser Extreme and Term next it, at least $= n$. Take this Series, or the former Series if $b - a$ is $=$ or $\urcorner n$, and continue the two first Terms into a Series $\div l$ to the proposed Number of Terms; it is the Series sought.

A . B . C . D . E . F, &c. X.
A : B : L : M : N : O, &c. Y.

DEMONSTR. Let A . B . C . D, &c. be a Series $\div l$ of the given Number of Terms, whose Extremes, A : X, are in the given Ratio, $a : b$, and B — A ($= b - a$) not less than n: Then the 2d Series, A . B . L . M, &c. being $\div l$, of the same

ſame Number of Terms, and the 1ſt and 2d Terms being the ſame, two Conditions of the *Problem* are anſwered, *viz.* the Number of Terms, and Difference of the 1ſt and 2d Terms: What remains to be ſhewn then is only this, that the Ratio of the Extremes Y : A is not leſs than the given Ratio $b : a$, or X : A; which is thus proved: In the Series $\div l$, A : B : C, &c. the Ratios of every Term to the preceding leſſer Term do conſtantly decreaſe (*Theor.* 16.), and X : A is in the compound Ratio of all the intermediate Ratios; but in the Series $\div l$, A : B : L, &c. the Ratios being equal, and Y : A in the compound Ratio of the intermediate ones, it muſt be greater than X : A. Or thus; In the Geometrical Series the Differences from the leſſer Extreme do conſtantly increaſe (by *Theor.* 13.), but in the Series $\div l$ they are the ſame; therefore Y — A is greater than X — A, and $Y \urcorner X$, conſequenly $Y : A \urcorner X : A$.

Scholium. If the Problem is propoſed ſo that the Extremes of the Series to be found ſhall be in a Ratio not exceeding the given one, then proceed thus: Having found a Series $\div l$, whoſe Extremes are in the given Ratio $a : b$, and the Difference of the leſſer Extreme and that next it $= b - a$, as that repreſented in the Margin, A . B . C, &c. U . X, wherein X : A :: $b : a$, and the common Difference $= b - a$; then I contrive a Series $\div l$, downwards from X : U to as many Terms as the other Series, as X : U, &c. N : M : L; and becauſe the Differences decreaſe from the greater Extreme X, therefore M — L is leſs than X—U; for the ſame Reaſon, and the Equality of the Differences in the other Series, it's plain that L is greater than A, and conſequently X : L is leſs than X : A (or $b : a$, the given Ratio): But, laſtly, becauſe $M - L \angle b - a$, I multiply the whole Series by ſuch a Number as makes $nM - nL$ not leſs than $b - a$; and ſo theſe Products make the Series ſought.

A . B . C, &c. U . X
L . M . N, &c. U . X
nL . nM . nN, &c. nU . nX

CHAP.

CHAP. VI.

Of Harmonical Proportion; in which I use this Sign, hl, *for the Words* Harmonically Proportional.

§. 1. *Contains the* THEORY, *considered purely as an* Arithmetical Doctrine.

PROBLEM I.

HAVING three Numbers, to find a fourth *hl* with them, taken in a certain given Order.

Rule. Take the Product of the first and third, divide it by the Difference of the second and double the first, the Quote is the Number sought. So to these, A, B, C, a fourth *hl* is $\frac{AC}{2A-B}$

Exa. To these 3, 4, 6, a fourth *hl* is 9; thus $3\times6=18$; then $2\times3\ (=6\ )-4=2$. Lastly, $18\div2=9$; so that these 4 are *hl*, *viz.* 3, 4, 6, 9, for these are geometrically Proportional, *viz.* $3:9::4-3\ (=1):9-6\ (=3)$

Demon. Let four Numbers *hl* be represented by these Letters, A, B, C, D; then if A is greater than B, these are :: *l*, *viz.* $A:D::A-B:C-D$. Hence $AC-AD=AD-BD$, and adding A D to both these Equals, it is $AC=2AD-BD=\overline{2A-B}\times D$, and dividing equally by $2A-B$, it is $D=\frac{AC}{2A-B}$, which is the Rule.

Or if A is less than B, then $A:D::B-A:D-C$; hence $AD-AC=BD-AD$, and adding A C to both, it is $AD=BD-AD+AC$. Again, substracting $BD-AD$ from both, it is $AC=2AD-BD\ (=\overline{2A-B}\times D)$ and dividing by $2A-B$, it is $D=\frac{AC}{2A-B}$

COROLLARIES.

1st. We can by the same Method find a third *hl* to two given Numbers, if we take the second Term twice to make three given Numbers; and then the Rule will be plainly thus; divide the Product of the two given Numbers by the Difference betwixt the second and double the first, the Quote is the Number sought.

Exa. To these 3, 4, a third *hl* is 6; and to these 6, 4, it is 3; as you'll find by the Rule universally to these A, B, a third *hl* is $\frac{AB}{2A-B}$

SCHOLIUM (1°.) A third or 4th *hl* is always possible when 2A, (or double the first Term) is greater than the second B; but not otherways. The Reason is plain, because the Divisor $2A-B$ is then some real Number; but if 2 A is less or only equal to B, then there is no real Divisor, so that a third or fourth *hl* to the given Numbers is impossible.

Exa. To these 3, 6, or 3, 6, 7, there is no third or fourth *hl*, because double the first Term is equal to the second; nor to these 2, 5, or 2, 5, 6, because double the first Term is less than the second.

2°. Obſerve alſo theſe other Characters of two or three Numbers, to which a third or fourth *hl* is poſſible, *viz.* If to two Numbers, a third arithmetically proportional (which two laſt Words are marked thus $:l$) can be found either increaſing or decreaſing (as it can always be increaſing) then a third *hl* to the ſame two Numbers can be found contrarily decreaſing or increaſing; becauſe in this Caſe double the firſt Term of the Harmonicals is always greater than the ſecond. Thus if A, B, C, are $:l$, then $2\,B = A+C$; conſequently a third *hl* to B, C is poſſible. Again, if a third $:l$ to C, B, is poſſible, a fourth *hl* to B, C, and any other Number, as B, C, D, is alſo poſſible; for the ſame Reaſon, *viz.* becauſe 2 B is greater than C. *Laſtly*, If three Numbers are $::l$, and if to the middle Term with either extreme, a third *hl* is poſſible; to the ſame three Numbers a fourth *hl* is poſſible in this Order, *viz.* If A, B, C, are $::l$, and if to B, C, a third *hl* is poſſible, then alſo to A, B, C a fourth *hl* is poſſible; for ſince $A : B :: B : C$, hence $2\,A : B :: 2\,B : C$; but 2 B is greater than C, elſe a rhird *hl* to B C would not be poſſible; therefore 2 A is greater than B, which makes a fourth *hl* to A, B, C poſſible.

The Reverſe of all theſe are alſo true and neceſſary to be here remarked; thus, if to B, C, or B, C, D, a third or fourth *hl* is poſſible, then reverſely, to C, B, a third $:l$ as A is alſo poſſible; for by Suppoſition 2 B is greater than C, which is all the Condition neceſſary to make A poſſible, ſince $A = 2\,B - C$. Again, if A, B, C, are $::l$, and if to A, B, C, a fourth *hl* is poſſible, then a third *hl* to B C is alſo poſſible; for this requires only that 2 B be greater than C, which it will be when 2 A is greater than B, as it is in caſe a fourth *hl* to A, B, C, is poſſible.

3*d.* From any given Number a Progreſſion or Series *hl* may be found decreaſing in Infinitum, but not increaſing; for it will ſtop whenever the laſt found Term is equal to or exceeds the Double of the preceding Term; ſo this *hl* Series 12, 15, 20, 30, can be continued no further increaſing, becauſe 30 is leſs than $40, = 2 \times 20$; yet it's poſſible to find two Numbers from which any aſſigned Numbers of Terms *hl* may proceed increaſing, as you'll learn in *Theorem* ſecond, or to find a Series *hl* of any Number of Terms, all Integers; which cannot be done from any given Number, though we take a Series decreaſing, becauſe $2\,A - B$ will not be in every Caſe an aliquot Part of A B.

PROBLEM II.

To find a Mean hl *betwixt two Numbers.*

Rule. Divide double their Product by their Sum, the Quote is the Mean ſought; thus, betwixt A, B, a Mean *hl* is $\frac{2\,A\,B}{A+B}$

Exa. Given 3, 6, the Mean is 4; thus, $3 \times 6 = 18$, then $2 \times 18 = 36$ and $36 \div 9 = 4$. For theſe are *hl* 3, 4, 6, becauſe $3 : 6 :: 4-3\ (=1) :: 6-4\ (=2.)$

Demon. If A, B, C, are *hl*, decreaſing from A, and increaſing from C, then it is $A : C :: A-B : B-C$, and reverſly $C : A :: B-C :: A-B$, whence $A\,B - A\,C = A\,C - B\,C$; and adding B C to each, it is $A\,B - A\,C + B\,C = A\,C$; again, adding A C to each, it is $A\,B + B\,C\ (= \overline{A+C} \times B) = 2\,A\,C$; and, *Laſtly*, $B = \frac{2\,A\,C}{A+C}$; which is the Rule.

THEOREM I.

If four Numbers *hl* be equally multiplied, or divided, the Products or Quotes are alſo *hl*.

Exa.

Exa. 1*st*. If these, 3, 4, 6, 9, are multiplied by 2, the Products 6, 8, 12, 18, are *hl*, which again divided by 2 Quote the former.

Demon. The Products or Quotes of the Extreams are still in the same Ratio with the Extreams; and the Differences of the Products of the middle Term and Extreams, though they are greater or lesser than the Differences of these Terms themselves, yet are so proportionally, and that in the Ratio of the Multiplier or Divisor, which is the Ratio of the Products of the Extreams; therefore there is still a geometrical Proportion in the Products or Quotes, betwixt the Extreams and the Differences of these from the mean Terms, *i. e.* these Products or Quotes are *hl*.

Or, see this Truth also in universal Characters thus, 1°. Let these be *hl*, *viz.* A, B, C, D, that is A : D : : A—B : C—D; then are these *hl*, Ar, Br, Cr, Dr, That is Ar : Dr : : Ar—Br : Cr—Dr, for these are the plain Effects of multiplying *r* into the preceding : : *ls*, *viz.* A : D : : A—B : C—D, therefore the Products are : : *l*, *viz.* Ar : Dr : : Ar—Br : Cr—Dr, *That is*, Ar, Br, Cr, Dr, are *hl*, according to the Definition. As for Division, it is but the Reverse of the first, and so its Demonstration is contained in it.

COROLLARIES.

1*st*. Any three Terms *hl* being equally multiplied or divided, the Products or Quotes will be *hl*; because by taking the middle Term twice, they make four Terms *hl*, for 2, 3, 6, is the same as 2, 3, 3, 6, as to *hl*.

2*d*. As any three Numbers *hl*, so by equal Reason any Series continually *hl* being equally multiplied or divided, the Products or Quotes are still *hl*.

3*d*. By this *Theorem* and *Problem* first, we learn how to find a Series *hl*, consisting of any Number of Terms, all Integers; thus, begin with any two Integers, and find a third *hl* decreasing; if it's a Fraction, or mix'd Number reduced to a Fraction, then multiply the two given Numbers by the Denominator of this Fraction, the Products with the Numerator are three Terms *hl*. To these join another Term *hl* decreasing; and if it's Fractional, multiply all the preceding by its Denominator, and these Products with the Numerator are four Terms *hl* continually; after this manner go on to any Number of Terms. But you'll learn an easier Way of solving this Problem afterwards.

4*th*. We learn here also, how to find a Series $\div l$ or $\because l$ in Integers, betwixt every two adjacent Terms of which, there falls a *hl* Mean also Integral. *Thus*, take any Series $\div l$, or $\because l$ all Integers; then find the *hl* Means, which being all or part of them mix'd Numbers, reduce the whole Series first found, together with the Means, to a common Denominator, and the Numerators give the Numbers sought, and also the Means.

THEOREM II.

If there is a Series of Numbers, $\div l$, as, *a*, *b*, *c*, *d*, &c. increasing or decreasing, their Reciprocals, *viz.* $\frac{1}{a}$ $\frac{1}{b}$ $\frac{1}{c}$ $\frac{1}{d}$, &c. is a *harm.* Progression, contrarily decreasing or increasing; and reversely, if *a*, *b*, *c*, *d*, &c. are *hl.* $\frac{1}{a}$, $\frac{1}{b}$, *&c.* are $\div l$.

Demon. (1°) If *a*, *b*, *c*, *d*, &c. is a Series increasing or decreasing, their Reciprocals $\frac{1}{a}$, $\frac{1}{b}$, *&c.* must contrarily decrease or increase. This follows evidently from the Nature of Fractions (V. *Lemma* 5th, Chap. 1st, B. 2d.)

2°. If the Thing asserted in the *Theorem* be true of any three Numbers in Progression $\div l$ or *hl*, it must necessarily be so, how many Terms soever be in the Progression; because for a Series to be continuedly : *l*, or *hl*, is no other Thing than to have every three Terms in the continued Order of the Series : *l* or *hl*.

 3°. What

3°. What remains to be proved then is only this, *viz.* That if any three Terms, a, b, c, are $\div l$ or hl, their Reciprocals must be contrarily hl: or $\div l$; which I shew thus, (1°) Suppose a, b, c, are $\div l$, then $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are hl, that is, $\frac{1}{a} : \frac{1}{c} :: \frac{1}{a} - \frac{1}{b} \left(=\frac{b-a}{ab}\right) : \frac{1}{b} - \frac{1}{c} \left(=\frac{c-b}{bc}\right)$; for since a, b, c, are $\div l$, then $b-a=c-b$, and hence $\frac{c-b}{abc}=\frac{b-a}{abc}$. *That is*, $\frac{1}{a} \times \frac{c-b}{bc} = \frac{1}{c} \times \frac{b-a}{ab}$, whence it is, $\frac{1}{a} : \frac{1}{c} :: \frac{b-a}{ab} : \frac{c-b}{bc}$. 2°. Suppose a, b, c, are hl, then, $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are $\div l$, for since a, b, c, are hl, i. e. $a : c :: b-a : c-b$. Therefore $\frac{b-a}{a} = \frac{c-b}{c}$, and dividing equally by b it is $\frac{b-a}{ab} = \frac{c-b}{cb}$. But $\frac{b-a}{ab} = \frac{1}{a} - \frac{1}{b}$, and $\frac{c-b}{cb} = \frac{1}{b} - \frac{1}{c}$ hence $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are $\div l$.

Exam. 1. 2, 3, 4, are $\div l$, and $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, hl; for $\frac{1}{2} : \frac{1}{4} :: \frac{1}{2} - \frac{1}{3} \left(=\frac{1}{6}\right) : \frac{1}{3} - \frac{1}{4} \left(=\frac{1}{12}\right)$
Exam. 2. 3, 4, 6, are hl, and $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$, are $\div l$, for $\frac{1}{3} - \frac{1}{4} \left(=\frac{1}{12}\right) = \frac{1}{4} - \frac{1}{6} \left(=\frac{1}{12}.\right)$

SCHOLIUM.

1*st*. This Truth is universal, whether the supposed Numbers are Integers or Fractions; because $\frac{1}{a}$ expresses the Reciprocal of a, whatever a be, from the Nature of Division. If a is Integer the Thing is plain. If $a = \frac{n}{m}$, then is $1 \div a = 1 \div \frac{n}{m} = \frac{m}{n}$ the Reciprocal of $\frac{n}{m}$ $\left(=a.\right)$

2*d*. If any Series consists of Fractions (either all, or only some of the Terms) then if the whole Series is reduced to a common Denominator, the new Numerator is a Series of Integers of the same Kind of Progression ($\div l$, hl or $\div\div l$) with that reduced; because the new Fractions are so, since they are equal to the former ones, and the Numerators are Equimultiples of the Fractions (for they are their Multiples by the common Denominator, since $\frac{a}{b} \times b = a$;) and, *Lastly*, the Equimultiples of any Series, $\div l$, $\div\div l$ or hl, are of the same Kind; as has been shewn of each in their Places.

3*d*. If any Series consists all of Integers, the most convenient way of reducing the Series of their Reciprocals (which are all Fractions) to a common Denominator, is this,

Arithmeticals,	2,	3,	4,	5
	3,	2		
		4,	3	
Harmonicals,	12,	8,	6	
			5,	4
Harmonicals,	60,	40,	30,	24

Of the given Series of Integers take every Couplet reversely in Order from the beginning, and continue them as so many Geometrical Ratio's (by *Probl.* I. *Chap.* 4.) as in the annex'd Example; the Operation of which you'll easily perceive to be the same as that which finds the new Numerators where the Reciprocals of these Numbers are reduced to a common Denominator; thus, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, reduced, the new Numerators are $3 \times 4 = 12$; $2 \times 4 = 8$, $2 \times 3 = 6$. And $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, reduced, the Numerators are $3 \times 4 \times 5 = 60$; $2 \times 4 \times 5 = 40$; $2 \times 3 \times 5 = 30$; $2 \times 3 \times 4 = 24$, and so on; which is yet more evident in this Example in Letters. But yet again,

4°. With-

a,	b,	c,	d
b,	a		
	c,	b	
bc,	ac,	ab,	
		d,	c
bcd,	acd,	abd,	abc

4°. Without regard to the Rule for reducing Fractions to one Denominator, we can easily demonstrate that these Products have the Quality asserted in the *Theorem*, *viz.* that if a, b, c, d, &c. are $\div l$ or hl, the Products so made are contrarily hl or $\div l$; *Thus*, these Products are Series in the reciprocal Ratio's of the given Series (as has been demonstrated in the *Problem* referr'd to) but the Reciprocals of the given Series are Numbers also in the reciprocal Ratio's of the same Series, therefore both these Series (*viz.* the Series of Products and of Reciprocals) have the same Quality; which is thus proved. Let a, b, c represent the Reciprocals of any three Numbers, and A, B, C, other three Numbers, in the same Ratio's, *viz.* $a : b :: A : B$ and $b : c :: B : C$, then comparing these, it is $a : A :: c : C$ or $a : c :: A : C$. Also, $a : A :: b—a : B—A$, and $c : C :: c—b : C—B$. Again, comparing the last three Proportions, it is $b—a : B—A :: c—b : C—B$; or $b—a : c—b :: B—A : C—B$. These Preparations being made, suppose now that a, b, c, are hl, *i. e.* $a : c :: b—a : c—b$, then are A, B, C, hl; or, $A : C :: B—A : C—B$; for because $a : c :: A : C$, therefore $A : C :: b—a : c—b$, and because, $b—a : c—b :: B—A : C—B$, hence A, B, C, are hl: Again, suppose a, b, c, are $\div l$, or $b—a = c—b$, so are A, B, C, or, $B—A = C—B$. For it is above shewn that $b—a : c—b :: B—A : C—B$, and therefore if $b—a = c—b$ so must $B—A = C—B$. In the last Place, since the Thing is true of three Terms, it's true of any Number in a Progression.

COROLLARIES.

1*st*. Here then we learn a very easy Way of finding a Series hl, consisting of any Number of Terms, all Integers; *thus*, take any Series $\div l$ of the same Number of Terms, all integral, and multiply their Terms together, according to the Direction in the second *Scholium*, and you have a Series hl of as many Terms, all Integers.

2*d*. If it were required to assign two Numbers, to which any Number of Terms hl are possible, increasing, without finding the whole Series; it's done by taking any Series $\div l$ consisting of the given Number of Terms, and multiplying continually into one another all the Terms of the Series $\div l$, except the greater Extreme; for their Product is the lesser of the Numbers sought, and the continual Product of all the Terms, except that next the greatest Extreme, is the greater of the two Numbers sought.

But if it's also required that both the Terms sought, and also all that are proposed as possible to be added to them, be Integers; then let the assumed Series $\div l$ be all Integers.

3*d*. Further, though a Series hl may be continued infinitely decreasing from any given Number, yet the Terms will not all be Integers; therefore if it's required to find two Numbers, to which a given Number of Terms hl may be found decreasing, and all Integers, its done *thus*; take a Series $\div l$, consisting of the given Number of Terms, all Integers, and the continual Product of all the Terms, except the lesser Extreme, is the greater of the Numbers sought, and the continual Product of all, except the Term next the lesser Extreme, is the lesser of the Terms sought. All this is plain from the preceding Operation, wherein every Term of the Harmonicals is the continual Product of all the Terms of the Arithmeticals, except the Correspondent in order, *i. e.* the 1*st*, 2*d*, or 3*d*, &c. if it is the 1*st*, 2*d*, or 3*d*, of the Harmonicals.

4*th*. If any Numbers, a, b, c, &c. which are $\div l$ or hl, are applied as Divisors to the same Number, N, the Quotes are contrarily hl or $\div l$; because Quotes are in the reciprocal Ratio's of the Divisors when the Dividend is common.

5th. Hence we learn the Solution of this Problem, *viz.* to find a Progreſſion of any Number of Terms *hl*, and whoſe Extremes ſhall be in any aſſigned Ratio; which is ſolved *thus:* Find a Series ÷*l* of the ſame Number of Terms, and whoſe Extremes are in the given Ratio (by *Probl.* II. Ch. 5.) then the Reciprocals of theſe or their Numerators, being all reduced to a common Denominator, are the Numbers here ſought.

6th. We have alſo the Solution of this Problem, *viz.* to find the firſt of any Number of Harmonical Means betwixt two given Numbers, *thus*, Take the Reciprocals of the given Numbers, and find the firſt of the propoſed Number of —*l* Means betwixt them (as directed in Article 2*d* of the *Schol.* after *Coroll.* 8. *Probl.* 4. Ch. 5.) the Reciprocal of this Mean is the *hl* Mean ſought.

7th. We have alſo the Solution of this Problem, *viz.* to find two Integers, betwixt which a propoſed Number of *hl* Means, all Integers, may be found; *Thus*, take any Series ÷*l* betwixt whoſe Extremes there is the propoſed Number of Means; the continual Product of all the Antecedents, and the continual Product of all the Conſequents are two Numbers ſuch as required; this is manifeſt from *Scholium* 3*d*.

Theorem III.

If four Numbers are : : *l*, A : B : : C : D; and if betwixt each Couplet you take a *hl* : Mean, as *x* betwixt A : B, and *y* betwixt C : D; then the two Antecedents or two Conſequents with theſe two *hl* Means, are alſo : : *l*; thus A : *x* : : C : *y*, and B : *x* : : D : *y*.

Demon. Let it be A : B : : Ar : Br. (inſtead of C, D) then A, *x*, B, being multiplied by *r*, the Products Ar, xr, Br, are *hl* (*Theo.* I.) alſo A : *x* : : Ar : xr. But Ar=C and Br=D, therefore xr=y. conſeq. A : *x* : : C : *y*. But again, A : C : : B : D, therefore, *Laſtly*, B : *x* : : D : *y*.

Coroll. If betwixt every two adjacent Terms of a Geometrical Series a *hl* Mean is taken, theſe Means make alſo a Geometrical Series in the ſame Ratio.

Geom. *a*, *b*, *c*, *d* | For *a* : *l* : : *b* : *m*; or *a* : *b* : : *l* : *m*; alſo, *b* : *c* : : *m* : *n*,
Hl. Means *l*. *m*, *n*. | therefore *l* : *m* : : *m* : *n*; and ſo on.

Theorem IV.

In an Harmonical Progreſſion, any three Terms, whereof the Middle is equally diſtant from the Extremes, are alſo *hl*.

Exam. In this Series 10, 12, 15, 20, 30, 60, theſe three are *hl*, 10, 15, 30; alſo theſe 12, 20, 60.

Demon. Since by *Schol.* 3d. to *Theor.* II. an Arithmetical Series can be found, each immediate Couplet whereof is in the reciprocal Ratio's of the Correſpondent Harmonicals, and of the Series : *l*, any three Terms, whereof the Middle is equally diſtant from the Extremes, are : *l*; therefore the Correſpondent to theſe in the Series *hl* muſt alſo be *hl*.

Theorem V.

If there are four Numbers ſo ſtated, that the two middle Terms are : *l* with the one Extreme, and *hl* with the other, theſe four Numbers are : : *l*.

Exam. 2 : 5 : : 8 : 20, are : : *l*; 2, 5, 8, are : *l*, and 5, 8, 20, *hl*. *Univerſally*, if *a*, *b*, *c*, are : *l*, and *b*, *c*, *d*, are *hl*, then, *a*, *b*, *c*, *d*, are : : *l*.

$a,\ b,\ c,\ \frac{bc}{2b-c}=\frac{bc}{a}$ | *Demon.* Suppoſe *a*, *b*, *c*, are : *l*, then to *b*, *c*, a third *hl* is $\frac{bc}{2b-c}$ (by *Probl.* 1ſt, *Cor.* 1ſt.) but to *a*, *b*, *c*, a 4th : : *l* is $\frac{bc}{a}$, and this is equal to the other; for *a*, *b*, *c*, being : *l*, therefore $a=2b-c$, hence

$\frac{bc}{2b-c}=\frac{bc}{a}$, i. e. a, b, c, being $: l$, the ſame Number d which is a 3d hl, to b, c, is a 4th $:: l$, to a, b, c.

$2b\text{-}c : b :: c : \frac{bc}{2b-c}$

Or alſo thus, ſuppoſe any two Numbers, b, c, a 3d $: l$ to c, b, is $2b-c$, and a 3d hl to b, c, is $\frac{bc}{2b-c}$, and theſe 4 are manifeſtly $:: l$; for by the common Rules a 4th $:: l$, to $\overline{2b-c}, b, c$, is $\frac{bc}{2b-c}$.

The *Reverſe* of this Theorem is alſo true, *viz.* that if four Numbers are $:: l$, and if the two Means with one of the Extremes are $: l$, or hl, they are contrarily hl or $: l$ with the other Extreme. The Demonſtration of which is contained in the former; for whether we ſuppoſe three Terms $: l$, $\overline{2b-c}$, b, c, or three Terms hl, b, c, $\frac{bc}{2b-c}$ the 4th $:: l$ will be $\frac{bc}{2b-c}$, or $2b-c$, which is contrarily hl: or $: l$ with the other Extreme.

SCHOLIUMS.

1ſt. As either the Theorem, or its Reverſe, are demonſtrated independently of one another, ſo the one being ſuppoſed true, the other may be demonſtrated by its Means.

1°. Suppoſe the Theorem true, the Reverſe is demonſtrated thus, if a, b, c, d, are $:: l$, and $a, b, c, : l$, then ſuppoſe a third hl to b, c, is n; by the Theorem $a : b :: c : n$; but $a : b :: c : d$, hence $n=d$, and b, c, d, are hl. The Demonſtration will proceed the ſame way by firſt ſuppoſing $a, b, c.$ to be hl.

2°. Suppoſe the Reverſe is true, the Theorem is demonſtrated thus; let a, b, c, be $: l$, and b, c, d, hl; then to a, b, c, let a 4th $:: l$ be n, by Suppoſition b, c, n, are hl, (becauſe a, b, c, are $: l$, and a, b, c, n, are $:: l$.) But ſo alſo are b, c, d, conſequently, $n=d$, and $a : b :: c : d$. If a, b, c, are ſuppoſed hl, the Demonſtration proceeds the ſame way.

2d. We may find Examples of this Theorem in Integers, by taking any three Integers, which are $: l$, and to them find a 4th $:: l$. If this is an Integer, you have what's ſought; but if it's fractional, multiply all by the Denominator. Thus if a, b, c, are three Numbers $: l$, the 4th $:: l$ is $\frac{bc}{a}$, and if this is not an Integer, then multiply by the Denominator a, and the four Numbers ſought are $aa : ab : ac : bc$.

THEOREM VI.

If there are four Numbers ſo ſtated that the Extremes with one Mean are $: l$, and with the other hl, (*i. e.* if betwixt any two Numbers you put an Arithm. and alſo an Harm. Mean) the four will be $:: l$.

Exam. $6 : 8 :: 9 : 12$ are $:: l$, $6, 9, 12$, are $: l$, and $6, 8, 12$, hl. *Univerſally*, if a, c, d, are $: l$, a, b, d, hl : then are $a, b, c, d, :: l$.

$a : \frac{a+b}{2} :: \frac{2ab}{a+b} : b$

Demon. Betwixt a and b, a Mean $: l$ is $\frac{a+b}{2}$, and a Mean hl is $\frac{2ab}{a+b}$, but the Products of the Extremes and Means are evidently equal, therefore the four are $:: l$.

The Reverſe of this Theorem is alſo true, *viz.* if four Numbers are $:: l$, and if the 2 Extremes with the one Mean are $: l$ or hl, they are contrarily hl or $: l$ with the other Mean;

Mean; the Demonstration of which is contained in the former, for a, $\frac{a+b}{2}$, b represent any three Numbers : l, and a, $\frac{2ab}{a+b}$, b any three hl; and if a, b, are the Extremes of four Numbers : : l, and either of these other Expressions one of the Means, it's shown that the other of them will be the other Mean, which is an : l or hl Mean betwixt the Extremes a, b, if the former is contrarily hl, or : l.

Scholium.

1. Suppose either the Theorem or its Reverse true, the other may be demonstrated by means of it, after the same manner as was done in the last Theorem.

2. We may find as many Examples of this Theorem as we please in Integers, after this manner; take any two Integers, and betwixt them find a mean hl, which being a mixt Number, reduce all the three by its Denominator to other three Numbers, which shall be all Integers, and still hl (by *Theor.* 1st) then if the half Sum of the Extremes is an Integer, it is the Arithm. Mean, but if it's not Integral, double the three Terms already found, and then the half Sum of the Extreams will be an Integer.

Exam. If I take 3 and 5, a mean hl is $\frac{30}{8}$, therefore multiplying all by 8, I have 24, 30, 40, hl; and a mean : l betwixt 24 and 40 is 32; therefore 24, 30, 32, 40, is an Example of what's required. But if I take 2, 3, their harm. Mean is $\frac{12}{5}$, and I reduce them to this Series 10, 12, 15; but here I cannot have a mean : l in Integers, therefore I double these Numbers, making 20, 24, 30, and the mean : l is 25; and so 20, 24, 25, 30, is an Example of what's required.

Hence again, If it were proposed to find out two Numbers, the half of whose Sum is an Integer, and also whose Sum is an aliquot Part of double the Product, it's plain from the preceding Demonstration, that if we find any Example of this Theorem, the Extremes are Numbers such as are here required; for the Extremes being a, b, the two Means are $\frac{a+b}{2}$ and $\frac{2ab}{a+b}$.

3. This Theorem and the preceding may coincide, *viz.* there may be four Numbers : : l, whereof the two Means may be : l with the one Extreme, and hl with the other; and also the two Extremes may be : l with the one middle Term, and hl with the other, as in this Example 2 : 3 : : 4 : 6; but this Coincidence does not always happen; for either of the Parts may be found by it self without the other. So these 6, 8, 9, 12, is an Example of this Theorem, but not of the other; as these 3, 6, 9, 18, or these 3, 4, 6, 8, belong to the former Theorem, neither of which belongs to this.

Coroll. If betwixt two Numbers, A, E, are put three Means, an Arithmetical (B), Geometrical (C), and Harmonical Mean (D), these Means are in Geometrical Progression; the Geometrical Mean being the middle of the three; for $A\times E=B\times D$, also $A\times E=C\times C$, therefore $B\times D=C\times C$, or $B : C : : C : D$.

Theorem VII.

If four Numbers are so stated that the two Means with the one Extreme are hl, and with the other : : l, these four are hl.

Exam. 3, 4, 6, are hl; 4, 6, 9, are : : l, and 3, 4, 6, 9, are hl. *Universally*, if a, b, c, are : : l, and b, c, d, hl, then a, b, c, d, are hl.

$$a,\ b,\ c,\ \frac{bc}{2b-c}=\frac{ac}{2a-b}$$

Demon. Since a, b, c, be : : l, and to b, c, a third hl is by Supposition possible; this by *Problem* first is bc

$\frac{bc}{2b-c}$; alſo to a, b, c a fourth *hl* is poſſible (*Sch. Coroll.* I. *Prob.* I.) and it is $\frac{ac}{2a-b}$; which I demonſtrate to be equal to $\frac{bc}{2b-c}$; *thus*; $a : b :: b : c$ by Suppoſition; hence $2a : 2b :: b : c$, and $2a-b : 2b-c :: b : c$. But $b : c :: bb : bc$; hence $2a-b : 2b-c :: bb : bc$; and again, $bc : 2b-c :: bb : 2a-b$, therefore $\frac{bc}{2b-c} = \frac{bb}{2a-b}$; but $bb = ac$, therefore $\frac{bc}{2b-c} = \frac{ac}{2a-b}$.

Or alſo thus: To any two Numbers, as b, c, a third *hl* is $\frac{bc}{2b-c}$, and a third $::l$ to c, b, is $\frac{bb}{c}$, and theſe four are *hl*, *viz.* $\frac{bb}{c}$, b, c, $\frac{bc}{2b-c}$; a fourth *hl* to $\frac{bb}{c}, b, c$, being $\frac{bc}{2b-c}$ by *Probl.* I. for $\frac{bb}{c} \times c = bb$, and $\frac{2bb}{c} - b = \frac{2bb-bc}{c}$, and $bb \div \frac{2bb-bc}{c} = \frac{bbc}{2bb-bc} = \frac{bc}{2b-c}$.

The Reverſe of this Theorem is alſo true, *viz.* if four Numbers are *hl*, and the two Means with one Extreme $:l$, or *hl*, it will be contrarily *hl* or $:l$ with the other Extreme.

The Demonſtration is contained in the former: For whether we ſuppoſe three Numbers $::l$, which may be expreſſed $\frac{bb}{c} : b : c$; or three Numbers *hl* which may be expreſſed $b, c\ \frac{bc}{2b-c}$; the fourth *hl* will be $\frac{bc}{2b-c}$ or $\frac{bb}{c}$, contrarily *hl* or $:l$, with the other Extreme.

Schol. Either the Theorem or the Reverſe being ſuppoſed true, the other may be demonſtrated by means of it in the Manner ſhown in *Theorem* V.

PROBLEM III.

To find a fourth Contra hl *to three given Numbers.*

Caſe 1ſt. If the firſt Term is leſs than the ſecond, the Rule is this; from the Product of the firſt and ſecond Terms ſubſtract the Square of the firſt Term; and to the Difference add the fourth Part of the Square of the third Term; out of this Sum extract the ſquare Root, to which add the half of the third Term; this laſt Sum is the fourth Term ſought.

Exam. To theſe d, c, b (d being leſs than c) ſuppoſe a fourth Contra *hl* is a, then is $a = \frac{b}{2} + \overline{dc - dd + \frac{bb}{4}}\big|^{\frac{1}{2}}$.

Exam. in Numbers. To theſe three Numbers, 2, 8, 4, a 4th Contra *hl* is 6; for $2 : 6 :: 6-4 : 8-2$; which I find thus; $2\times8=16$, and $2\times2=4$, then $16-4=12$; again, $4\times4=16$, whoſe fourth Part is 4, which added to 12, the Sum is 16, whoſe ſquare Root is 4, to which add 2 (the half of 4) the Sum is 6.

Caſe 2d. If the firſt Term is greater than the ſecond Term, the Rule is this: From the Square of the firſt Term ſubſtract the Product of the firſt and ſecond; and this Difference ſubſtract from the fourth Part of the Square of the third Term; then extract the

Square Root of this last Difference; which add to or substract from the Half of the third Term; the Sum or the Difference (one of the two) will be the fourth sought: And you must chuse that which makes the Proportion with the given Numbers.

Exam. To these a, b, c, (a being greater than b) a fourth Contra hl; is $\frac{c}{2}+\overline{\frac{cc}{4}-aa-ab}\big|^{\frac{1}{2}}$; or $\frac{c}{2}-\overline{\frac{cc}{4}-aa-ab}\big|^{\frac{1}{2}}$.

Example in Numbers. To these 6, 4, 8, a fourth Contra hl is 2, thus found; $6\times4=24$, and $6\times6=36$, then $36-24=12$; again, $\frac{8\times8}{4}=\frac{64}{4}=16$, and $16-12=4$, whose Square Root is 2, then $\frac{8}{2}=4$; Lastly, $4+2=6$, and $4-2=2$, which last is the Number sought.

Demon. Let these be Contra hl, a, b, c, d; and suppose a greater than b, and c than d; then it is $a : d :: c-d : a-b$; and multiplying the Extremes and Means, it is $aa-ab=dc-dd$. But it has been demonstrated (*Probl.* VI. *Book* III.) that if $aa-ab=p$, then is $a=\frac{b}{2}+\overline{p+\frac{bb}{4}}\big|^{\frac{1}{2}}$, which is the Rule of *Case* 1st, supposing $dc-dd=p$. Again, it is also demonstrated, that if $dc-dd=p$, then is $d=\frac{c}{2}\pm\overline{\frac{cc}{4}-p}\big|^{\frac{1}{2}}$, which is the Rule of *Case* 2d, supposing $aa-ab=p$. It is demonstrated also that if $\frac{cc}{4}$ be greater than p, so will $\frac{c}{2}$ be greater than $\overline{\frac{cc}{4}-p}\big|^{\frac{1}{2}}$. But if p is not less than $\frac{cc}{4}$, the *Problem* is impossible. Also in both Cases, if the Square Root to be extracted is surd, there is no fourth Contra hl in rational Numbers.

Coroll. If to two given Numbers a third Contra hl is required, the preceding Rules are applicable, by supposing the second and third Terms to be the same. Thus; to these a, b, a third Contra hl, when a is less then b, is $\frac{b}{2}+\overline{ab-a^2+\frac{b}{4}}\big|^{\frac{1}{2}}$; and if a is greater than b, then it is $\frac{b}{2}\pm\overline{\frac{bb}{4}-aa-ab}\big|^{\frac{1}{2}}$, as you will find in this Example; 3, 5, 6, where $3 : 6 :: 6-5 : 5-3$, or $3 : 6 :: 1 : 2$.

PROBLEM IV.

To find a Contra hl *Mean betwixt two given Numbers.*

Rule. Divide the Sum of their Squares by their Sum; the Quote is the Mean sought.

Exam. Betwixt a, b, a Contra hl Mean is $\frac{aa+bb}{a+b}$.

Example in Numbers. Betwixt 3, 6, a Contra hl Mean is $5=\frac{9+36}{9}=\frac{45}{9}$.

Demon.

Demon. If these three are Contra hl, a, b, c, that is, if $a : c :: b-c : a-b$, then multiplying the Extremes and Means, it is $a^2-ab=bc-c^2$; and adding c^2 to both Sides, it is $bc=a^2-ab+c^2$; and again, adding ab to both Sides, it is $ab+bc=a^2+c^2$; and dividing both Sides by $a+c$, it is, $b=\frac{a^2+c^2}{a+c}$.

THEOREM VIII.

If four Numbers are so stated that the Extremes with one Mean are hl, and with the other Contra hl, *i. e.* if betwixt two Numbers you put a Mean hl, and another Contra hl, these four Numbers are $: l$.

Exam. 3, 4, 6, are hl, and 3, 5, 6, Contra hl, and 3, 4, 5, 6, $: l$. *Universally*; if a, b, c, d are hl, and a, c, d Contrà hl, then are a, b, c, d, $: l$.

$$a . \frac{2ab}{a+b} : \frac{a^2+b^2}{a+b} . b$$

Demon. Betwixt a, b, a Mean hl is $\frac{2ab}{a+b}$ (*Probl.* II.) and a Mean Contra hl is $\frac{a^2+b^2}{a+b}$ (*Probl.* IV.) and these four are $: l$; *viz.* a; $\frac{2ab}{a+b}$; $\frac{a^2+b^2}{a+b}$; b: for $a-\frac{2ab}{a+b}=\frac{a^2+b^2}{a+b}-b=\frac{a^2-ab}{a+b}$, by the common Rules: Or thus, $a+b$ (the Sum of the Extremes) is $=a^2+b^2+2ab$, ($=a+b$ squar'd) divided by $a+b$, the common Denominator; which Quote is the Sum of the middle Terms.

The *Reverse* of this *Theorem* is also true, *viz.* That if four Numbers are $: l$, and the Extremes with one Mean are hl, or Contra hl; with the other Mean they will be contrarily Contra hl, or hl. The Demonstration is contained in the former: Thus; any three Numbers hl may be represented $a : \frac{2ab}{a+b} : b$, and any Contra hl, $a : \frac{a^2+b^2}{a+b} : b$; and if a, b are the Extremes of four Numbers $: l$, where one of the Means is $\frac{2ab}{a+b}$, or $\frac{a^2+b^2}{a+b}$, an hl, or Contra hl Mean betwixt the Extremes; it's demonstrated that the other of the two Means will be the other of these two Expressions, *viz.* contrarily an hl or $: l$ Mean betwixt the same Extremes: Or thus, let a, b, c, d, be $: l$, and a, b, d, be hl, i. e. $a : d :: a-b : b-d$; then because $a-b=c-d$, also $a-c=b-d$, therefore $a : d :: c-d : a-c$, i. e. a, c, d, are Contra hl; or if a, b, c, are $\div l$, the Demonstration will proceed the same Way.

COROLLARIES.

1. Hence we have another Method for finding a Mean Contra hl: Thus, find an Arithm. Mean; then from the Sum of the Extremes take this Mean; the Remainder is the Mean Contra hl: Because the four being $: l$, the Sum of the Extremes and Means are equal.

2. The hl, $: l$, and Contra hl Means, betwixt two Numbers, are in Arithm. Progression; for if betwixt A, E, the Mean hl is B, the $: l$, C, and Contra hl, D; then because A, B, D, E and also A, C, E are $: l$; hence $A+E=B+D$, and $A+E=2C$; therefore $B+D=2C$, or B, C, D, are $\div l$.

THEOREM IX.

Of the mean Proportionals betwixt two Numbers already explained, the Order is this: The direct Harmonical is the least Number; then follow in Order the Geometrical, the Arithmetical, and the Contra hl.

2	:	$3\frac{1}{5}$	:	4	:	5	:	$6\frac{2}{5}$	:	8
10	·	16	·	20	·	25	·	32	·	40
A	:	B	:	C	:	D	:	E	:	F

Or thus,

$$a : \frac{2ab}{a+b} : \overline{ab}|^{\frac{1}{2}} : \frac{a+b}{2} : \frac{a^2+b^2}{a+b} : b$$

hl.   *Geom.*   *Arith.*   *Con. hl.*

Demon. 1°. It's already shewn in *Theo.* XXI. Chap. V. that the Arithmetical Mean D is greater than the Geometrical C; and by *Coroll.* to *Theorem* VI. preceding, it's shewn, that B, C, D are $\div l$, therefore D and C are both greater than B, *i. e.* the Harmonical Mean is less than either the Geometrical or Arithmetical, and so these three Means are in the Order here stated.

2°. The Contra *hl* Mean is greater than the Arithmetical; for by *Coroll.* 2. to the last *Theorem*, the *hl*, $: l$, and Contra *hl* Means are $\div l$, the Arithmetical being the Middle of the three; and by the Expressions of these Means it will be easily shewn, that the Contra *hl* Mean $\frac{a^2+b^2}{a+b}$ is greater than the *hl*; $\frac{2ab}{a+b}$; from which it is plain, that the Contra *hl* is greater than the Arithmetical. The former is proved thus, the Denominators are equal, therefore we have only the Numerators $2ab$, and a^2+b^2 to compare; now $a^2 : ab :: ab : b^2$ (both being as a, b) and $a^2+b^2 > 2ab$ (*Theo.* XXI. Chap. V.) therefore the Contra *hl* Mean is greater than the *hl*, and consequently than the Arithmetical; so that the four Means are in the Order proposed.

General Scholium I.

Comparing the three Kinds of Proportions, Arith. Geom. and harm. we have this very remarkable Difference to observe, *viz.* That a Progression Geom. can be continued from a given Number, upwards or downwards, in Infinitum; an Arithmetical Upwards but not Downwards; a direct harmonical Downwards but not Upwards; and a Contra *hl* neither Ways.

II.

As the Harmonical Proportions of three Numbers already explained, proceed from a Comparison of the Ratio of the Extremes with the Ratio's of the Differences betwixt the Extremes and the middle Term; so there are other Comparisons that may be made in the same general Way; that is, by comparing the Ratio of any two of three Numbers, with the Ratios betwixt the Differences of any one of the three, and the other two. But as I design no particular Consideration of these (since after what is done of this Kind, any may do what more they please) I shall only lay down Examples of the Comparisons wherein it is possible there may be a Proportionality, and shew where it cannot be.

Let a, b, c, be three Numbers, whereof a is the least, and c the greatest; then is it possible to invert them so as the following Numbers be $:: l$, *viz.*

		a · b · c
$a : c :: b-a : c-b$	*Examples*	3 · 4 · 6
$a : c :: c-b : b-a$		3 · 5 · 6
$a : c :: b-a : c-a$		6 · 8 · 9
$a : c :: c-b : c-a$		6 · 7 · 9

		a · b · c
$a : b :: b-a : c-b$	*Examples*	4 · 6 · 9
$a : b :: c-b : b-a$		6 · 9 · 11
$a : b :: b-a : c-a$		4 · 6 · 7
$a : b :: c-b : c-a$		3 · 4 · 7

$b : c :: b-a : c-b$	*Examples*	4 : 6 : 9
$b : c :: c-b : b-a$		1 : 4 : 6
$b : c :: c-b : c-a$		8 : 9 : 12
$b : c :: b-a : c-a$		*Impossible.*

Upon

Upon these Examples *observe*, That they are distinguished into three Cases, according as $a : c$ or $a : b$, or $b : c$, are made the first and second Terms: Then in every Class there are four Cases; and in the two first of these, the third and fourth Terms are the same, only reversed; but of the third and fourth Cases the fourth Terms only are common; nor can a Proportion be stated with these reversed, while the first and second Terms keep the same Order, for then the Comparison would be dissimilar. Thus it is impossible, that it should be $a : c :: c—a : c—b$, because a is less than c; but $c—a$ is greater than $c—b$, since b is greater than a. Then the last Case of the third Class is marked impossible; for if the same Number a is taken from each Term of any Ratio of Inequality, the Remainders cannot be in the same Ratio; since the Remainders cannot be in the same Ratio unless the Numbers taken away be in the Ratio of the whole, (*Coroll.* II. *Theo.* V. Ch. III.)

Observe also, That in the first Case of the second and third Classes, the Examples are always $\div l$, as will be manifest from the Consideration of *Theorem* V. Chap. 3. and so any three Numbers $\div l$ are Examples of these Cases.

III.

Of harmonical Progressions there is another Kind than that already explained, wherein every three adjacent Terms are directly *hl*; for a Series of direct Harmonicals may be found such that every four adjacent Terms are *hl*, as in this Example, $6 : 8 : 10 : 15 : 20 : 40$.

But then *observe* this great Difference betwixt the Nature of these harmonical Progressions, and the Geometrical and Arithmetical; *viz.* that in these last, because every three Adjacent are Proportional, therefore so are every four either Adjacent or taken two and two at equal Distance; but it's not so in the harmonical Kinds, for tho' every three Adjacent are *hl*, yet every four will not be so, neither will the Conclusion hold from four adjacent Terms to three.

§. 2. *Of the Name and Application of* Harmonical Proportion.

I HAVE already said, that the Name comes from the Application of this kind of Proportion found in *Musick*: Not as if this were the only Proportion found among musical Sounds; but because its Effects are the most perfect. It would be out of my Road to say much on that Subject here; for I might as well pretend to explain all the Subjects to which the other Proportions are applicable: Yet this being a Thing little considered (though the Writers on Musick have fully explained it) and of great Use and Curiosity; I shall say as much upon it as may serve to give a distinct Idea of this Application, to such, at least, who have made the following Observations, or can distinctly conceive them.

1st. The Thing in Sounds, upon which what we call Harmony in Musick depends, is that Property, of them whereby they are distinguished into High and Low, called also Acute and Grave; the Idea of which we get by a Series of Notes or Sounds raised one after another upon a musical Instrument, or by a Voice. And observe also, That this Highness or Acuteness is very different from the Strength or Loudness of a Sound; for the Voice of a Boy may be Acuter, though not so Strong or Loud as that of a Man.

2d. Take two Strings (fit for a musical Instrument) which are equal, or the same in all respects, except the Lengths [*i. e.* of the same Matter, and Dimensions, and equally stretched: And this will necessarily happen, if we take a String stretched to any Degree fit for sounding: Then divide it into any two unequal Parts, which may be sounded separately; which is done by having the String fix'd at both Ends, and a little raised

over

over the Surface of any Inftrument or Table; and fetting under it, in any Point that divides it unequally, a Bridge, fo that the String is not the more ftretched by it, but the two Parts fo feparated that they can be founded each by it felf.] The longer String will give the Lower (or Graver) Sound; and the fhorter, the Higher (or Acuter) Sound; fo that a Number of Strings of different Lengths (all other Circumftances being the fame) will give a Series of different Sounds or Notes, Rifing or Falling, in Acutenefs and Gravity, as the Strings become fhorter or longer.

3d. A String may be made of fuch a Length, and fo ftretched, as that its Sound fhall have the fame Degree of Acutenefs (or Gravity; it's no Matter which we fay, fince they are only Words expreffing a Relation of one Sound to another) with any other Sound; and confequently, any two Sounds may be expreffed by two Strings, the fame in all Refpects but the Lengths; and then the Relation of their Lengths may very fitly be confidered as expreffing the Relation of thefe two Sounds, as to Acutenefs or Gravity, which we alfo call the Relation of their Tones; for every Relation muft have fome real abfolute Foundation; and this in the Relation of Acutenefs and Gravity among Sounds we call the Tone. For though every Sound is both Acute and Grave in refpect of different Sounds; yet every Sound muft have its own determinate Degree and Meafure of that upon which Acute or Grave depend; which are but relative Names for different Degrees of it compared to one another. What this Tone depends upon more immediately we fhall next confider.

4th. Though the different Lengths of Strings (other Circumftances being the fame) produce Sounds of different Tones; or Acute and Grave in refpect to one another; yet the Relations and Degrees of Tone are not meafured by the fimple Differences of the Lengths of the Strings; fo that though feveral Strings have an equal Difference of Lengths, as if they were 8, 6, 4, 2; yet their Sounds do not exceed each other by equal differences of Tone; but the Relation of the Tone is the Geometrical Relation of the Lengths of the Strings; fo that to make a Series of Sounds rife or become Acuter by equal Differences, the Lengths of the Strings muft be in Geometrical Progreffion, as 8 . 4 : 2 : 1. Now for the Reafon of this, obferve, that when a String is founded it is put into a Motion, which we call Vibratory, *i. e.* to and again; and the Vibrations, or Motions to and again, are quicker or flower as the String is fhorter or longer (other Circumftances being alike.) And the Mathematicians have demonftrated that the Number of Vibrations of two fuch unequal Strings made in the fame Time, are in the Ratio of their Lengths reciprocally; thus, one String being one Foot long, and another two, the former makes two Vibrations in the Time that the other makes one. And fince all Sound is produced by the vibratory Motion of the Parts of Bodies, they conclude, that the Tone of every Sound depends immediately upon the Number of Vibrations made in any Time; and fince we can exprefs the Tone of any Sound by a String, and confequently of any Number of different Sounds, by as many Strings differing only in Length; the Vibrations of thefe Strings are the fame as the Vibrations of the Parts of other Bodies, whofe Tones they are equal to; and becaufe the Vibrations are reciprocally as the Lengths, therefore the Ratios of the Lengths of Strings (*cæteris paribus*) are true Expreffions of the Ratios of Tone; fo that where-ever betwixt any two fuch Strings, there is the fame Ratio of Lengths, there muft be the fame Ratio of Tone, *i.e.* the fame Excefs of the one above the other: For *Example*, If four Strings are 8 : 4 : : 6 : 3, as far as the Tone of 4 is above that of 8, fo far is the Tone of 3 above that of 6; becaufe they are in the fame Ratio 2 : 1. And fo in thefe, 8 : 4 : 2, where if we judged by the fimple Differences of the Numbers, the Tone of 2 would exceed that of 4, only by the Half of what the Tone of 4 exceeds that of 8; whereas the Excefs of Tones is equal, the Ratios being fo.

In

In the preceding Observations you have the general Grounds of the Arithmetical Theory of Musick; or the Foundation upon which musical Sounds fall under Arithmetical Calculation: The following shall finish what I have to say upon this Subject, and shew you the Application of Harmonical Proportion.

5th. Sounds differing in Tone are applied in Musick two Ways, *viz.* In Succession and Consonance; *That is*, by raising distinct Notes one after another; and by mixing or joining them together so that they fall upon the Ear all at once. But then *observe*, That every Difference of Tone is not fit for Musick; or, any Notes taken in any Relations of Tone, and in any Order, cannot please the Ear, either in Succession or Consonance; but there are certain Relations upon which Musick depends, and without which it has no Being; and these Experience has discovered and approved; *that is*, it is found that Sounds in certain Relations of Tone, being heard together, or one after another, have such an Agreement or Union as to please the Ear, but in different Degrees, according to the Relations. Every Relation that produces an agreeable Consonance will also make an agreeable Succession, but not always the contrary; and therefore the former are reckoned the fundamental and essential Principles of Musick; and such Sounds are particularly called *Concords*; the contrary Effect being called Dissonance or Discord. Though there be different Degrees of Concord, according as the Relations differ, yet to our present purpose it's enough to take Notice of what Musicians call the Simple, Primitive, or Original Concords, of which there are only seven; expressed by the Ratios (of Numbers) and Names in this Table; to be understood thus,

2 : 1 *Octave*,
3 : 2. *Fifth*,
4 : 3 *Fourth*,
5 : 4 *Third greater*,
6 : 5 *Third lesser*,
5 : 3 *Sixth greater*,
8 : 5 *Sixth lesser*.

If two Strings differ only in Length, then their Lengths being in any of these Ratio's; for *Example*, as 2 : 1 or 3 : 2, *&c.* their Tones are Concord, and the Agreeableness is according to the Order here expressed; the *Octave*, 2 : 1 being the most perfect, then the *Fifth* 3 : 2, and so on (tho' there is some Question, Whether the *Third lesser*, or *Sixth greater*, is preferable.) Or if we would express all these Concords in relation to one fundamental Sound, which we may express by 1, the Series of Sounds having these gradual Concord-Relations to that Fundmental, is expressed thus,

$$1 : \frac{1}{2} : \frac{2}{3} : \frac{3}{4} : \frac{4}{5} : \frac{5}{6} : \frac{3}{5} : \frac{5}{8}$$

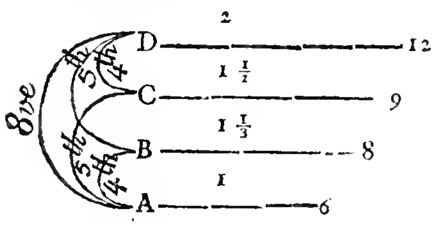
Fundam, *Octave*, *Fifth*, *Fourth*, *Third greater*, *Third lesser*, *Sixth greater*, *Sixth lesser*.

When two Sounds have the same Tone, they are said to be Unisons; which certainly is the first and most perfect degree of Concord; yet more commonly Concord is applied to Sounds of different Tone. The Reason of the Names *Octave*, &c. you'll find below.

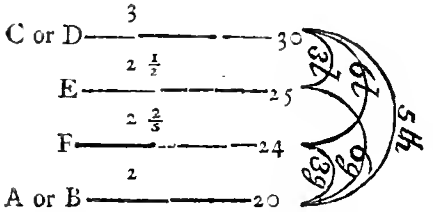
Now as these Concord-Relations are the fundamental and essential Principles of Musick; so the Thing remarkable to our Purpose here is, their Connection and Dependance upon one another; in which the Application of the Proportions Arithmetical, Geometrical and Harmonical is to be found. *Thus*,

Let

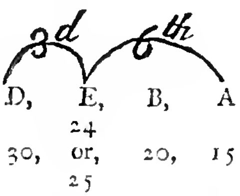
Let any two Strings, D and A be in length, as 2 : 1 (*cæteris paribus*) they make the Concord *Octave*, as above. Betwixt 2, 1, take an Arithmetical Mean C, which is $1\frac{1}{2}$; this String will make with D the Concord, Fourth, (for $2 : 1\frac{1}{2} :: 4 : 3$) and with A the Concord, Fifth, (for $1\frac{1}{2} : 1 :: 3 : 2$.) Again, take an Harmonical Mean betwixt D and A, it is B $= 1\frac{1}{3}$, which makes with D a Fifth (for $2 : 1\frac{1}{3} :: 3 : 2$) and with A a Fourth, (for $1\frac{1}{3} : 1 :: 4 : 3$.) And the Lengths of these four Strings being reduced to Integral Expressions, are 12, 9, 8, 6; which are in Geometrical Proportion; for 12 : 9 :: 8 : 6, or 12 : 8 :: 9 : 6, in which 12 : 9 and 8 : 6 are both Fourths; and 12 : 8, 9 : 6 both Fifths.



Again, take the Fifth 3 : 2; an Arithmetical Mean, E $= 2\frac{1}{2}$, makes the Third lesser with D, and Third greater with B. But take an Harmonical Mean, F $= 2\frac{2}{5}$, it makes a Third greater with D, and a Third lesser with B. And being reduced to Integers, they are 30, 25, 24, 20, which are in Geometrical Proportion; for 30 : 25 :: 24 : 20, or 30 : 24 :: 25 : 20; in which 30 : 25 and 24 : 20 are both Thirds lesser; and 30 : 24, 25 : 20 both Thirds greater.



Now, as the two Concords next in Perfection to the *Octave* arise immediately from the Division of that Concord, *i. e.* putting an Arithmetical or Harmonical Mean betwixt the Extremes of the *Octave*; so the Fifth being the same way divided, produces the two next Concords of Third greater and lesser. Then for the Sixths greater and lesser; they are the Consequences of the preceding Divisions; for having divided the *Octave* into the Fifth and Fourth; then the Fifth into the two Thirds; we have also the Sixths: *Thus*. If D, A are *Octave*, and D, B a Fifth, also D, E a Third greater or lesser; then E, A will be contrarily a Sixth lesser or greater, as the Numbers annexed do shew; for if E=24, then, as 30 : 24 :: 5 : 4 a Third greater; so 24 : 15 :: 8 : 5 a Sixth lesser. And if E=25, then as 30 : 25 :: 6 : 5 a Third lesser; so 25 : 15 :: 5 : 3 a Sixth greater.



I shall go one short Step further, and shew the Reason of these Names, *Octave*, *Fifth*, &c. which will shew a further Application of the Harmonical Proportion. Let A : H be an *Octave*; A : D a Fourth, and A : E a Fifth; then A : C a Third greater; also D : F a Third greater, (from the Division of the Fifth D : H) then will F : H be a Third lesser, and consequently A : F a Sixth greater.

A	B	C	D	E	F	G	H
1	$\frac{8}{9}$	$\frac{4}{5}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{8}{15}$	$\frac{1}{2}$

Again, A : C being a Third greater, as 5 : 4, take B a Harmonical Mean, then will A to B be as 9 : 8, and B to C as 10 : 9. *Lastly*, Let G be taken in the same Ratio to F, as B to A, and its Length will be $\frac{8}{15}$ for $9 : 8 :: \frac{3}{5} : \frac{8}{15}$. And then we have eight Sounds in such Relations of Tone to one another, as make the Series or Succession of Sounds, which is called the *natural Scale* of Musick, which contains in it all the Principles of *Harmony*; wherein, besides the Concord-Relations, these are also very con-

considerable, which are betwixt the several intermediate Sounds, as A · B, B · C, *&c.* which are called the Degrees of the *Scale*; of which there are but three different ones, *viz.* that of A to B, as 9 : 8; of B to C, as 10 : 9; of C to D, as 16 : 15. That the rest are the same, and in what Order they are, you see by comparing their Expressions: Or see them here, where the Lengths of the Strings are set above them, and their mutual Relations betwixt them below. So B is $\frac{8}{9}$ of A, C is $\frac{9}{10}$ of B, D is $\frac{15}{16}$ of C, and so on.

$$\begin{array}{ccccccccccccccc} 1 & & \frac{8}{9} & & \frac{4}{5} & & \frac{3}{5} & & \frac{2}{3} & & \frac{3}{5} & & \frac{8}{15} & & \frac{1}{2} \\ A & \cdot & B & \cdot & C & \cdot & D & \cdot & E & \cdot & F & \cdot & G & \cdot & H \\ & \frac{8}{9} & & \frac{9}{10} & & \frac{15}{16} & & \frac{8}{9} & & \frac{9}{10} & & \frac{8}{9} & & \frac{15}{16} & \end{array}$$

Now the Relations of each of these Sounds to the fundamental A being named according to the Number of Notes from A in the Scale, hence are the Names of Third, Fourth, *&c.* So the Relation 5 : 4 is called a Third, because it's that betwixt A : C, which having one Note betwixt them make in all three Notes of the Scale; for the same Reason 6 : 5 is called also a Third, being the Ratio betwixt C and E. But 5 : 4 is a greater Ratio, and therefore called the Third greater, as the other 6 : 5 is called the Third lesser. For the like Reason, 4 : 3, which is betwixt A : D, is called a Fourth; 3 : 2, which is betwixt A : E, is called a Fifth. 3 : 5 betwixt A : F is called a Sixth greater; 8 : 5 betwixt C : H is called a Sixth lesser; and 2 : 1 betwixt A : H is called an *Octave.* Then for the intermediate Degrees, they are called Seconds, whereof 9 : 8 is the greatest, 10 : 9 next, and 16 : 15 least. And they are otherwise called, particularly, 9 : 8 a greater Tone, 10 : 9 a lesser Tone, 16 : 15 a Semi-Tone, (the Word Tone being here taken in another Sense than we formerly used it in.) And the Ratio 15 : 8 betwixt A : G is called a Seventh. But I shall insist no more upon this Subject, having done all that was proposed, or could be expected in this Place.

APPENDIX
TO
BOOK IV.

Containing ſome further CONSIDERATIONS concerning the Doctrine of *Ratios* and *Proportion* (Geometrical.)

§. 1. *Of Quantities Commenſurable and Incommenſurable, and their* Ratios : *Shewing how the whole Doctrine of* Ratios *and* Proportion *is reduced to the Science of Numbers.*

THE two grand Branches of *Pure Mathematicks* are *Arithmetick* and *Geometry*. As the Object of the firſt is *Number*, called alſo *Quantity* diſcontinued ; ſo the Object of the other is called *Magnitude*, or alſo *Quantity* continued ; the Species of which are *Lines*, *Surfaces* and *Solids*. Theſe two Branches are ſo connected, that the firſt is neceſſarily ſubſervient to the other, which without it would be uſeleſs, or rather could have no Being at all ; for the Conſideration of Numbers runs through the whole Science of Geometry. Among the Principles that are common to both Subjects, the moſt remarkable are contained in the Doctrine of Ratios and Proportion, whoſe Truths hold equally in Arithmetical and Geometrical Quantities : So that this is an univerſal Doctrine in Mathematicks, equally applicable to both its Branches ; but the Way of explaining it, ſo as it may comprehend both, has been Matter of Controverſy among the Mathematicians ; while ſome have approved *Euclid*'s Method (in the fifth Book of his Elements) and others have cenſured it ; ſome of them objecting againſt the very Foundations of his Method, and others complaining only of the Tedioùſneſs and Difficulty of it. To my preſent Purpoſe ; I have only theſe two Things to obſerve :

1ſt. That as I have never been able to help thinking *Euclid*'s Method tedious and difficult (without Regard to other Objections ; for I enter no further into the Controverſy : Thoſe who are curious may ſee Dr. *Barrow*'s learned Defence of *Euclid*, in his Mathematical Lectures) ſo I readily own they have been juſtly blamed, who for the univerſal Doctrine of Proportion, have only given us that of Numbers ; without explain-

explaining how the ſame is applicable to all Kinds of Geometrical Quantities. And,

2*d*. As I think this may be done in a very reaſonable Senſe; ſo it will make much eaſier Work than *Euclid*'s Method; and reduce the whole Doctrine of Ratios and Proportion to the *Science* of Arithmetick; by bringing the Relations of all Quantities under the Notion of Numeral Relations, in ſuch a Senſe that the ſame Truths may be applied in the ſame Demonſtration to all Kinds of Quantities; whereby the Method is truly univerſal.

In order to this I ſhall firſt conſider the Nature of Geometrical Quantities, as they are diſtinguiſhed into *Commenſurable* and *Incommenſurable*: Thus,

One *Magnitude* (*Line*, *Surface*, or *Solid*) may be equal to a certain Part or Parts of another (of the ſame Kind;) and hence 'tis plain that the Definition of Geometrical Relation, applied to Numbers, is equally applicable to ſuch two Magnitudes, and reſolves into the ſame, as the Relation of one Number to another; ſo if the one is $\frac{a}{b}$ Parts of the other, they are to one another as a to b; for the one being divided into as many Parts as a expreſſes, the other is diviſible into a Number of the ſame Parts equal to b: And theſe two Magnitudes are hence ſaid to be *Commenſurable*; becauſe the ſame Magnitude is an *aliquot* Part to both, and therefore is contained in each of them a certain Number of Times without a Remainder; from which it is ſaid to meaſure them both, and they to be Commenſurable. Thus in the preſent Caſe, the ſame Magnitude is $\frac{1}{a}$ part of the one, and $\frac{1}{b}$ part of the other.

But every two Geometrical Quantities (of the ſame Kind) are not Commenſurable, or have not a common *aliquot* Part; (as the *Geometers* have found and demonſtrated in many Caſes.) Therefore the Definition of Geometrical Relation that agrees to Numbers and Commenſurable Quantities, can not be accurately and ſtrictly apply'd to Incommenſurable Quantities: For though there is a Relation of containing and being contained betwixt two ſuch Magnitudes, yet it is not ſuch that we can ſay the leſſer is preciſely ſuch a Part or Parts of the other; and therefore is not the ſame preciſely as that of one Number to another; for if it were ſo, the Magnitude repreſented by the Unit compoſing theſe Numbers, would be an *aliquot* Part of both; and ſo they were not Incommenſurable.

But though Incommenſurable Quantities are not accurately as Number to Number, yet they come infinitely near to that Condition. To underſtand this, I ſhall firſt obſerve, That the ſimple Idea of being Incommenſurable, or having no common *aliquot* Part, though it ſufficiently diſtinguiſhes them from Commenſurable Quantities, yet does not of it ſelf give us any ſuch Idea of their Relation or Manner of containing one another, as to diſtinguiſh the Relations of ſeveral Incommenſurables; becauſe they are all equal and alike in this Reſpect: Yet from this general Notion of Incommenſurability we have a plain Conſequence which furniſhes ſuch an Idea of their Relations, as ſufficiently diſtinguiſhes them, and contains a Character or Mark of their Equality and Inequality, upon which to ground an Idea of Proportionality; which Conſequence is this:

If two Quantities A, B, have not a common *aliquot* Part, then this neceſſarily follows, that the leſſer, A, being taken out of the greater, B, as oft as poſſible, there is a Remainder; which Remainder being taken out of A, as oft as poſſible, there is alſo a Remainder; and this Remainder being taken out of the laſt Remainder, as oft as poſſible, there is yet a Remainder: And going on in the ſame Manner, taking the laſt Remainder out of the preceding, there will ſtill be a Remainder for ever, which grows leſs and leſs *ad Infinitum*: For if ever we come to a Diviſion that leaves no

Remainder, then the Divisor must be a common Measure to the given Quantities, A, B; the Reason of which you see plainly in the Demonstration of the Rule for finding the greatest common Measure of two Numbers (in *Probl.* V. *Chap.* II. *Book* II.) which is equally applicable to any two Quantities.

Now here is one way of conceiving the Relation betwixt these two Magnitudes, *viz.* By the infinite Series of Quotes arising from these alternate Divisions; and therefore, if any other two Incommensurables, as C, D, being the same Way divided, give the same Series of Quotes *ad Infinitum*, their Relation is like and equal to the former; and consequently we may say these four Magnitudes are Proportional, *viz.* A : B : : C : D. But now, though we can thus in general conceive of the Equality and Inequality of the Relations of different Incommensurables (and have reduced them, in one Sense, to Arithmetick;) yet still we want some more particular Exponent of these Relations; by Means of which we may bring them more directly under the Notion of the Relations of Number to Number; so that the whole Doctrine of Proportion may be comprehended in the same Principles and Method of Demonstration, already used for Numbers and commensurable Quantities: And this may be done different Ways in Consequence of the preceding Principles.

1°. In the alternate Division of B by A, and A by the Remainder, and so on; the farther the Operation is supposed to be carried on, the Remainder becomes the lesser *ad Infinitum*; and consequently we come the nearer and nearer for ever to a Quantity which is a common Measure or *aliquot* Part to both A and B; for if we should come at last to a Division without a Remainder, the Divisor would be a common Measure to A, B; and consequently the lesser the Remainder is, the nearer is the Divisor to such a common Measure; and as the Remainders diminish infinitely, it follows that they approach infinitely to the Condition of a common Measure of A, B: Wherefore supposing A, B, both divided by these Remainders successively one after another, the Quotes will approach nearer and nearer *ad Infinitum* to true and complete Expressions of these Magnitudes; for as the last Remainder may be supposed less than any assignable Quantity, so will the Quotes of A, B, by the last Divisor, express two Quantities that shall want less than any assignable Quantity of A and B; and consequently A and B are infinitely near in the Relation of these two Quotes, *i. e.* infinitely near, as Number to Number. Or we may conceive it also thus: Suppose any of the two Quantities, as B, divided into any Number of equal Parts, each of which is less than the other, A; then will A contain a certain Number of the same Parts, but with some Remainder over; else A, B were Commensurable: And the more Parts B is divided into, as that Part is smaller, so A contains the more of them, with a lesser Remainder: And by supposing the Number of Parts of B increased gradually *ad Infinitum*, the Number of the same Parts contained in A, does also increase, and the Remainder decreases *ad Infinitum*; so that what we take of A becomes nearer and nearer, *ad Infinitum*, equal to A; and consequently the Relation of these Numbers becomes nearer, *ad Infinitum*, to a true Expression of the Relation of these two Magnitudes, A, B; for they express the Relation of two Quantities, one of which is equal to B, and the other equal to A, infinitely near, or within less than any assignable Difference: Which shews us also this remarkable Truth, that what hinders any two Magnitudes to be perfectly Commensurable, is a Magnitude infinitely little or less than any assignable one; which being neglected as nothing comparatively, the two Quantities are Commensurable, infinitely near.

In this Manner then all Quantities are reduced to the Relations of Number to Number, by distinguishing Relations into such as are accurately so, and such as are so infinitely near. And the valuable Use and Application of this is, that whatsoever Conclusions can be drawn from the Proportion of Numbers, the same must hold true in the Proportion

portion of Incommenſurables, ſince they are infinitely near as Number to Number: So that whatever four proportional Quantities theſe Letters repreſent A : B : : C : D, we may argue with them the ſame Way as if they were Numbers. For ſince there are Numbers which expreſs any two Quantities by equal Parts, either accurately or infinitely near, the Concluſion can never be falſe, while theſe Expreſſions remain indefinite as to Numbers: Since whatever Error might be in ſuppoſing them determinate to any certain Degree of Approximation, it is corrected by ſuppoſing the Approximation carried further on; and becauſe this can be done without End, the general Concluſion from theſe indefinite Expreſſions muſt be accurately true. So that the ſame Demonſtrations reach to all Kinds of Magnitudes, Commenſurable and Incommenſurable; and by this Means the univerſal Doctrine of Proportion is reduced to the Science of Arithmetick.

2°. There is alſo another, and perhaps, a better Way of conceiving and expreſſing the Relations of Quantities Incommenſurable. Thus; Suppoſe two ſuch Quantities, A and B; if B, the greater, is divided by A, let the Quote be q, and the Remainder m; ſo that B contains A, q Times, and the Quantity m over; which is Incommenſurable to B (for elſe A and B would be Commenſurable:) Again, we can conceive A divided into a Number of equal Parts, each of which is a leſſer Quantity than m; ſo that m contains a certain Number of theſe Parts, or is equal to a certain Fraction of A, with a Remainder n, Incommenſurable to m; conſequently B contains A, q Times, and that certain Fraction of a Time, with the Remainder n over: In like Manner we can conceive n equal to a certain Fraction of A, with an Incommenſurable Remainder o; ſo that B contains A, q Times, and the Sum of thoſe two certain Fractions of a Time, with the Remainder o over. In this Manner we may proceed *ad Infinitum*, conſidering the laſt Remainder as a certain Fraction of B, with a new Incommenſurable Remainder, ſtill decreaſing infinitely; ſo that B is equal to q Times A, and the Sum of that infinite Series of Fractions of a Time (*i. e.* of A.) Therefore B may be expreſſed by Ar; r repreſenting the Sum of q, and that infinite Series of Fractions. Now, if the greater of two other Incommenſurables being divided by the leſſer, there ariſes the ſame Quote q, and alſo the ſame Series of Fractions, by dividing the leſſer and the Remainders in the Manner above mentioned; then the Relation is the ſame; ſo that the leſſer being called B, the greater is Br; and theſe are : : *l*, A : Ar : : B : Br. Thus then we have an univerſal Method of repreſenting all Quantities and their Proportion: For whatever Quantity A repreſents, Ar will repreſent another greater; which is either *Commenſurable* to it, if r is a determinate Number, Integral or Fractional, or *Incommenſurable*, if r expreſſes a Number mixt of a whole Number, and an infinite Series of Fractions decreaſing. Or A may be the greater of the two, and Ar expreſs the leſſer: In which Caſe r will repreſent either a certain determinate Fraction, or infinite Series of Fractions decreaſing; and in both Caſes, that Series carried, *ad Infinitum*, can never be equal to Unity.

Thus alſo we ſee the univerſal Doctrine of Ratios reduced to Arithmetick, under the Diſtinction of determinate and indeterminate Ratios; whoſe Equalities conſtitute *Proportion*; and being expreſſed in a general and uniform Manner (as A : Ar : : B : Br.) The Concluſions drawn from the Equality of r (whatever this is in it ſelf) are alike true and good.

I ſhall finiſh this Section with a few Conſequences from the Nature of Commenſurable and Incommenſurable Quantities, and their Arithmetical Expreſſions.

1. If A : B : : C : D, the one Ratio being determinate or ſurd, ſo is the other, becauſe they are equal. Or, as A : B are Commenſurable or Incommenſurable, ſo are C : D.

2 Quan-

2. Quantities A, B, that are both Commensurable to the same Quantity C, are Commensurable to one another.

3. If A is Commensurable to C, and B Incommensurable to C, then A, B are Incommensurable; for if they were Commensurable, then C and B were also Commensurable, by the last, contrary to Supposition.

4. If A, B are Commensurable, they are both Commensurable, or both Incommensurable to the same, C; for supposing A Commensurable to C, so is B, by the second.

5. As A, B are Commensurable or Incommensurable, A+B is so to them both. And if A+B is Commensurable or Incommensurable to A or to B, it is so also to the other; and so also is A to B.

SCHOL. A certain or determinate Number, and a Surd, are in the true absolute Sense Incommensurable; yet there is in Arithmetick another more limited Sense of Commensurability and Incommensurabilty, which is also among determinate Numbers, and depends upon their having or not having another common *aliquot* Part but Unity; though all these are Commensurable in the absolute Sense. The Theory of Numbers depending upon this limited Distinction of Commensurable and Incommensurable, which is very considerable, you have in the next Book; in the mean Time we pass on to another Consideration of Ratio's.

§. 2. *Concerning the* Arithmetick *of* Ratios.

AMong Authors there are some who talk of Ratios as a particular Kind of Quantities different from pure Numbers; and hence they ascribe to them the common Affection of Quantity, *viz.* a Capacity of more and less, or of Increase and Decrease: But as they imagine them to be of a Nature essentially different from pure Numbers, either Integers or Fractions, so they pretend to an Idea of the Addition of Ratios, and other Arithmetical Operations about them, quite different from these about pure Numbers.

How chimerical, and void of all solid and reasonable Foundation, this Notion is, the very ingenious and learned Doctor *Barrow* has sufficiently shewn in his Mathematical Lectures. I shall only mention one principal Argument of his, *viz.* That pure Relations (for it is the abstract Relation of the Antecedents being after a certain Manner contained in, or containing the Consequent, which they call the Ratio; and which Name I apply to the Exponent of the Relation) cannot be any absolute Things; else there is no Difference betwixt Things absolute and relative; which is a manifest Absurdity. He shews, that it is impossible to make any Comparison of two Ratios, unless they have a common Consequent, or be reduced to that State, *i. e.* four Quantities of one Kind must be found (or supposed) whereof the two several Couplets have the same or an equal Consequent; and their Ratios the same as those in the Question; and then to say, that the one Ratio is greater or lesser than the other, can have no other Meaning, says he, but that the one Antecedent is greater or lesser than the other, for there is no other real Thing in this Case capable of being compared as to more and less. And so the Quantity of the Relation, or Ratio, can be nothing else but the Quantity of the Antecedent, when the Ratios are reduced to a common Consequent.

Now, though the Doctor censures the groundless Notion of a real and absolute Quantity of Relations (as such) yet he allows, as useful and convenient, the received Way of speaking of Ratios being equal or unequal, provided it be taken in the only true Sense and Meaning, which has a rational Foundation, as he explains it; and which is the same Sense in Effect that I have followed in the Foundation I have laid of the Doctrine of Proportion Geometrical, in Chap. I.

All that I ſhall add further upon this Queſtion, is in ſhort this, *viz.* That as we can form no diſtinct particular Idea of the Ratio of any two Quantities, which are not as Number to Number; ſo in the Proportion of Numbers, or of any Quantities which are as Number to Number, though the abſtract Relation of any Number or Quantities being equal to a certain Part or Parts of another (which is the Geometrical Relation of two Numbers, or Quantities expreſſed by Numbers) is no real and abſolute Quantity, yet the Exponent of that Relation (which is the Thing I call the Ratio, in diſtinction from the pure abſtract Relation it ſelf, of which it is the Exponent) being a Fraction proper or improper, different Exponents, or Ratios, as I take the Word, are capable of more and leſs, and of being compared in Quantity the ſame Way as different Fractions are; *Thus*, for *Example*, if A is $\frac{4}{5}$ of B, and C $=\frac{2}{3}$ of D; then we may ſay, that the Ratio of A to B is greater than that of C to D; meaning no other Thing, than that A is a greater Fraction of B than C is of D; for being reduced to a common Denomination they are $\frac{4}{5}=\frac{12}{15}$ and $\frac{2}{3}=\frac{10}{15}$, which is the ſame Senſe of the Quantity of a Ratio that we have heard above in the Doctor's Reaſoning: For theſe four Quantities are either four Numbers, or expreſſible, according to the ſuppoſed Ratios, by four Numbers, *viz.* 2 : 3, and 4 : 5; and by reducing the Fractions, or Exponents of the Relations, to a common Denominator, which is reducing the Ratios to a common Conſequent, they are 10 : 15 and 12 : 15; and from the Compariſon of the Antecedents 10 and 12, the Compariſon of the Fractions or Ratios is made, and their Quantities determined.

Obſerve alſo, That we don't compare $\frac{4}{5}$ of B to $\frac{2}{3}$ of D; for if B, D are Things of different kind, no ſuch Compariſon can be made; and though they were of one Kind, yet $\frac{2}{3}$ of D might be greater than $\frac{4}{5}$ of B, according as the Quantities of B and D happen to be. But we ſimply compare the two abſtract Fractions $\frac{4}{5}$ and $\frac{2}{3}$ which expreſs the Relations without regard to any Difference or Likeneſs of the Things; becauſe the immediate Subject of the Relation is mere Quantity; which requires no more, but that the Terms of the Relation be of one Kind in each Couplet; and then it is plain, That A's being $\frac{4}{5}$ of B is being a greater Part of it, than C's being $\frac{2}{3}$ of D is of it; whatever kind of Things A, B, and C, D are.

If the Authors, or their Followers, whom *Barrow* oppoſes, were content with this Notion of the Quantity of a Ratio, then they ſhould be forced to own, that the Arithmetick of Ratios is in all reſpects coincident with that of Fractions (which are relative Numbers, or Expreſſions of Quantities relatively to others) and ſo they would have no Reaſon to talk of a new Species of Quantities; or of any different Notion of Addition, and other Operations about theſe Quantities. Some of theſe Authors, however abſtractly they pretend to think about Ratios, as Quantities of a particular Species; yet as they repreſent them no other way than as Quotes of the Antecedent divided by the Conſequent (which are Fractions, or reducible to ſuch) ſo they perform all Operations with them as Fractions: But there are others who form a quite different Notion of the Arithmetick of Ratios; and though the Ground of the Application is perhaps arbitrary and whimſical, yet as it is capable of being propos'd in other more

more reasonable Terms, and is, indeed, no other but the Application of some of the preceding Propositions, I shall here briefly explain it.

I. *Of the Addition of* Ratios.

Suppose any Numbers, $a : b : c : d : e$, containing any Ratios; the Ratio of the Extremes $a : e$ is, by the Authors I have now in view, called the Sum of the intermediate Ratios, *viz.* $a : b$, $b : c$, $c : d$, $d : e$; for no other Reason, I know, than that they are continued into one Series, and so exhibit a certain Kind of Adjunction of the Ratios. This Operation or Problem is plainly then no other than this, *viz.*

Having two or more Ratios given, to find the Ratio of the Extremes of a Series, whose intermediate Terms are in the given Ratio; the Solution of which see in *Coroll.* IV. *Probl.* I. Chap. IV.

Exam. To add 2 : 3 and 4 : 5 the Sum is 8 : 15; for 2 : 3 and 4 : 5 continued make this Series 8 : 12 : 15; and by the Rule referred to, the Sum or Ratio sought is thus found, *viz.* $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$: So that, in the last Place, you may observe, that this Addition of Ratios is the same as Composition of Ratios; and is Multiplication and not Addition of Ratios considered as Fractions.

Schol. Though in the true and proper Notion of the Addition of Quantities, the Sum is greater than any of the Parts; yet in this Application the Sum will always be a lesser Ratio, considered as a Fraction, than any of the Parts, if they are all proper Fractions; but it will be greater when they are improper Fractions. And in the last Place, if we consider only the Distance of the Terms in a Series of Numbers, then the Extremes being at greatest Distance, in this respect, the Ratio of the Extremes is the greatest; but to take it in this Manner is a very fantastical Notion of its Quantity.

II. *Of the Substraction of* Ratios.

This depends upon the former, and is to be conceived thus: Let there be any three Numbers $a : b : c$; then as the Ratio $a : c$ is called the Sum of $a : b$ and $b : c$, so either of these is call'd the Difference betwixt the other and $a : c$; and so is reducible to this Problem, *viz.*

Having two Ratios, to find a third, which being continued into one Series, with one of the given Ratios, the Extremes shall be in the other given Ratio. The Solution of which is opposite to that of Addition, and is therefore done by dividing the Substrahend as a Fraction, by the Substractor.

Exam. To substract 2 : 3 from 8 : 15, the Remainder is 4 : 5; for $\frac{8}{15} \div \frac{2}{3} = \frac{24}{30} = \frac{4}{5}$, as this Series shews; 16 : 24 : 30; where 16 : 24 : : 2 : 3; 24 : 30 : : 4 : 5, and 16 : 30 : : 8 : 15; or this Series, 24 : 30 : 45; where 30 : 45 : : 2 : 3, and 24 : 45 : : 8 : 15. From all which it's easy to observe, that this is the Effect of reducing the two given Ratios to one common Antecedent or Consequent (by *Probl.* II. Ch. IV.) the Ratio of the other two Terms found, being what is here called the Remainder or Difference, in the Sense opposite to the Sum in Addition.

III.

III. *Of the Multiplication of* Ratios.

Case 1. The Multiplier being an Integer, this is nothing but a repeated Addition of the ſame Ratio, or the Continuation of the ſame Ratio to a given Number of Terms; and ſo reſolves into this, *viz.*

Finding the Ratio of the Extremes of a Series $\div l$, *which is in a given Ratio, and the Number of Terms* 1 *more than a given Number (which is the Multiplier.)* Which is the Application of *Cor.* IV. *Prob.* I. Ch. IV. Thus, to multiply the Ratio $a : b$ by 3, the Product is $a^3 : b^3$; and univerſally $a : b$ multiplied by n is $a^n : b^n$. For the Extremes of a Series $\div l$ in the Ratio $a : b$, and whoſe Number of Terms is $n+1$, (or the Number of whoſe intermediate Ratios is n) is by *Cor.* IV. *Probl.* 1. $a^n : b^n$.

Exam. The Ratio $2 : 3$ continued to 5 Terms is $16 : 24 : 36 : 54 : 81$. And ſo $16 : 81$ $(=2^4 : 3^4)$ is the Product of $16 : 24$ $(=2 : 3)$ by 4; becauſe this Ratio is repeated in a Series 4 Times.

Schol. A Ratio may be alſo multiplied by a Fraction (which is the ſecond Caſe.) But as the multiplying by a Fraction, and dividing by its Reciprocal, are the ſame Thing, and the Operation depends upon the Diviſion by a whole Number; I muſt refer this till the firſt Caſe of Diviſion be explained; where you'll ſee both explained together.

IV. *Of the Diviſion of* Ratios.

Caſe 1. To divide a Ratio by a whole Number. As Diviſion is oppoſite to Multiplication, ſo is the Senſe and Work of the Diviſion of Ratios to that of Multiplication. Therefore as in a Series $\div l$, $a : b : c : d : e$, the Ratio of the Extremes $a : e$ is called the Product of the common Ratio $a : b$ by the Number of Terms leſs 1; ſo that common Ratio $a : b$ is oppoſitely the Quote of the Ratio of the Extremes $a : e$ divided by the Number of Terms leſs 1; ſo that this Work reſolves into this Problem, *viz.*

Having one Ratio given, to find another, which being continued into a Series $\div l$ *of a given Number of Terms* (viz. 1 *more than the propoſed Diviſor*) *the Extremes ſhall be in the given Ratio.* The Solution of which is plainly had by *Coroll. Theor.* VI. Ch. IV. *viz.* finding the ſecond Term of a Series $\div l$, whoſe Number of Terms is $n+1$ (n being the propoſed Diviſor) and whoſe Extremes are the Terms of the given Ratio. For this ſecond Term with the adjacent Extreme, or that Term of the given Ratio which we chuſe to call the firſt Term of the Series, contain the Ratio ſought.

Exam. The given Ratio, $16 : 81$, and the Diviſor 4, the Ratio ſought is found by the *Coroll.* referred to, $2 : 3$, which is equal to $16 : 24$; as in the Series $16 : 24 : 36 : 54 : 81$.

Caſe 2. Both for Multiplication and Diviſion. To multiply a Ratio by a Fraction, or divide it by the reciprocal Fraction. Let any Series $\div l$, be $a : b : c : d : e : f$; then as $a : f$ is the Product of $a : b$ by 5, (the Number of Terms leſs 1) and $a : b$ the $\frac{1}{5}$ of $a : f$; ſo $a : c$ is $\frac{2}{5}$, and $a : d$, $\frac{3}{5}$ of $a : f$; or $a : f$ is $\frac{5}{2}$ of $a : c$, and $\frac{5}{3}$ of $a : d$; *that is*, $a : c$ is the Product of $a : f$ by $\frac{2}{5}$, or the Quote of $a : f$ by $\frac{5}{2}$. Wherefore, as Multiplying by a Fraction is a mix'd Operation of multiplying and dividing by an Integer; ſo this Caſe is ſolved by applying the former Caſes of multiplying and dividing Ratios by a whole Number. Thus univerſally, to multiply by $\frac{n}{m}$ (or divide by $\frac{m}{n}$) multiply by n, and then divide the Product by m; or firſt di-

vide by m, and then multiply by n, (which will make the Work easier.) For 'tis plain, that the Ratio $a:f$, divided by 5, gives $a:b$; and this multiplied by 2, gives $a:c$, the $\frac{2}{5}$ of $a:f$. Or, multiply $a:f$ by 2, and call the Product $a:l$; then is $a:f::f:l$; therefore betwixt $f:l$ there fall 4 Means in the same Ratio, as that of $a:b$, making in all this Series, $a:b:c:d:e:f:g:h:i:k:l$. Now here $a:c:e:g:i:l$ are $\div\!\!\div l$; therefore $a:c$ is the $\frac{1}{5}$ of $a:l$, *that is*, the $\frac{1}{5}$ of double of $a:f$; the same as the Double of $a:b$, which is the $\frac{1}{5}$ of $a:f$. And this shews the Advantage of beginning first with the Division.

For a particular *Example.* Suppose $8:27$ multiplied by $\frac{2}{3}$ (or divided by $\frac{3}{2}$) the Product is $4:9$; as in this Series $8:12:18:27$, where $8:12::2:3$, and $8:18::4:9$; and as $2:3$ is $\frac{1}{3}$ of $8:27$, so $4:9$ is $\frac{2}{3}$ of it: But had we first multiply'd, the Question would have appear'd in this Series, $64:96:144:216:324:486:729$; wherein $64:729::8^2:27^2$, the Double of $8:27$; and $64:144$ (or $4:9$; for $64:144::4:9$) is the $\frac{1}{3}$ Part of it; because $64:144:324:729$ are $\div\!\!\div l$. To conclude,

The Sense of this Case may be resolved into this Problem, *viz. To find the Ratio of the Extremes of a Series* $\div\!\!\div l$, *containing a given Number of Terms*, (*viz.* 1 more than the Numerator of a given Multiplier, or the Denominator of a Divisor) *and whose Ratio is such, that another Series may be found in the same common Ratio, whose Number of Terms is another Number given*, (*viz.* 1 more than the Denominator of the given Multiplier, or Numerator of the Divisor) *and whose Extremes are in a given Ratio*, (*viz.* that propos'd to be multiplied or divided.) The preceding Series and Explication shew manifestly that this is the true and proper Meaning of multiplying or dividing a Ratio by a Fraction.

Case 3. To divide one Ratio by another, both of one Species, *i. e.* having two Ratios of one Species given, to find how oft the one is contained in the other; or to find that Number by which the one being multiplied (according to *Case* 1, *Multiplication of Ratios*) the Product shall be equal to the other given Ratio; and if the Divisor is not an *aliquot* Part of the Dividend, we are to find the greatest Number of Times it is contained in it, and also the Ratio that remains over.

Rule. Substract the Divisor from the Dividend (by Substraction of Ratios) and the same Divisor from the Remainder, and the same Divisor again from the last Remainder, and so on continually, till the Remainder be a Ratio of Equality; and then the Number of Substractions is the Number sought; in which Case the Divisor is an *aliquot* Part of the Dividend: Or, till the Species of the Ratio in the Remainder is different from that of the given Ratios; and then the Number of Substractions less 1 is the greatest Number of Times the Divisor is contained in the Dividend; and the last Remainder but one, is the Ratio that is contained in the Dividend more than so many Times the Divisor.

Thus, if the Series $a:b:c:d:e$ is $\div\!\!\div l$, then is $a:b$ contained in $a:e$ four Times: For $a:b$ taken from $a:e$, leaves $b:e$; and from this taking $a:b$ (or $b:c$) the Remainder is $c:e$; and from this take $a:b$ (or $c:d$) the Remainder is $d:e$; and from this take $a:b$ (or $d:e$) the Remainder is $1:1$; and the Number of Substractions being 4, this is the Quote.

Again:

Again: Suppose $a : b : c : d$ are $\div l$, but $d : e$ a different Ratio; then supposing the Series to increase from a to e; 'tis plain that $d \div e$ must be greater than $c \div d$; or e must be less than a third Proportional to $c : d$; for if it were greater, we should have another Term betwixt d and e in the preceding Ratio. Now then, $a : b$ from $a : e$, leaves $b : e$ of the same Species. Again, $a : b$ (or $b : c$) from $b : e$ leaves $c : e$ of the same Species; $a : b$ (or $c : d$) from $c : e$, leaves $d : e$ of the same Species. Lastly, $a : b$ from $d : e$ makes the Remainder $db : ae$, which is of a different Species; for $a : b$ and $d : e$, being both Ratios of the lesser to the greater; and $\frac{a}{b}$ a proper Fraction lesser than $\frac{d}{e}$, the Quote $\frac{db}{ae}$, is an improper Fraction. If the Series decrease from a to e, then is e greater than a true third Proportional to $c : d$; so that when $a : b$ is taken from $d : e$, the Remainder $\frac{db}{ae}$ is a proper Fraction, because $\frac{a}{b}$ is in this Case greater than $\frac{d}{e}$. Wherefore the Number of Substractions, less 1, *viz.* 3, is the Quote; and the Remainder of the Division is the Ratio $d : e$.

BOOK V.

Containing these following SUBJECTS,

VIZ.

CHAP. I.

Of the Composition and Resolution of Numbers: Or, The Doctrine of Prime and Composite Numbers; with that of the Commensurability and Incommensurability of Numbers.

§. 1. Containing the General *Principles* and *Theory*.

DEFINITIONS.

I. ONE Number is said to *measure*, or be a *Measure* of another, when it is contained in it a certain Number of Times precisely; so that being taken out of it as oft as possible, there shall nothing remain over. Thus, 4 measures 12; because it is contained in it precisely 3 Times. *Observe* also, that one Number is said to measure another by that Number which is the Quote: So 4 measures 12 by 3; and reciprocally, 3 measures 12 by 4: And hence any Number with the Quote, by which it measures another, may be called the reciprocal Measures of that Number.

COROLLARIES.

1. Every *aliquot* Part of a Number measures it; and every Number which measures another, is an *aliquot* Part of it.

2. Unity measures every Number by that Number it self; and every Number measures it self by Unity; and these are the greatest and least Measures of any Numbers; which are also reciprocal Measures.

II.

II. A Number is called the *Common Measure* of two or more Numbers, when it measures each of them: So 3 is a Common Measure of 6, 9, 12. And if it's the greatest Number that measures them, it is called their *Greatest Common Measure*; as Unity is their least.

III. A Number is called a *Prime Number*, which has no Measure, but it self and Unity; as 2, 3, 5, 7; and which consequently is the Product of no other Numbers.

IV. A Number is called a *Composite Number*, which has some Measure besides it self and Unity; and which consequently is the Product of some two other Numbers: For every Measure has its Reciprocal, and their Product is the Number measured by them, from the Nature of Division. Thus, 3 measures 12 by 4, and 3 Times 4 is 12.

V. Two or more Numbers are said to be *Commensurable*, when they have some common Measure besides 1. Thus, 6 : 9 are Commensurable, because 3 measures them both; and 5, 10, 15, because 5 measures them all.

VI. Two or more Numbers are said to be *Incommensurable*, when they have no common Measure besides 1, as 3, 4; or 4, 5, 6. Such Numbers are also said to be *Prime* to one another, or among themselves; though none of them be really *Prime* in it self.

Corol. Two or more Prime Numbers are *Incommensurable*, because they have no Common Measure but 1. And hence again; if several Numbers, A, B, C, *&c.* are Commensurable, no two of them can be Prime Numbers; and if one of them is a Prime, it must be the common Measure of the Whole, else they have no Measure, since that Prime has no other Measure besides 1.

Schol. Tho' Unity is a common Measure of all Numbers, yet the Notion of *Composition* and *Commensurability* is limited so as to exclude 1 from being a Measure: For since 1 measures all Numbers, if this were admitted, there would be no such Distinctions as Prime and Composite, Commensurable and Incommensurable. If we take *Composition* in the largest Sense, then Unity is the only Number which we can call *Simple*; all others being Collections or Compositions of Units: But this Consideration is too general and simple to be of any Use in discovering the Properties of Numbers; and therefore the Composition here treated of, is that particular Kind which depends upon Multiplication, taken in its more proper and strict Sense, as applied to the Repetition of Numbers, or multiplying them by a Number greater than Unity; because Unity apply'd as a Multiplier, makes no Alteration of the Number to which it is apply'd.

Observe also, That *Integral* Numbers only make the Subject of this Chapter: For if Fractions were admitted, then there is no Number, either Integral or Fractional, but some Fraction will measure it. For Example; Let A be any whole Number; take any Fraction whose Numerator is 1, as $\frac{1}{n}$, it will measure A; the Quote being nA. Again, let $\frac{A}{B}$ be any Fraction, 'tis measurable by this Fraction $\frac{A}{Bn}$, the Quote being $\frac{ABn}{AB}=n$. Again *observe*, that the Distinction of Fractions into Simple and Compound, explained in Book II. is nothing like this Distinction of Prime and Composite; even though Multiplication is concerned in that Composition; for that is merely a Distinction of two different Forms of expressing the same Quantity: Thus, $\frac{2}{3}$ of $\frac{4}{5}$, and $\frac{8}{15}$, are but the same Thing differently conceived and expressed: So that if we take the Notion of Composition in General, as the Effect of Multiplication,

tiplication, then $\frac{8}{15}$ is a compound fractional Number, tho' expressed in a simple Form: And in this Sense we can call that only a Simple Fraction which is not the Product of two real Fractions, proper or improper (excluding such improper ones whose Value is an Integer, and not a mix'd Number) and such only are all Fractions, whose Denominators are Composite Numbers. Example: $\frac{4}{5}$ is a Simple Fraction, both in Form and in its Nature; because no two Fractions produce it, or it is not the Fraction of a Fraction; but this $\frac{7}{15}$ is Simple in its Form, and in its Nature Compound; for it is $\frac{7}{5}$ of $\frac{1}{3}$, and $\frac{8}{15}=\frac{2}{5}$ of $\frac{4}{3}$, or $\frac{4}{5}$ of $\frac{2}{3}$: And this Simplicity and Composition is what belongs to Fractions, but has nothing to do in the Subject of this Chapter.

VII. A Number is called *Even*, which is Measurable by, or is a Multiple of 2; as 2, 4, 6, *&c.*

VIII. A Number is called *Odd*, which is not Measurable by, or is not a Multiple of 2; as 3, 5, 7, *&c.*

COROLLARIES.

1. An odd Number divided by 2, leaves 1 of Remainder.

2. Take the natural Progression 1 . 2 . 3 . 4, *&c.* and beginning at 2, take every other Number, *i. e.* take one and leave the next continually, and so you have the whole Series of *even* Numbers, 2 . 4 . 6 . 8 . 10, *&c.* for the Series, 1, 2, 3, *&c.* having 1 for the common Difference, the Difference of any Term, and the next but one, is 2; consequently beginning at 2, and taking every other Term, we have a Series differing by 2; which beginning with 2, is therefore the Series of Multiples of 2, *i. e.* of all *even* Numbers. Hence again, if we begin at 3, and take every other Term, as 3, 5, 7, 9, *&c.* we have the Series of *odd* Numbers; which proceeds also by the common Difference of 2.

3. If we take the natural Progression, 1 . 2 . 3, *&c.* and double each Term of it, the Series of Products is the Series of *even* Numbers; because it is the Series of Multiples of 2. And taking the same natural Progression, if we take the Sums of every two adjacent Terms, thus, $1+2 : 2+3 : 3+4$, *&c.* these make the Series of *odd* Numbers 3 : 5 : 7, *&c.* for the first, $1+2=3$ is the first *odd* Number, and the Series proceeds by the constant Difference of 2; because every two adjacent Sums have one Part common, and the other Parts are either two adjacent *odd* Numbers, or two adjacent *even* Numbers; which differing by 2, therefore the Series of Sums differ by 2; and because the first is 3, they must make the Series of *odd* Numbers.

4. 1 added to any *even* Number or substracted from it, makes the Sum or Remainder the next greater or lesser *odd* Number; and 1 added to or substracted from any *odd* Number, makes the Sum or Difference the next greater or lesser *even* Number. Again, 2 added to or substracted from any *even* or *odd* Number, gives the next greater or lesser Number which is also *even* or *odd.*

5. All *even* Numbers have 2, 4, 6, 8, or 0, in the Place of Units if they exceed 8; for they proceed from the continual Addition of 2 to it self, and to every succeeding Sum; but the first of them are these 2 . 4 . 6 . 8 . 10, and consequently the same Figures must circulate continually in the Place of Units. Again, all *odd* Numbers above 9, have in the Place of Units, one of these Numbers, 3, 5, 7, 9, or 1; for all *odd* Numbers proceed from the constant Addition of 2, first to 1, and then to the Sum, making the first 5 *odd* Numbers these, 3, 5, 7, 9, 11, ; whence it's plain, that the same Figures will continually circulate in the Place of Units.

5. All

5. All *even* Numbers, except 2, are *Composite*. But of *odd* Numbers some are *Prime*, as 3, 5, 7, and some *Composite*, as 9, 15, 21. And since the two Series of *even* and *odd* Numbers comprehend all Numbers, it follows, that,

6. All *Prime* Numbers are *odd*, except the *Prime* 2.

SCHOL. An *odd* Number may measure an *even*, as 3 measures 12: But an *even* cannot measure an *odd*. Also the Product of two *even* Numbers, or an *odd* and *even*, is always *even*; as the Product of two *odd* is *odd*; the Reasons of which you will learn afterwards. And upon these Things are founded the following Definitions; whereby all *Composite* Numbers are divided into *evenly even*, *oddly even*, and *oddly odd*. Thus:

IX. An *even* Number is called *evenly even*, which an *even* Number measures by an *even* Number, or is produced by two *even* Numbers, as $12=2\times6$, and $24=4\times6$.

X. An *even* Number is called *oddly even*, which an *odd* Number measures by an *even*; or is produced by an *odd* and *even*, as $18=3\times6$.

XI. An *odd Composite* Number is called *oddly odd*; because an *odd* Number measures it by an *odd*, as $15=3\times5$; or it is produced by 2 *odd* Numbers.

Observe, Because no *even* Number measures an *odd*; therefore *odd* Composites are but of one singular Species, *viz. oddly odd*; therefore to call an *odd* Number *Composite*, implies *oddly odd*; but of *even* Numbers there is a Variety: Also besides the preceding two general Distinctions, it's remarkable, that some of them are *evenly even* only, *i. e.* they are not also *oddly even*; as $8=2\times4$, which no *odd* Number can measure. Some of them are *oddly even* only, *i. e.* which are not the Product of two *even* Numbers, as $14=2\times7$. Lastly, some are both *evenly* and *oddly even*, as $12=2\times6=3\times4$.

Again, *observe*, That though 1 may answer to the general Definition of an *odd* Number; yet it's excluded in all that follows especially in what relates to the three last Definitions; because these Names imply Composite Numbers, in which 1 is no Component Factor in a proper Sense. It's true indeed, that if we apply 1 as an *odd* Number, in some of the following *Theorems* they will still be true; but then it is to no Purpose, because they will coincide with some other Thing.

XII. A Number is called *Perfect*, which is equal to the Sum of all its *aliquot* Parts; as $6=3+2+1$, which are all the *aliquot* Parts of 6.

XIII. A Number is called *Abundant*, the Sum of whose *aliquot* Parts exceeds it; as 12, whose *aliquot* Parts are $1+2+3+4+6=16$.

XIV. A Number is called *Deficient*; the Sum of whose *aliquot Parts* is less than it; as 8, whose *aliquot* Parts are $1+2+4=7$.

AXIOMS.

1*st*. If a Number, A, measures each of the several Numbers, B, C, D, *&c.* it will measure their Sum. And if it measure them all but one, it cannot measure the Sum.

2*d*. The Number A, which measures the Sum of two Numbers, B+C, if it measures one of these Numbers, it will measure the other also; or, if it measures the Sum of several Numbers, and also each of the Parts to one, it must measure that one also.

COROL. The Sum of two Numbers is Commensurable with each of them; or it is Incommensurable with each of them; and cannot be Commensurable with the one, and Incommensurable with the other: And the Commensurability or Incommensurability of the Sum with each of them, is according as they are to one another Commensurable or Incommensurable; and reversely, as the Sum is Commensurable or not to each of them, so are they Commensurable or not to one another.

3*d*. The

3*d*. The Number A, that meaſures another, B, meaſures all the Numbers that B meaſures, *i. e.* all the Multiples of B; and what is meaſured by A, is ſo by all the *aliquot* Parts of A.

COROLLARIES.

1. No *even* Number can meaſure any *odd* Number; for then 2, which meaſures all *even* Numbers, would meaſure an *odd* Number, contrary to the Definition of an *odd* Number.

2. If a Number, A, is a Multiple or *aliquot* Part of another, B, which is a Multiple or *aliquot* Part of another C, which is ſo of another, D, and ſo on; then is each leſſer of theſe Numbers, a Multiple or *aliquot* Part of all the greater, *i. e.* A of B, C, D, *&c.* B of C, D, *&c.*

3. From this and *Ax.* 2. follows, that if one Number, A, meaſures each of two other Numbers, B, C, it will alſo meaſure the Remainder after B is taken out of C, as oft as poſſible.

4*th*. Of whatever Factors any Number is compos'd by Multiplication, it is reſolvable into the ſame by Diviſion; *i. e.* it is meaſurable by each of theſe Factors, or the Product of any two or more of them; and the Quote is always the Product of all the reſt of them: Thus, if $N = a \times b \times c \times d$, then $N \div a = bcd$, and $N \div ab = cd$.

Schol. A Number may be diſtributed into Parts, ſo that though another Number can meaſure none of theſe Parts; yet it may meaſure the Whole: But if it meaſures the Whole, it's always poſſible to ſeparate it into Parts, each of which that Number will meaſure. Again, if a Number meaſures none of the Parts, or not all of them, and yet meaſures the Whole, it muſt meaſure the Sum of all the Remainders that happen upon the Diviſion of the ſeveral Parts; for theſe being taken away, it meaſures what remains; and meaſuring the whole, therefore it meaſures the Sum of theſe Remainders, which is the other Part of the Whole.

PROBLEM I.

Of all the *odd* Numbers, not exceeding a given one, to diſtinguiſh which of them are *Prime*, and which *Compoſite*; and conſequently to find whether any *odd* Number is *Prime* or *Compoſite*.

Rule. Begin with the Number 3, and take the Progreſſion of *odd* Numbers 3, 5, 7, 9, 11, *&c.* till you have the given Number: Then beginning at 3, the firſt *Prime odd* Number, ſet ſome Mark, as a Point or Daſh over the third Term after it, and over the third Term after this, and ſo on till you have not 3 Terms within the given Limit. Then begin at 5, and number 5 Terms after it, ſetting the ſame Mark over the 5th, and over the 5th after this, and ſo on continually as long as you have 5 Terms; do the ſame from all the following Terms 7, 9, *&c.* till you come to one after which you cannot find as many Terms as it expreſſes. And obſerve that where, in the Courſe of the Work, you find a Mark already, you need not put a new one. The Numbers thus marked, are all *Compoſites*, the others not marked being all *Primes*.

Exam. To find all the *Primes* from 3 to 79, they are theſe; 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79; as you ſee them marked in this Scheme:

3 · 5 · 7 · 9 · 11 · 13 · 15 · 17 · 19 · 21 · 23 · 25 · 27 · 29 · 31 · 33 · 35 · 37 · 39 · 41 ·

43 · 45 · 47 · 49 · 51 · 53 · 55 · 57 · 59 · 61 · 63 · 65 · 67 · 69 · 71 · 73 · 75 · 77 · 79 ·

Demon.

Demon. Since an odd Number cannot be measured by an even (*Coroll.* I. *Ax.* IIId.) therefore if an odd Number is composite, it is a Multiple of some lesser odd Number by some other; and hence 'tis plain, that if we can distinguish all the odd Numbers within the Limits of the Question, which are the Multiples of each odd Number by every other, we have all the Composites within the Limits of the Question. Now that these are truly found by the Rule, I thus prove:

The common Difference in the Series of odd Numbers is 2; therefore a Term distant from any Term, as far as this Term expresses (*i. e.* the third Term after 3; or fifth after 5, &c.) is the Multiple of that former Term by 3; for it exceeds that former by as many Times 2, as its Distance from that former, *i. e.* by the Multiple of that former by 2, or of 2 by that; and consequently it is 3 Times that former. Going one Period further, according to the Rule, the next Term we mark exceeds the Term last marked, by the same Difference as it does the first Term, *i. e.* by 2 Times that first Term (because equidistant Terms taken out of an Arithmetical Progression, are equidifferent.) But the last marked is equal to three Times the first, and that now marked exceeds the preceding by two Times the first, and therefore it is equal to 3+2, or 5 Times the first: For the same Reason the next marked will be 5+2, or 7 Times the first; and so on in the Progression of odd Numbers, *i. e.* the several Terms marked in numbering from every Term, are the Multiples of this Term by all the Terms of the odd Series from 3. Wherefore we have found all the Multiples, not exceeding the Limits of the Question, of every odd Number by every odd Number, *i. e.* all the odd composite Numbers required; and consequently what are not marked are all prime.

Scholiums.

That the Rule and Demonstration of this Problem might not be too embarassed and difficult, I have left some Things to be explained here, by which the Work is made easier.

1st. Of the Series of Composites numbered from 3, mark the second, which is 15, with a double Point or Dash; then from that one begin the Numbering by the next odd Number 5, and mark the second of this new Series with a double Mark; then from this one begin the Numbering by the next odd Number 7, and so on through all the rest. These Terms doubly marked will all follow one another in order, and therefore 'tis always the last double Mark at which we begin for the next Step. Thus you see them marked in the preceding Example.

The *Reason* is this: Take any Term of the odd Series, its Multiples by each of the preceding Lesser coincide with the Multiples of each of these by this one. *Example:* The Multiples of 7 by 3 and 5 are the same as the Multiples of 3 and 5 by 7; whence it is plain, that if all the Multiples of all the Terms preceding any given one are marked, then we have so many of the Multiples of this one already marked, as do not exceed its Product into the preceding Term; and so we need only to begin at this Product in numbering by this Term. But again, it is plain, that the Product of any Term into the preceding, and its Product into the following, will have but one Composite betwixt them, *viz.* the Product of that Term into it self; therefore the Product of any Term into the preceding is the second after the Product of that preceding into its preceding; hence, *Lastly*, if we mark all the Multiples of 3, the first odd Number, and begin numbering by 5 at the second Composite from 3, we shall have all the Multiples of 5; and beginning the Numbering by 7 at the second of these numbered by 5, we shall have all the Multiples of 7; and so on.

2d. We may yet save a good deal of trouble in writing down the Series of odd Numbers, by this Method:

Suppose the given Limit be 99, write down the Numbers, 1, 3, 5, 7, 9 in one Line, reckoning each of them as simple Units; in a Column on the left Hand of these, write the Series 0, 1. 2. 3. 4. 5. 6. 7. 8. 9, reckoning each of these as so many 10's; then

	1	3	5	7	9
0					·
1			··		
2	·		·	·	
3		·	··		·
4			·		·
5	·		·	·	
6		··	·		·
7			·	·	
8	·		·	·	
9	·	·	·		··

draw Lines ſeparating them, as in the annex'd Scheme: the Spaces made by the croſſing of theſe Lines, taken in the continued Order of the odd Numbers from the firſt Space in the upper Corner on the left, againſt 0, and under 1, and numbered to the Right through that Line, and ſo ſucceſſively through the reſt of the Lines, are the Places in which all the odd Numbers, ſet down one after another in order from 1 to 99, would ſtand: And without writing them down, their Places are known thus; take the Figure in the Place of 10's of any Number in the Column on the left Hand, and the Figure in the Place of Units in the upper Line; then the Space where the Line of Spaces from the Figure in the Place of 10's, and the Column of Spaces from the Figure in the Place of Units, do meet, is the Place of that Number. Wherefore if we begin at the Space under 3 in the firſt Line of Spaces, and number the Spaces in order from this towards the right Hand through every Line, and mark the Spaces as before we did the Numbers repreſented by theſe Spaces, we ſhall have the Primes and Compoſites the ſame Way diſtinguiſhed; with this plain Advantage of Eaſe in the Work, that we have not the Trouble of writing down all the odd Numbers ſeparately. There are other Advantages of this Method, which you'll learn afterwards.

Again, If the Limit be 999, carry on the odd Numbers upon the upper Line to 99; and the Series 0 . 1 . 2 . 3 . 4 . 5 . 6 . 7 . 8 . 9 on the Left, repreſents 100's.

By this you'll eaſily perceive, how the Table may be made to any Limit. Afterwards (ſee *Probl.* III.) you'll find a Table carried to 999, with other Work upon it, to ſerve other uſeful Purpoſes.

3*d.* When you begin to number by any Term, ſee firſt if it be a Prime or Compoſite, which the Table will ſhew, according as the Place of it is marked or not; for all the Compoſites within the Limits aſſigned, which do not exceed the Product of this Term into the preceding (and in ſome Caſes ſome that do exceed this Product) are already marked; and therefore if this Term is Compoſite it is already marked: Then if the ſuppoſed Term is a Prime, you muſt go on by the Rule; but if it is Compoſite, all its Multiples are already marked, becauſe they are alſo Multiples of any of its component Parts, and all the Multiples of theſe are already marked: Only it will be neceſſary to number out the firſt two Periods, that the ſecond Compoſite in numbering by this Term may be doubly marked, in order to know where to begin for the next Term.

4*th.* If it be propoſed to find, Whether any given odd Number be Prime or Compoſite, it will not be always neceſſary to find all the Primes and Compoſites within that Limit; for if in the Courſe of the Work the Mark of a Compoſite falls upon the given Number, we have the Queſtion ſolved, and there we may ſtop. And if you have already a Table made, there you have the Queſtion ſolved for all Numbers not exceeding the Limits of the Queſtion. Afterwards (ſee *Coroll. Probl.* III.) you'll ſee another Method of finding whether a Number be Prime or Compoſite, without carrying a Table ſo far as the given Number.

5*th.* The laſt Remark I make on this Queſtion is, That all Numbers which have 5 in the Place of Units, as 15, 25, 35, *&c.* are Compoſite Numbers meaſurable by 5; for if that 5 in the Place of Units be taken away, what remains has 0 in the Place of Units, and ſo is a Multiple of 10, and conſequently a Multiple of 5; therefore 5 which meaſures both the Parts does alſo meaſure the whole; ſo 5 meaſures 140 and 5, therefore it meaſures 145. Again, a Number which has any Figure, except 1, in all its Places, is a Compoſite meaſurable by that Figure; ſo $333=111\times3$. $7777=1111\times7$; but if 1

is in all its Places, the Number is Prime in some Cases, and in some Composite; so 11 is Prime, 111 is Composite, for it is $=3\times37$.

PROBLEM II.

To find if two or more given Numbers are Commensurable or Incommensurable; and what is their greatest common Measure.

Case I. *For two Numbers.*

The Rule and Reason of this Case we have already explain'd [*Book* II. *Chap.* II. *Probl.* V.] where it is taught how to reduce a Fraction to its lowest Terms by first finding the greatest common Measure to its Numerator and Denominator. I shall therefore refer to that Place, and draw from it these *Corollaries.*

COROLLARIES.

1*st.* Whatever Number measures any two Numbers A, B, it will also measure their greatest common Measure. If you look back to the Rule and Operation referred to, this Consequence will be evident. For whatever measures the given Numbers, *i. e.* the first Dividend and Divisor, measures also the first Remainder (*Cor.* III. *Ax.* III.) and consequently it measures the second Divisor (which is the first Remainder) and the second Dividend (which is the first Divisor) and so on through all the succeeding Divisors and Dividends to the last Divisor, which is the greatest common Measure.

Exam. The greatest common Measure of 84 and 156 is 12; and because the Numbers 2, 3, 4, 6 do measure both 84 and 156, therefore they measure 12.

2*d* Two Numbers, whose Difference is 1, are Relative Primes, for their greatest common Measure is 1, this being the very first Remainder. And for the same Reason any Series of Numbers differing continually by 1, are relative Primes, since no two of them, whose Difference is 1, can have any other common Measure.

3*d.* Two odd Numbers differing by 2 are Incommensurable; for the first Remainder is 2, and the second is 1. Hence also any Series of odd Numbers differing by 2 are Incommensurable.

4*th.* If two Numbers, A, B, are Incommensurable, then when any Multiple of the lesser, A, as nA, is taken out of the greater, B, the Remainder B—nA is either 1, or a Number Incommensurable to A: For if any Number measure A, it will measure nA; and if it also measure B—nA it will measure B; and so A, B would be Commensurable, contrary to Supposition.

Case II. *For more than two Numbers.*

Rule. Find the greatest common Measure to any two of them; then find the same for the Number now found, and any other of the given Numbers; and again, for the last found, and another of the given Numbers; and so on, till you have gone through them all, and the last found is the Number sought.

$$\begin{array}{ccccc} 24 & : & 40 & : & 52 \\ & & 8 & & 4 \end{array}$$

Exam. The greatest common Measure of 24, 40, 52, is 4, found thus; the greatest common Measure of 24 : 40 is 8, and that of 8 and 52 is 4.

$$\begin{array}{l} \text{A . B . C . D . \&c.} \\ \quad\;\; m \,.\, n \,.\, o \,.\, \text{\&c.} \end{array}$$

DEMON. 1°. Since m Measures A, B, and n measures m, C, therefore n measures A, B, C (*Ax.* III.) Again, o measures n, D, and n measures A, B, C, therefore o measures A, B, C, D; and so it proceeds for ever, *i. e.* each Number found in the Operation is a common Multiple to all the given Numbers.

2°. The Numbers found are the greatest common Measures of the given Numbers; for, what measures A, B measures m, and what measures m, c, measures n, (*Cor.* I.

Case I.) therefore what measures A, B, C, measures n, and consequently it is not greater than n, which is therefore the greatest common Measure of A, B, C. Again, what measures A, B, C measures n (by the last Step) and what measures n, D measures o (*Cor.* I. *Case* I.) therefore what measures A, B, C, D, measures o, and consequently is not greater than o, which therefore is the greatest common Measure of A, B, C, D.

The same Reasoning is manifestly good from one Step to another for ever; from which we have plainly gained the following Truth, *viz.*

Coroll. 5*th*. Whatever measures any Numbers A, B, C, *&c.* measures their greatest common Measure; so that all their other common Measures are *aliquot* Parts of the greatest.

Schol. An Integer being divided by a mixt Number less than it self may quote an Integer, and upon that Account we may say, that the mixt Number measures the other; so that a mixt Number may be the common Measure of two or more Integers. For *Example*, 18 and 24 being divided by $1\frac{1}{2}$ or $\frac{3}{2}$ quote 12, 16. But from the preceding Demonstrations we learn these Truths:

COROLLARIES.

6*th*. A mixt Number can never be the greatest common Measure of two Integers; for it's shewn, that this must be an Integer, *viz.* the last Remainder of a Division of Integers: Hence again,

7*th*. No mixt Number can be the greatest common Measure of any Number of Integers, for then it might also be the greatest common Measure of two Integers.

8*th*. Two Integers which have not another common Measure in Integers but 1, have not one of any kind, except, perhaps, an *aliquot* Fraction, (*viz.* such as has 1 for its Numerator) or some equivalent one; for 1 being their greatest Measure, no other Number can measure them, except it be a proper Fraction; but no other than an *aliquot* Fraction can do it: For suppose any other, as $\frac{a}{n}$, if it measures A, B, it measures also their greatest Measure 1, which is impossible; because the Quote of 1 by $\frac{a}{n}$ is $\frac{n}{a}$, which cannot be equal to an Integer precisely, for then $\frac{a}{n}$ would be equal to an *aliquot* Fraction, *i.e.* suppose $\frac{n}{a}=r$ or $\frac{r}{1}$, then is $\frac{a}{n}=\frac{1}{r}$, contrary to Supposition

9*th*. Integers that have a mixt common Measure, have also an Integral one greater than 1; for their greatest must be an Integer, and it must be greater than 1, because a mixt Number, which is always greater than 1, is supposed to measure them.

But *observe*; Two Numbers may have an integral common Measure greater than 1, and yet have no mixt common Measure.

Theorem I.

Every prime Number, A, is Incommensurable with every Number, B, which it does not measure.

A=5 . B=8 | Demon. If A, B, are Commensurable, then either A or some other Number measures them both; either of which is contrary to the Supposition.

Coroll. Of several Numbers A, B, C, *&c.* if one of them, as A, is a Prime, it is the common Measure of the whole, else they are Incommensurable. And if it do measure the whole, it is their greatest and only common Measure, because it self has no other Measure but 1.

The-

THEOREM II.

If a Number, A, meaſures one, B, of two Numbers, B, C, that are Incommenſurable; it is Incommenſurable with the other, C.

A=3
B=6. C=7

DEMON. If any Number meaſures both A and C, it will alſo meaſure B, which A doth meaſure (*Ax.* 3.) therefore B, C are Commenſurable, contrary to Suppoſition.

Obſerve, The Reverſe will not always hold; for tho' A is Incommenſurable with C, it does not follow that it will meaſure A; becauſe it may be Incommenſurable to both A and C; as in Caſe A be =5.

THEOREM III.

Numbers a, b, c, &c. that are leaſt in their Ratios, are like *aliquot* Parts of, or do equally meaſure, any other Numbers, A, B, C, *&c.* that are in the ſame Ratios reſpectively, *that is*, a Meaſures A, and b meaſures B, *&c.* equally. Alſo, the Number by which they meaſure them is the greateſt common Meaſure of A, B, C, *&c.* *Reverſely*, The greateſt common Meaſure of certain Numbers A, B, C, *&c.* meaſures them by Numbers a, b, c, &c. that are leaſt in the ſame Ratios.

	A	B	C	D
Dividends,	6	9	15	24
Diviſors,	2	3	5	8
	a	b	c	d

Common Quote $q=3$

DEMON. 1°. Since, $a:b::A:B$, and $b:c::B:C$, *&c.* then alternately $a:A::b:B$, alſo $b:B::c:C$, *&c.* hence $\frac{A}{a}=\frac{B}{b}=\frac{C}{c}$, *&c.*

Now if we ſuppoſe that theſe equal Quotes are not Integral (or if a does not meaſure A, *&c.*) yet becauſe they are equal, therefore the integral Part is the ſame, and ſo alſo is the Fraction. Let the Remainders of the Diviſions be r, s, t, *&c.* then theſe Fractions are, $\frac{r}{a}=\frac{s}{b}=\frac{t}{c}$, *&c.* but being proper Fractions the Numerators r, s, t, *&c.* are leſs than their Denominaators a, b, c, *&c.* and being equal Fractions they are reſpectively in the ſame Ratios, *i. e* $r:s::a:b$, and $s:t::b:c$; hence a, b, c, *&c.* are not leaſt in their Ratios, contrary to Suppoſition; and therefore a, b, c, *&c.* muſt meaſure A, B, C, equally. Again, the Quote q is the greateſt common Meaſure of A, B, C, *&c.* for whatever it be, it meaſures A, B, C, *&c.* by the ſame Numbers which were before the Diviſors, *viz.* a, b, c, &c. (from the Nature of Diviſion.) But if A, B, C, *&c.* are divided by any Number, the Quotes are alſo in the ſame Ratio (from the Nature of Ratios.) Alſo the greateſt common Meaſure muſt give leſſer Quotes than any other Meaſure, and therefore either q is the greateſt common Meaſure, or a, b, c are not the leaſt in their Ratios: But a, b, c, &c. are leaſt in their Ratios, therefore q is the greateſt common Meaſure.

The *Reverſe* of the Theorem is manifeſt from the Nature of Diviſion.

SCHOL. Though Numbers a, b, c, d, do equally meaſure others, A, B, C, D, and ſo are like *aliquot* Parts of them, it does not follow that they are leaſt in their Ratios; for in order to this, they muſt meaſure them by their greateſt common Meaſure.

COROLLARIES.

1*ſt.* Numbers that are not leaſt in their Ratios are Equimultiples of ſuch as are ſo, becauſe theſe are like *aliquot* Parts of the others.

2*d.* Here we have another Proof that a Fraction can never be the greateſt common Meaſure of ſeveral Integers; for then the leaſt in their Ratios would not meaſure other Numbers in the ſame Ratios, as they muſt do by what has now been ſhewn.

3*d*

3*d*. We learn here how to find Numbers that are least in the same Ratios with other given Numbers, *viz.* by finding the greatest common Measure to these given Numbers, and by it dividing them, the Quotes are the Numbers sought; so the least in the same Ratios with 9 : 15 : 21, are 3 : 5 : 7, for the greatest common Measure of the former is 3, and the Quotes are 3 5. 7.

THEOREM IV.

Numbers *a*, *b*, *c*, *d*, &c. that are least in their Ratios, are Incommensurable; and the *Reverse*, Incommensurables are least in their Ratios.

a, *b*, *c*, *d*, 2, 3, 6, 7,	DEMON. If *a*, *b*, *c*, *d*, are Commensurable, then being divided by their common Measure, the Quotes will be in the same Ratios, and also lesser Numbers; therefore *a*, *b*, *c*, *d*, are not the least, contrary to Supposition.

For the *Reverse*, If the least Numbers in the same Ratios with *a*, *b*, &c. are lesser than they, then will these Numbers equally measure them by their greatest common Measure (*Theo.* I.) But *a*, *b*, &c. being Incommensurable, have no common Measure; therefore they are least in their Ratios.

COROLLARIES.

1*st*. Since Numbers being least in their Ratios, and being Incommensurable, do necessarily follow from one another, we may put any of these in place of the other in any Propositions; particularly in the preceding Theorem; so that if Numbers *a*, *b*, *c*, &c. are Incommensurable, they do equally measure any others in the same Ratios A, B, C, *&c.* and hence all the following Corollaries.

2*d*. If Numbers *a*, *b*, *c*, &c. are Incommensurable, others in the same Ratios A, B, C, *&c.* are Commensurable, and also all *Composites*; for *a*, *b*, *c*, &c. do equally measure them, and the common Quote does reciprocally measure each of them; which therefore are both Commensurable, and all *Composites*. Hence again,

3*d*. If four Numbers are : : *l*, *a* : *b* : : A : B, no three of them can be prime Numbers, nor consequently the whole; [or thus, to three prime Numbers a fourth : : *l* can't be a whole Number] for if *a*, *b*, are Primes, they are Incommensurable, and so A, B, are both Composites. Whence again, if three Numbers are : : *l*, A, B, C (or A : B : : B : C) none of the Extremes with the middle Term, nor consequently all three, can be Primes [*i. e.* to two Primes, A, B, a third : : *l* can't be an Integer] for if A, B, are Primes, therefore B, C, are Composites, so that B is both Prime and Composite; which is absurd.

In another Place (see *Theo.* XXII. *Cor.* III.) you'll see it demonstrated, that betwixt two Primes there can't be a geometrical Mean, either in whole Numbers or Fractions.

THEOREM V.

If any Numbers A, B, C, *&c.* Incommensurable, are measured by other Numbers *a*, *b*, *c*, *&c.* (*i. e.* A by *a*, and B by *b*, &c.) these last are neither in the same Ratios with the former, nor do they measure them equally, nor, *Lastly*, are they Commensurable.

DEMON. *a*, *b*, *c*, can't be in the same Ratios with A, B, C, *&c.* for since they measure them, it would follow, that they measure them equally; because if *a* : A : : *b* : B then $\frac{A}{a}=\frac{B}{b}$; and whatever the common Quote, is it will reciprocally measure A, B, C, which consequently are not Incommensurable. Again, whatever be the Ratios of *a*, *b*, *c*, if they measure A, B, C, equally, the Quote will reciprocally measure A,B,C, which

which therefore are not Incommensurable. *Lastly*, If *a*, *b*, *c*, are Commensurable, their common Measure must measure A, B, C, which they measure; consequently A, B, C, are not Incommensurable. And observe, that this last Article does also prove the first; for if *a*, *b*, *c*, and A, B, C, are both Incommensurable, they can't be in the same Ratios.

THEOREM VI.

If a Number, A, measures the Product of two others, B, C, and is Incommensurable to one of them, it will not only be Commensurable with the other, but also measure it.

DEMON. Let A measure BC by D, then is it A : B : : C : D; but if A, B, are Incommensurable (or least in their Ratio) therefore A measures C, and B measures D equally, by the last.

COROLLARIES.

1st. If a Number, A, measures the Product of two Factors B, C, (*i. e.* if four Numbers are : : *l*, A : B : : C : D) then will it either measure one of these Factors, B or C, or be Commensurable to each of them; for if it's Incommensurable to any one of them, it measures the other. But if A is a prime Number, and measures BC it will necessarily measure B or C; for if it do not measure B, it is Incommensurable to it (by *Theor.* I.) and consequently measures C, by this Theorem.

2d. If a Number, A, is Incommensurable to each of two Factors B and C, or if it's Incommensurable to the one, and does not measure the other, it cannot measure the Product (*i. e.* A, B, C, being Integers, and A : B : : C : D, then A being Incommensurable to both B and C, or Incommensurable to the one, and not measuring the other, D is not an Integer;) for if it measures the Product (*i. e.* if D be also an Integer) and is Incommensurable to one of the Factors, it measures the other, and so is not Incommensurable to it: Both contrary to Supposition. Hence again,

3d. If A is Incommensurable to B, it can't measure the Square of B, *i. e.* a third : : *l* to two Numbers A, B, that are Incommensurable, can't be found in Integers.

SCHOL. As A's being Incommensurable to both B and C is a certain Consequence of their being all three prime Numbers, so *Coroll.* 3*d.* *Theo.* IV. is in effect comprehended in the preceding *Coroll.* 2*d.*

Again, though three Numbers be Incommensurable (which is a different Thing from one of them being so to each of the other two) yet D may be integral, as in these *2 : 3 : : 4 : 6.*

Further, in the various Circumstances of A, B, C, being all Composite Numbers, or only Commensurable, *Observe* (1°) That though they are all Composites, yet a fourth in Integers may be impossible, because A may be Incommensurable to both B and C, which is consistent with their being all Composites, as in these 4, 6, 9. (2°.) Though they are all Composites, and also Commensurable, yet a 4th Integral may be impossible; as in these, 6 : 10 : : 14, to which a 4th is 23 $\frac{1}{3}$. But to understand the general Reason of this, and what Circumstances of their Composition makes it so, depends upon some other Principles than we have yet heard, and which you will find afterwards (see *Schol.* III. *Theo.* XXIII.) where I shall shew you the general Character upon which depends the 4th's being Integral or not *Lastly*, observe, That the three given Numbers being Commensurable, if the first of them, A, is a prime Number, then is D always an Integer; for in this Case A is the common Measure, and because it measures B and C it will also measure BC so that D, which is the Quote, is an Integer.

THEOREM VII.

The least Number A, which measures any composite Number B, must be a prime Number.

$A=3. \quad B=15$ | DEMON. If A is a Composite, then its component Parts will measure B (*Ax.* 3.) and consequently A is not the least which measures B, contrary to Supposition.

COROLLARIES.

1*st*. Some prime Number measures every Composite; or, every Composite is measured by (*i. e.* is the Multiple of) some Prime; for there must be a least Measure, and that must be prime.

2*d*. Every Number is either a Prime, or may be measured by some Prime.

3*d*. Commensurable Numbers have some prime common Measure; for their common Measure is either a Prime, or some Prime measures it, which therefore will measure these Composites: And hence also, if they have several common Measures, the least of them is a prime Number. *Reversely*, Numbers that have no prime common Measure are Incommensurable.

THEOREM VIII.

There are an infinite Number of prime Numbers: Or thus, no Number of Primes can be assigned, but another may be found different from all the given ones.

$A\times B\times C$, &c. $=P$
$P+1$; z | DEMON. Let A, B, C, &c. be any Number of Primes, whose continual Product is P, to which add 1; then if $P+1$ is a Prime, 'tis plainly different from the given ones; but if it be Composite, some Prime, as z, measures it (*Corr.* 1*st. Theo.* VII.) and this is a different Number from any of the given Primes; for if it be the same with any of these, then it will measure P their Product (*Ax.* 3.) but 'tis supposed also to measure $P+1$, therefore it must measure 1 (*Ax.* 2.) which is absurd.

THEOREM IX.

Take the greatest Number of Factors, a, b, c, &c. of which any Number, N, can be composed (or to whose Product it is equal) they will be all prime Numbers.

$a\times b\times c=N$
$3\times 5\times 7=105$ | DEMON. If any of them be Composite, the component Parts of it are also Components of the given Composite N (*Ax.* 3.) and so the proposed Number of Factors is not the greatest whose Product makes N, contrary to Supposition. So if $a=n\times r$, then is $N=n\times r\times b\times c$.

COROLL. Every composite Number is equal to the Product of a certain Number of Primes, *viz.* the greatest Number of Factors by which it can be produced.

THEOREM X.

A Number, M, which is the Product of a certain Number of given Primes, a, b, c, d, &c. whether they are all different Numbers, or some of them oftner than once involved, can be measured only by one of these prime Factors, or the Product of any two, or more of them; *that is*, it cannot be measured by any other prime Number, nor by any Number which has in its Composition any other Prime, *i. e* which is the Multiple of any other prime Number; nor, *Lastly*, by any Composite, which, though it have in its Composition no Prime different from any of these that produce M, yet has any of the same Primes oftner involved, *i. e.* is the Multiple of a greater Power of any of these Primes than what M is; as, if M has in it only the Root or Square of a, and this other has in it the Cube of a.

Exam.

Exam. $2\times3\times7\times11=462$; which Composite 462, cannot be measured by 5, which is a different Prime; nor by 15, the Multiple of 5; nor by 9, the Square of 3.

Demon. 1*st*. That M is measurable by any one, or the Product of any Number of its own Component Primes, is plain, by *Axiom* 4. And,

2*d*. That it can be measured by none other, *i. e.* by none of these described in the *Theor.* which plainly comprehend all others, is thus demonstrated.

(1°.) It cannot be measured by any other Prime, as x.

For x is Incommensurable to a and b, because both are Primes different from x; therefore it cannot measure their Product ab (by *Cor.* 2. *Theor.* VI.) and so is Incommensurable to it (*Theor.* I.) and being also Incommensurable to another Prime c, it cannot measure the Product abc (*Cor.* 2. *Theor.* VI.) and hence is Incommensurable to it (*Theor.* I.) and being also Incommensurable to another Prime d, it cannot measure the Product, $abcd$. And so the Reasoning proceeds for ever.

Or thus also. Let d be any Prime different from x, or the Product of any Number of Primes all different from x; then, upon Supposition that x cannot measure d, I say it cannot measure the Product of one Prime more, *i. e.* dm (m being also a Prime different from x;) for if x measure dm, let it be by y, then is $x : d :: m : y$. But x being a Prime, which does not measure d, then are x, d, Incommensurable (*Theor.* I.) and so x measures m (*Theor.* IV. *Cor.* 1.) which is absurd, because m is a Prime, and different from x. But x cannot measure another Prime; therefore, by what is now shewn, it cannot measure the Product of two others, nor consequently the Product of three others, and so on, *i. e.* it cannot measure the Product of any Number of others.

(2°.) It cannot be measured by the Multiple of another Prime; for then that Prime would also measure it (*Ax.* 3.) which is contrary to the last Article.

(3°.) It cannot be measured by any Number N, which tho' it has in it no other Prime, yet has any one of the same Primes oftner involved. For suppose any one or more of them is oftner involved in N than in M; then imagine all the Prime Factors of N, that are also in M, to be taken out of both, *i. e.* let both of them be divided by the continual Product of all these common Prime Factors, and call the Quotes A, B, they will be in the same Ratio, or N : M : : A : B. But now, of those Primes that were not so oft involved in N as in M, what were more of any of them in M than in N, and what were not at all in N, will remain in B; and what were more in N than in M, will remain in A; (by *Ax.* 4.) but none of these will be in B; for because there were fewer of them in M than in N, therefore they were all taken out of M; consequently there will be some Prime in A, which is not in B, and therefore A cannot measure B; for then that Prime would measure B, contrary to what is shewn. Therefore lastly, N cannot measure M, because N : M : : A : B. And if A cannot measure B, neither can N measure M.

N) M
A) B

COROLLARIES.

1*st*. Of two Composite Numbers, A, B; if there is in the Composition of the one, any Prime which is not in the other, or any the same Prime oftner involved, these two Numbers cannot be equal: For in these Circumstances, the one cannot measure the other, and consequently they cannot be equal.

2*d*. M, the Product of a certain Number of Primes, $a, b, c, d,$ &c. cannot be equal to (or the same Number with) N, the Product of any greater Number of Factors, whatever they be; nor to the Product of any other Choice of an equal Number of Factors; nor lastly, to the Product of a lesser Number of Factors, which are all Primes. For (1.) A greater Number of Factors are either all Primes, or are resolvable into a greater Number of Primes; and therefore, among them there must necessarily be found some Prime

different from any of theſe in M, or ſome of theſe oftner involved; and ſo N cannot meaſure M, and ſo not be equal to it: (2.) For another Choice of an equal Number of Factors, they are either all Primes, and therefore muſt have ſome different Prime, or a greater Power of ſome Prime; and ſo they cannot be equal (by the firſt *Cor.*); or if any of them be Compoſite, then being reſolved into their Primes, there will be a greater Number of Factors; and ſo it coincides with the firſt Caſe. (3.) For the laſt Caſe, 'tis already demonſtrated in the firſt; where it's ſhewn, that a greater Number of Prime Factors cannot produce the ſame Number as a leſſer; the Reverſe of which is the preſent Caſe, which we may alſo prove in this Manner, *viz.* A leſſer Number of Primes muſt either have ſome different Prime, or a greater Power of ſome of the ſame Primes; and ſo N cannot meaſure M, and therefore cannot be equal to it: or, the Factors of N are a Part of the ſame Primes that compoſe M, and ſo N will be only a Part of M. Hence again reverſely,

3*d.* The ſame Number, M, cannot be reſolved into a different Number of Prime Factors. For *Exam.* It cannot be reſolved into 3, and alſo into 4 Prime Factors: Nor into any one Number of Prime Factors, with a Variety of Choice: But every Compoſite has a preciſe limited Number of determined Primes; ſo that neither in the particular Primes, nor in their Number, can there be any Variety.

4*th.* Two unequal Compoſite Numbers may be compoſed, either of a different Number, or the ſame Number of Primes: But in both Caſes theſe Factors are either all or part of them different Primes, or ſome Prime common to both, is oftner involved in the one than in the other. But then *obſerve*, That the leſſer Compoſite may have either the leſſer or greater Number of Factors; for that depends upon the Numbers themſelves; thus, $42=2\times3\times7$; and $221=13\times17$.

5*th.* A Number, M, which is the Product of any two or more Factors, whatever they be, as $A\times B\times C$, *&c.* being reſolved into its Primes, theſe can be no other than the Primes into which the Factors, A, B, C, *&c.* can be reſolved; for elſe the ſame Number could be compoſed of different Primes, contrary to *Cor.* 3. And hence again, There is no Prime in the Compoſition of any Power, but thoſe which compoſe the *Root*.

6*th.* No Numbers can meaſure any Power of a Prime Number, but either the Root it ſelf, or ſome other of its Powers; for every other Number has in it ſome other Prime.

7*th.* Whatever Prime Number, N, meaſures any of the Powers of any Number, A, as A^n, the ſame will meaſure the Root A, and all the other Powers; for ſince N meaſures A^n, it muſt be one of its Component Primes, *i, e.* one of the Primes that compoſe A, by *Cor.* 5. therefore N meaſures A, and all its Multiples, or all its other Powers. Hence again,

8*th.* If any Number, N, meaſures A^n, and does not meaſure A, it's a Compoſite Number; for if it were a Prime, it would meaſure A.

Theorem. XI.

Of all the Component Primes of any Number, N, only one (if there be one) can be a Number greater than the Root of the greateſt Integral Square, contained in that Number.

Demon. 1*ſt.* If the given Number, N, is a perfect Square, then it has no Prime in its Compoſition, but thoſe that compoſe the Root. (*Cor.* 5. *Theor.* 10.)

2*d.* If N is not a Square, let A be the Root of the greateſt Integral Square contained it; then is $\overline{A+1}^2$ greater than N: And if we take two Primes greater each than A, they muſt be, the one of them at leaſt equal to $\overline{A+1}$, if not greater; and the

the other greater than this one; consequently their Product will be greater than $\overline{A+1}^2$, *i. e.* greater than N: And therefore they are not both Component Primes of N, since they produce a greater Number.

PROBLEM III.

To find all the Component Primes of any Number.

Rule 1°. Find all the Prime Numbers, not exceeding the Root of the greatest Intergral Square, contained in it. Then,

2°. Beginning with 2, if the given Number is even; or with 3, if it's an odd Number, try if 2 or 3 measures it, and do the same with the Quotes, as long as the same Prime measures them; but when it does not measure, apply the next greater Prime in the same Manner; and go on so till you have tried all the Primes, not exceeding the Root mentioned, or till you find a Quote which is a Prime Number; then all these Primes, which were Measures to the given Number and to the succeeding Quotes, together with that Prime Quote, or the last Quote, to which there was no Measure among the Primes found, by the first Article (which is also a Prime) are the Component Primes of the given Number.

Exam. 1. To find the Primes of 42; the Root of the greatest Square contained in it is 6; and the Primes not exceeding this, are,

2, 3, 5.

2) 42

3) 21

7

And trying 42 by these, I find 2 measures it by 21; but this cannot be measured by 2, therefore I try 3, which measures 21 by 7, which is a Prime Number; as is also known according to the Rule, by this, that neither 3 nor 5 measures it; therefore 2, 3, 7, are the Component Primes of $42=2\times3\times7$.

2	68796
2	34398
3	17199
3	5733
3	1911
7	637
7	91
	13

Exam. 2. To find the Primes of 68796; the Root of the greatest Square contained in it is 262; and the Primes not exceeding this, are 2, 3, 5, 7, 11, 13, &c. and trying, I find 2 measures twice, 3 measures 3 Times; 5 does not measure 637, the last Quote, by 3; therefore I apply 7, which measures twice, and the last Quote is 13, a Prime Number; therefore the Component Prime Factors of 68796, are 2, 2, 3, 3, 3, 7, 7, 13.

Demon. As there is no Matter in what Order any Numbers are apply'd by continual Division, since the last Quote will still be the same: So if certain Primes apply'd by continual Division, in whatever Order, do measure out the given Numbers, then it's plain, from the Nature of Multiplication and Division, that the continual Product of these Divisors, will again produce the same Number: And if certain Prime Factors produce a Number, no other Variety or Choice whatever of Prime Factors, can produce the same Number, by *Cor.* 3. *Theor.* 10. What remains then to be shewn is this; that when we have got a Quote, which neither the Prime last apply'd, nor any greater, not exceeding the Root mentioned, do measure, that Quote is a Prime: The Reason is this; none of the preceding lesser Primes can measure it; for each of these are supposed to be taken out of the given Number as oft as possible; and since none of the Primes, not exceeding the Root mentioned, can measure it, none of these exceeding that Root can measure it, unless it self be one of these Primes; for if another could do it, the Quote would be a Number less than the Root, and must be either a Prime, or measurable by a Prime, which reciprocally would measure it; consequently none greater can do it, unless it self be one of these greater Primes; and therefore it must be a Prime Number.

 Corol.

Corol. By this Method we can know whether a Number is Prime or Compoſite, though we have a Table carried only as far as the Root of the greateſt Square contained in it; for if none of the Primes of this Table meaſures it, then it is a Prime.

Scholiums.

1ſt. If a Number has o's in the firſt Places on the Right-hand, cut them all off, and proceed with the remaining Figures, according to the Rule; and then among the Primes of this Number reckon as many 2's and as many 5's as the Number of o's cut off; becauſe $2\times5=10$; and all together are the Primes ſought.

Again, if an odd Number end with 5, try it with 5 before 3; becauſe 5 will certainly meaſure it, though 3 will not always.

2d. If you have a Table of Primes and Compoſites extending to the given Number, or to the laſt Quote after it is meaſured as oft as poſſible, by 2 and 5; (for which ſee the laſt Article) then ſeek every other Quote (which is not meaſurable by 2 or 5) in the Table; becauſe this ſhews whether it's a Prime or Compoſite: So that being a Prime, you know the Work is ended; and being Compoſite, proceed according to the Rule.

3d. But again; we will often have the Trouble of trying Primes that do not meaſure the given Number, or ſucceeding Quotes, and which would be ſaved, if we knew the leaſt Prime that meaſures any of theſe: Now this we may know by help of the Table of Primes and Compoſites; if the Spaces are ſo marked as to ſhew the leaſt Component Prime of each Compoſite; and how this may be eaſily done I ſhall here explain. Thus:

When you are to number Spaces by any odd Number, ſee firſt whether it's a Prime or Compoſite (by the Table;) if it's Prime, write it in the Places of its Compoſites, *i. e.* in the Spaces where Points are placed by the former Method, unleſs ſome leſſer Prime ſtand there already; as will certainly be, if the Compoſite belonging to that Place, has in it a leſſer Prime; for then it's a Multiple of that leſſer Prime, and is therefore already marked with its leaſt Prime. *Obſerve* alſo, that with this Prime you ſet ſome other Mark, as a Point or Daſh, upon the Place of the ſecond Compoſite, in numbering by this odd Number, in order to know where to begin for the next.

If the Number, by which the Spaces are to be number'd, is Compoſite, all its Multiples are already marked; only you muſt number out the firſt two Periods, that the ſecond Compoſite, in numbering by this Term, may be particularly marked (as I have done in the following Table by a Colon :) in order to know how to begin for the next.

The following Table, carried to 999, is made up in this Manner: The Uſe of which, for finding the Component Primes of any Number, is this;

If the Number is even, meaſure it by 2 as oft as poſſible: If it ends with o's cut them off, and reckon as many 2's and 5's among the Primes ſought; then proceed with the laſt Quote or remaining Number (or with the given Number, if it's odd) thus: See by the Table if it's a Prime or Compoſite; if Prime, the Queſtion is ſolved; if Compoſite, you have its leaſt Component Prime; by which, meaſure it, and ſeek the Quote in the Table; meaſure this by its leaſt Component Prime, and ſo on till you have a Quote which is a Prime; and that Quote, with the preceding Diviſors are the Primes ſought.

	0	1	2	3	4	5	6	7	8	9
1			3	7		3			3	17
3			7	3	13		3	19	11	3
5		3	5	5	3	5	5	3	5	5
7			3		11	3		7	3	
9	$\overline{3}$		11	3			3			3
11		3			3	7	13	3		
13			3		7	3		23	3	11
15	3:	5	5	3	5	5	3	5	5	3
17		3	7		3	11		3	19	7
19		7	3	11		3			3	
21	3	$\overline{11}$	13	3			3	7		3
23		3		17:	3		7	3		13
25	$\overline{5}$	5	$\frac{15}{3}$	5	5	3	$\frac{25}{5}$	5	3	5
27	3			3	7	17	3			3
29		3		7	3	$\overline{23}$	17	$\frac{27}{3}$		
31			3			3		17	3	7
33	3	7		3		13	3		7	3
35	5:	3	5	5	3	5	5	3	5	5
37			3		19	3	7	11	3	
39	3			3		7	3			
41		3		11	$\frac{21}{3}$			3	$\overline{29}$	
43		11:	3	7		3			3	23
45	3	5	5	3	5	5	3	5	5	3
47		3	13		3			3	7	
49	$\overline{7}$		3			3	11	7	3	13
51	3			3	11	19	3		23	3
53		3	11		3	7		3		
55	5	5	3:	5	5	3	5	5	3	5
57	3			3			3			3
59		3	7		3	17		3		7
61		7	3	$\overline{19}$		3			3	$\overline{31}$
63	3:			3			3	7		3
65	5	3	5	5	5	5	5	3	5	5
67			3			3	23	13	3	
69	3	$\overline{13}$		3	7		3		11	3
71		3		7	3		11	3	13	
73			3		11	3			3	7
75	3	5	5	3	5	5:	3:	5	5	3
77	7	3		13	3			3		
79			3			3	7	19	3	11
81	$\frac{9}{3}$			3	13	7	3	11		3
83		3			3:	11		3:		
85	5	5	3	5	5	3	5	5	3	5
87	3	11	7	3			3			3
89		3	$\overline{17}$		3	19	13	3	7	23
91	7		3	17		3		7	3	
93	3			3	17		3	13	19	3
95	5	3:	5	5	3	5	5	3	5	5
97			3		7	3	17		3	

Observe, As the Numbers of the first Column on the Left are carried to 99; so the Numbers on the Head of the Table are to be reckoned as Hundreds: So 1 is 100; 2 is 200, *&c.* And for any odd Number above 99, take the Number of its Hundreds on the Head, and what in it is less than a Hundred on the Side, and in the Angle where the Lines from each meet, is the Place of that Number: So that in composing of the Table, the Numbering is along the Columns from Top to Bottom, from the first Column under 0, and so on in Order, through the rest, (which is so far different from the little Table before given, that there the Series of odd Numbers was set on the Head, and so the Numbering was along the Lines, from left to right.) If we would make a larger Table, then continue the Series of Numbers on the Head, from 9 to 10, 11, 12, *&c.* as far as you please, keeping the same Column on the Left; and reckoning these Numbers on the Head always as expressing so many Hundreds; and continue the Numbering through the Columns in Order; and so the Table may be carried to any Length: For *Example*; If the Series on the Head is carried to 99, it is 9900; and so the greatest Number is 9999. If the Series on the Head is carried to 999, it is 99900; and the greatest Number is 99999.

Again, If there is a Line over any Number, in any Space, it shews that to be the Square Root, and also the least Prime of the Number belonging to that Space. If there are two Numbers in any Space, the greater is the Square Root, and the other is the least Prime.

Observe also, That if a Number is a perfect Square, then we may find all its Primes, by a Table carried no farther than to the square Root, or the odd Number next below it, if the Number is even. For if we take all the prime Factors of the Root twice, these are the prime Factors of the Square.

Another Use that may be made of this Table, is, That by it we can easily find, whether any Number, odd or even, is a perfect Square, and what is the greatest integral Square contained in it, if it is not; thus,

1°. Suppose the Number proposed is an odd Number; seek its Place in the Table, and if it's a composite Number then find the Place next before, and also after it, that is marked with a Colon; and take the Space that is in the Middle betwixt these two (for by the Construction of the Table, the Number of Spaces betwixt them is odd, and therefore has a middle Space.) If the given Number is in that middle

Space

Space, it's a square Number, because the Space next before it marked with a Colon is the Place from whence we begin to number Spaces by some odd Number, which is the Place of the Product of that odd Number by the preceding; and the next Space marked with a Colon is the Place of its Product, by the following; and therefore the middle Space betwixt them is its Square, or Place of its Product by it self. So that the middle Spaces betwixt every two marked with a Colon, are the Places of all the odd square Numbers. Wherefore if the given Number is not in one of these middle Spaces, it cannot be a Square, and the greatest Square contained in it is that belonging to the middle Space next it, towards the Beginning of the Table.

If the given Number is Prime, it's certainly not a Square, and you find the greatest Square contained in it the same way as before.

Observe, If you would find the Root of any Square found in the Table, number how many Spaces there are marked with a Colon from the Beginning of the Table, to that one next preceding the Place of that Square; number as many Terms after 3 in the Series of odd Numbers, the last of them is the Root sought; so if there are 5 pointed Spaces, the Root is 13, the 5th Term after 3, and the Root of 169. But as in making the Table, these Places of Squares are the first composite Spaces in numbering by the several odd Numbers, which are the Root of these Squares; if we not only mark these Spaces with the least Prime, but also with their Root, it will be the more convenient for this purpose; so that if any Space has two Numbers in it, the lesser is its least Prime, and the other the square Root of the Numbers belonging to that Place. And if its least Prime is also its square Root, we may either write it twice, or use some other Mark to show it, as a Line drawn over it, as I have done in this Table: And thus the Places of all odd Squares, and also their Root, are known by Inspection without any Trouble.

2°. If the given Number is even, seek in the Table the odd Number next lesser; if it's a square Number, the given Number can't be a Square, because the Difference of two integral Squares can never be 1, and that odd Square is the greatest contained in it; but if the next lesser odd Number is not a Square, seek by the Table the next odd Square; take its Root, and add 1 to the double of it; if the Sum is equal to the Difference betwixt that Square and the given even Number, then is this a square Number, whose Root is the even Number next above the Root of that odd Square: But if that Sum is either greater or lesser than that Difference, the given Number is not a Square; and if the Sum is greatest, that odd Square is the greatest Square contained in the given Number; but if the Sum is least, add it to the odd Square, and the Sum is the greatest Square contained in the given Number. The Reason of this is obvious from the Nature and Composition of Squares explained in *Book* III. particularly this, that $\overline{A+1}^2=A^2+2\,A+1$.

PROBLEM IV.

To find all the different Numbers that measure any given Number.

Rule. 1°. Find all the Component prime Factors of the given Number by the last Problem; then,

2°. Set them all in a Line; but those that are oftner than once involved, set them down but once, and instead of the rest of them set down the Series of their superior Powers, till you have a Power whose Index is the Number of Times that that one is involved in the given Number; these are so many of the different Measures sought. And though it is in Effect the same Thing, which Prime is first set down, yet one Order may prove more convenient than another for the following Part of the Work; there-

therefore set down first all those Primes, which are but once involved, and then those that are oftner, with their superior Powers as above, setting those first that are least involved; then,

3°. Beginning from the left Hand, multiply the first Number by the second, and set the Product under the second; then by the third (of those first set down) multiply all the preceding Numbers, setting the Products under the third; and so on, by every succeeding one (of the Numbers first set down) multiply all the preceding Numbers, setting the Product under their Multiplier.

But observe, That when you come to use for a Multiplier any superior Power of any of the Primes, you must not by it multiply any of the lesser Powers of the same Root, nor any of the Numbers standing under them; only multiply all the other Numbers preceding these, *i. e.* all the same Numbers which were multiplied by the Root. And to get these Products most conveniently, take the Root, and by it multiply all the Numbers standing under it, and set these Products under the Squares; then multiply all these Products set under the Square, by the same Root, and set the Products under the Cube, and so on.

Exam. 210; its Component prime Factors are 2, 3, 5, 7; for $2\times3\times5\times7=210$. And its several Measures are these following, disposed and found according to the Rule.

2	3	5	7
	6	10	14
		15	21
		30	42
			35
			70
			105
			210

First I set down 2, 3, 5, 7, then I multiply 2 by 3, and set the Product 6 under 3; next I multiply 2, and also 3, 6, by 5, and set the Products 10, 15, 30 under 5; then I multiply all the preceding Numbers by 7, and set the Products under 7; and all these Numbers are the Numbers sought.

Exam. 2*d.* 6552; its Component prime Factors are these, 2, 2, 2, 3, 3, 7, 13, whose continual Product is 6552, and its several Measures are these following; found thus,

7 :	13 :	3 :	9 :	2 :	4 :	8
	91	21	63	14	28	56
		39	117	26	52	104
		273	819	182	364	728
				6	12	24
				42	84	168
				78	156	312
				546	1092	2184
				18	36	72
				126	252	504
				234	468	936
				1638	3276	6552

The Primes 7, 13, are but once involved, and so I set them first down; 3 is twice involved, and I set down 3, 9; then 2 is thrice involved, and I set down 2, 4, 8. Then I begin and multiply 7 by 13, and set the Product 91 under 13. Next I multiply all the preceding Numbers by 3, and set the Products under 3; then I multiply by 9 all the Numbers preceding the Column, over which 3, the Root of 9, stands, and which I do, by multiplying all the Numbers standing under 3 by 3; then multiplying all the preceding Numbers by 2, I set the Products under 2; and for the following Numbers 4, 8, which are Powers of 2, by them I multiply all the Numbers preceding that Column over which the Root stands, *i. e.* all these which were multiplied by the Root; and this I do by multiplying all the Numbers standing under 2, by 2, and setting the Products under 4; then by the same 2 I multiply all the Numbers standing under 4, and set the Products under 8.

DEMON.

$$\begin{array}{rrrr} a : & b : & c : & d \\ & ab : & ac : & ad \\ & & bc : & bd \\ & & abc : & cd \\ & & & abd \\ & & & acd \\ & & & bcd \\ & & & abcd \end{array}$$

Demon. 1°. Suppose any different Primes, a, b, c, d, &c. their Product $a\,b\,c\,d$, &c. is measurable only by these Primes, or the Products of any two or more of them (*Theor.* X.) And to find all these Measures, we shall first suppose only one prime Number, a, that has no Measure but it self (standing alone in the first Column) but suppose another Prime, b, multiplied into it, then the Product ab has for Measures a, b, and ab (which make the first and second Column.) Again, let another Prime be involved, it's evident that Product abc has for Measures all the Measures of the preceding Product ab, with all these, in which the new Prime can be concerned, which plainly can be no other than c it self with its Products, into all the Measures of ab (which are all the Numbers of the preceding two Columns.) Join another primed d, the Product $a\,b\,c\,d$ has for Measures all the Measures of $a\,b\,c$, together with all these Products in which the new Prime d can be concerned, *i. e.* d it self, and its Products, by all the Measures of $a\,b\,c$ (which are all the Numbers of the preceding Columns) and so it goes on whatever Number of different Primes we suppose; which is all according to the Rule.

2°. If any of the different Primes are oftner than once involved, it's evident that all their Powers, to that one whose Index is the Number of Involutions of the Root, are Measures of the given Number. Then having by the Root multiplied all the Numbers standing already in the Columns preceding it, we have all the Measures of the given Number in which that Root is but once involved; and to have those in which it's twice or thrice involved, [or in which its Square or Cube, *&c.* are severally concerned, according to the different Powers of it involved in the given Number] it's plain we must multiply these several Powers into all the Numbers preceding the Root (*i. e.* those into which the Root was multiplied) but having done this we must not also multiply any of those Powers into any other of them, nor into the Numbers standing under these others, because those new Products would either contain a greater Power of the same Root, than the given Number contains, and so could not measure it (by *Theorem* X.) or would coincide with the Products of some of the higher Powers, by the Numbers preceding the Root. Thus, in the preceding *Exam.* 2d. if we multiply all the Numbers standing in the Column, which has 2 in the Top, by 4, this will be the same as the Column which has 8 on the Top; all which Numbers under 8 are the Products of 8, by all the Numbers preceding the Root 2, because the Numbers under 2 are the Products of all the preceding Numbers by 2, which Products therefore multiplied again by 4, will be equal to the Products of these preceding by 8, since $4\times2=8$. Again, we are not to multiply any of these in the Columns under 2 or 4 by 8, because the Numbers of the one of these Columns have 2 involved in them, (being the Products of all the preceding Numbers by 2) and the other has 4 involved (being the Product of the same Number by 4) and consequently if these were again multiplied by 8, they would have a greater Power of 2, than the given Number has, in which 8 is the greatest Power of 2. But then to find the Products of all the Numbers preceding the Root by the higher Powers, it's evident we can find them by multiplying gradually all the Numbers under each Power by the Root: For these under the Root are the Products of the preceding by the Root; therefore these Products multiplied again by the Root are the Products of the same preceding Numbers by the Square; and so on.

Coroll. If it's required to find all the *aliquot* Parts of any Number, find all the Measures of it; these (excluding the given Number it self) are the *aliquot* Parts sought.

Theo-

THEOREM XII.

If one Number, A, be Incommensurable to each of two or more other Numbers, B, C, D, *&c.* 'tis also Incommensurable to their Product B, C, D, *&c.* and reversely.

DEMON. Since A is Incommensurable to each of the Numbers B, C, D, *&c.* therefore A has in its Composition no Prime common with any in the Composition of any of these (*Cor.* 3d. *Theor.* VII.) and consequently none that's common to the Product of any two or more of these; because the Primes of these Products are no other than the Primes of their several Factors (*Cor.* 5. *Theo.* X.) The *Reverse* is plain from the same Principle; for if A is Incommensurable, or has no Prime common with the Product of any two or more of these Numbers B, C, D, *&c.* then it has no Prime common with any one of these; for if it had, it would also have a Prime common to the Product, since the Product has no other Prime than what belongs to the Factors.

SCHOL. Tho' this be the direct Demonstration, yet it may be proved very simply after *Euclid*'s Way, thus: If A and B C are Commensurable, then what measures A, one of two Incommensurables, is Incommensurable with the other, B (*Theo.* II.) and because it is Incommensurable with B, yet measures B C, therefore it measures C (*Theo.* VI.) and because it measures also A, hence A, C are not Incommensurable, contrary to Supposirion, and therefore A and B C are Incommensurable. For the same Reason A and B C D ($=B\,C\times D$) are Incommensurable; and so on, whatever Number of Factors you suppose to each of which A is Incommensurable.

For the *Reverse*; if any Number measures A, and any one of these others, it will also measure the Product of them all (which is a Multiple of that one) therefore A is not Prime with B C D E, *&c.* contrary to Supposition.

COROL. If any Number, A, is Incommensurable to another B, it's so also to all the Powers of that other, as B^2, B^3, *&c.* These Powers being the Products of Numbers to each of which A is Incommensurable; for they are all the same Number B, since $B^2=B\times B$, and $B^3=B\times B\times B$, and so on.

SCHOL. In forming the contrary to this Theorem, there must be some Limitations, thus,

1°. If A is Commensurable to B, and C, *&c.* its so also to their Product B C, *&c.* for it would be so, though it were only Commensurable to one of the Factors.

2°. If A is Commensurable to B C, *&c.* 'tis so also to one at least of the Factors (for else it were Incommensurable to the Product) but not necessarily so with them all; as here, 4 is Composite to 6, and to $6\times9=54$, but not to 9; so that as in the first Part, the contrary is larger, or requires fewer Conditions than the Theorem, in the second Part it extends not so far, or draws not so great a Consequence.

THEOREM XIII.

If any Numbers, A, B, C, *&c.* are Incommensurable, each to each of any other Numbers, M, N, O, *&c.* then the Product of any two or more of the first Set is Incommensurable to the Product of any two or more of the second Set. And the Reverse.

DEMON. This follows either from the same Principle as the last, *viz.* the Numbers, A, B, C, *&c.* having no common Prime with any of the Numbers M, N, O, *&c.* none of the Products of any two or more of the one, has any common Prime with any one, or the Product of any two or more of the other. Or we may deduce it from the last, *thus*, A is Incommensurable with M, N, O, *&c.* by Supposition, and therefore it's so with M N, N O, M O, or M N O (*Theor.* XII.) But the same is also true of B and C; consequently each of these last Products considered now as one Number, is Incommensurable

mensurable to the Products of any two or more of the former, that is, with AB, AC, BC, or ABC. The same Reasoning is equally good, how many Numbers soever there be in each Set.

For the Reverse, *viz.* That if all the Products of each two, or more Numbers, taken out of each Set, are Incommensurable to one another, so are the several Factors of the one Set to those of the other: This is plain from hence, that if we suppose any two of them, as A and M, to be Commensurable, then that Number which measures each of them, will measure any Product in which they are concerned, and so these Products will be Commensurable, contrary to Supposition.

Coroll. If two Numbers, A and B, are Incommensurable, any Power of the one, as A^n, is Incommensurable to any Power of the other, as B^n or B^m; and the Reverse, if A^n, B^m, are Incommensurable, so are A, B.

Exam. 3 and 5 are Incommensurable, and so are 9, and 25 their Squares; also 27, 25, the Cube of 3 and the Square of 5; wherefore if any Fraction, $\frac{A}{B}$, is in its lowest Terms, any of its Powers is so also, as $\frac{A^m}{B^m}$; and the Reverse.

Theorem XIV.

If any Number, A, measures another B, then will any Power of A, as A^n, measure the like Power of the other, B^n. And reversely, If A^n measure B^n, so will A measure B; and also, every other Power of A will measure the like Power of B.

Demon. A has no Prime but what's in B, nor any oftner involved, else it could not measure it; but the prime Factors of A being equally involved in A^n, as those of B are in B^n, it follows, that as there is no Prime in A^n but what is B^n, so there is none oftner involved, and consequently A^n measures B^n. The Reverse is plain from the same Principles. Or we may make the whole Demonstration as simply thus.

$A = 3$	measures	$B = 6$	By	2
$A^2 = 9$		$B^2 = 36$		4
$A^3 = 27$		$B^3 = 216$		8

Let A measure B by D, that is, $B \div A = D$; then is $A : B :: 1 : D$, and (by *Cor.* 11th. *Theo.* III. *Book* IV. *Chap.* IV.) $A^n : B^n :: 1 : D^n$; but 1 measures D^n, therefore so does A^n measure B^n. For the Reverse, Let $B^n \div A^n = D$, which is an Integer by Supposition; then $A^n : B^n : 1 :: D$. But the first 3 being Powers of the Order n, therefore (by *Theo.* XII. *Book* IV. *Chap.* IV.) D is a Power of the Order n: Suppose it $= X^n$, so that $A^n, B^n :: 1 : X^n$; wherefore (by *Cor.* 2. *Theo.* III. *Ch.* IV.) $A : B :: 1 : X$. But 1 measures X, and consequently A measures B, and hence every other Power of A will measure the like Power of B.

Theorem XV.

If any Composite Number measures another like Composite, so will the several Factors of the first measure the correspondent Factors of the other.

Thus, if A, B are like Composites, and a, b two of the similar Factors; then if A measure B, so will a measure b; and *Reversely*, If a measure b, so will A measure B.

Demon. Like Composites are in the Ratio of the like Powers of any of the correspondent Factors, that is, $A : B :: a^n : b^n$ (by *Theo.* V. *Book* IV. *Chap.* IV.) But by Supposition A measures B, therefore a^n measures b^n, and therefore (by the last) a measures b.

For the Reverse, If a measures b then does A^n measure B^n (by the last) but also $a^n : b^n :: A : B$, and a^n measures b^n, therefore A measures B.

THEOREM XVI.

If two Numbers, A, B, are composed of an equal Number of different Primes, they cannot be like Composites.

$A=3\times5\times13$, &c.
$B=2\times7\times11$, &c.

DEMON. 1°. They cannot be so, by taking these Primes for the similar Factors, because if four Numbers are : : *l* they cannot be all Primes (*Cor.* 3d. *Theo.* V.) consequently no two of the prime Factors of the one Compound are : : *l* with any two of the other; therefore, *Lastly*, Though these Compounds be similar, yet the Similarity does not depend on the Factors being taken in that manner.

2°. They are not Similar by taking any Products of these prime Factors for the similar Factors; for any one, or the Product of any two or more of the Primes of the one Compound, is Incommensurable with any one, or the Product of any two or more of the Primes of the other (*Theo.* XIII.) therefore let us take any two of such Products out of the Primes of each Compound, or a single Prime, and a Product out of each, or two Primes out of the one, and two Products out of the other, or, *Lastly*, a Prime and a Product out of the one, and two Primes or two Products out of the other; yet these cannot be : : *l*, because each Couplet of the compared Terms are Incommensurable, which is inconsistent with Proportionality; for if *a* : *b* : : A : B, then if *a*, *b* are Incommensurable, A, B, are Commensurable (by *Cor.* 2d. *Theo.* IV.)

Lastly, Since Numbers can be but one particular Way composed of prime Factors, and no other Numbers can measure them, but these or their Products; and since, as we have now seen, neither these nor their Products can make similar Factors, they can have none such, *i. e.* they cannot be like Composites, or they cannot be resolved into an equal Number of similar Factors.

PROBLEM V.

To find the least common Multiple to any given Numbers, A, B, C, D, &c. which are all different.

N. B. For brevity we shall put $\times le$, for Multiple; *co* : $\times le$, for common Multiple; and *l* : $\times le$, for least common Multiple.

Case I. *For two Numbers*, A, B.

Rule (1^mo^.) If they are Incommensurable, as 4, 7, their Product A B=28 is their *l* : $\times le$. But, (2°.) If they are Commensurable, find the two least in their Ratio, as *a* : *b* (by *Cor.* 3. *Theor.* III.) so that these are : : *l*, *viz.* A : B : : *a* : *b*, then is A*b*=*a*B, the Number sought.

Exam.
A =4 . B=18
a=2 . *b*= 9
A*b*=*a*B = 36

A : B
AB
N
C : D

DEMON. 1°. If A, B are Incommensurable, then is A$\times$B their *l* : $\times le$. It is a *co* : $\times le$; and to shew that it is the *l* : $\times le$, let A and B measure any other Number, as N, and let the Quotes be C, D, thus, A (N=C, and B) N=D; but the Dividend N being common, the Divisors and Quotes are reciprocally : : *l*, *that is*, A : B : : D : C; and A, B being Incommensurable, they measure D, C equally (*Theor.* II.) Again, N=A C and A B : A C : : B : C, therefore A B : N : : B : C. But B measures C, therefore A B measures N, which is therefore greater than A B, or its equal; consequently any other Number than A B, which is a *co* : $\times le$ to A and B, is greater than A B, therefore this is the *l* : $\times le$.

2°. If A, B are Commensurable, and *a* : *b*, the least in the same Ratio with A : B, then is A*b*, or *a*B, a *co* : $\times le$ to A, B; for A*b* is $\times le$ of A, and *a*B of B, also A*b*=*a*B. Again, I say, A*b*=*a*B is their *l* : $\times le$, for let them measure any other Number, as N,

 by

by C, D, fo that as before, A : B : : D : C ; then becaufe A : B : : *a* : *b*, therefore *a* : *b* : : D : C ; but *a* : *b* being Incommenfurable, *a* meafures D, and *b* meafures C equally ; alfo, A *b* : A C : : *b* : C, and A C=N, therefore A *b* : N : : *b* : C ; and fince *b* meafures C, fo will A *b* meafure N, which is therefore either the fame Number, or greater then A *b* ; hence A *b* is the *l* : ×*le* fought.

Coroll. The leaft *co* : ×*le* of two Numbers, A, B, meafures all their other *co* : ×*les*, which therefore are the Multiples of it ; for it's proved that A B, or A *b*, meafures any other Number, N, which is fuppofed to be a *co* : ×*le* to A, B.

Cafe II. *For more than two Numbers, as* A, B, C, D, *&c.*

Rule. Find the *l* : ×*le* to any two of them ; and then the *l* : ×*le* to the Number laft found, and another of them ; and fo on till you go through them all ; and the laft found is the Number fought.

Exam.

4 :	6 :	7 :	8
	12 :	84 :	168
A :	B :	C :	D
	m :	n :	o

Demon. Let A, B, C, D, be any Numbers ; and the *l* : ×*le* of A, B be *m* ; of *m*, C, be *n* ; and of *n*, D, be O. I fay *n* is the *l* : ×*le* of A, B, C ; and O, that of A, B, C, D ; for,

1°. A, B meafure *m*, and *m*, C meafure *n*, therefore A, B, C meafure *n*. Again, *n*, D meafure O, and A, B, C, meafure *n*, therefore A, B, C, D, meafure O ; and fo it proceeds for ever, *i. e* each Number found in the Courfe of the Operation is a *co* : ×*le* to all the given Numbers fo far.

2°. They are their leaft *co* : ×*les* ; for what is a ×*le* of A, B, is fo of *m*, and what is a ×*le* of *m*, C, is fo of *n* (*Coroll. Cafe* I.) therefore what is a ×*le* of A, B, C, is fo of *n*, and confequently it is not lefs than *n* ; which is therefore the *l* : ×*le* of A, B, C. Again, what is a ×*le* of A, B, C, is fo of *n* (by the laft Step) and what is a ×*le* of *n*, D, is fo of O (*Coroll. Cafe* I.) therefore what is a ×*le* of A, B, C, D, is fo of O, and confequently is not lefs than O, which is therefore the *l* : ×*le* of A, B, C, D. The fame Reafoning is evidently good from one Step to another for ever ; from which we have gained the following Truth, *viz.*

Gen. Coroll. The leaft common Multiple of any Numbers, A, B, C, *&c.* is an *aliquot* Part of all their other *co* : ×*les*, or thefe are Multiples of that.

Scholium.

1*ft*. The preceding general Corollary may be demonftrated independently of any Cafe of this Problem. *Thus*, Take any Number, N, which *l*, the *l* : ×*le* of A, B, C, *&c.* does not meafure, I fay it can be none of the *co* : ×*les* of A, B, C, *&c.* for fince *l* does not meafure N, (the Quote of *l*) N is a mix'd Number, fuppofe, $A+\frac{r}{l}$. Hence, by the Nature of Divifion, N=A*l*+*r*. Now A, B, C, *&c.* do each by Suppofition meafure *l*, and confequently they meafure its Multiple A*l* ; but *l* is the leaft Number they meafure, therefore they cannot meafure *r*, which is lefs than *l*, being the Remainder of a Divifion in which *l* is the Divifor. *Laftly*, What meafures one Part, and not the other Part *r*, cannot meafure the whole A*l*+*r*, which being equal to N, confequently A, B, C, *&c.* cannot each of them meafure N, a Number which their *l* : ×*le* *l* does not meafure.

2*d*. Though it be true that *m* meafures *n*, yet we cannot hence conclude, that the *l* : ×*le* of certain Numbers is a greater Number than that of a Part of thefe Numbers, becaufe they may happen to be equal ; fo *m* may be equal to *n*, as in this Example. The *l* : ×*le* of 3, 4 is 12, which is alfo the *l* : ×*le* of 3, 4, 6. This however is certain, that the *l* : ×*le* of the whole given Numbers can never be lefs than that of a Part of them ; for it's fhewn, that *m* muft meafure *n* ; or, for this obvious Reafon, that the

$l:\times le$ of the Whole is a $co:\times le$ to any Part of them, and can't be less than their $l:\times le$, which would be absurd.

Again, the Case in which it happens that the $l:\times le$ of the Whole is equal to that of a Part, is this, *viz.* when one of the given Numbers is equal to, or an *aliquot* Part of the least $co:\times le$ of a Part; for then it's manifest from the Manner of the Operation, that this Equality must happen, as you'll see by examining these Examples:

Given Numbers	3 . 4 . 6	3 . 4 . 5 . 12
$l:\times les$	12 . 12	12 . 60 . 60

3*d.* If Numbers are given, to find their $l:\times le$; and if several of these Numbers are the same, or equal, as A, A, B, C, It's plain that we have no more to do, but find the $l:\times le$ to all that are really different Numbers. But if we should apply the Rule to all the given Numbers, without considering that some of them are the same, the same Number must necessarily answer for the $l:\times le$; the Reason of which will also appear from the Nature of the Operation; as here, the $l:\times le$ of 3, 4 is 12; and if to the given Numbers you join another 4, the $l:\times le$ is not thereby changed, because 12 being a $\times le$ of 4, their Ratio in lowest Terms is 3 : 1; so that the Number found upon joining of the new 4, must necessarily be the same as the last, for 12 : 4 : : 3 : 1, and according to the Rule 12×1 is the Number sought, which is the last found $co:\times le$.

4*th.* This Problem is to the same Purpose with this, *viz.* To find the least Number that has Parts denominated by certain given Numbers; for a Number which has a Part of the Denomination A, B, *&c.* must be measurable by A and by B, *&c.* and the least Number which has such Parts must be the least Number measurable by (or the least common Multiple of A, B, *&c.*

Theorem XVII.

The prime Number which measures the $l:\times le$ of certain Numbers, A, B, C, D, *&c.* will also measure some one of these Numbers.

Demon. 1°. Take two Numbers, A, B, if they are Incommensurable, then is AB their $l:\times le$; and if a Prime p measures A B, it measures A or B (*Coroll.* 1. *Theo.* VI.) If A, B are Commensurable, let a, b, be Incommensurable (or least in the same Ratio) then is $Ab=aB$, the $l:\times le$ of A, B; and if p measure Ab and aB, and does not measure A, nor B, it must measure both b and a (*Coroll.* 1. *Theor.* VI.) which therefore are not Incommensurable, contrary to Supposition.

A . B . C . D
m . n . o

2°. If there are more then two Numbers, as A, B, C, D, suppose m the $l:\times le$ of A : B, and n the $l:\times le$ of m : C (*i. e.* of A, B, C.) Again, o the $l:\times le$ of n : D (*i. e.* of A, B, C, D); then if p measures o, it must measure n or D (by the first Article) if not D then n, and consequently it measures either m or C; if not C then n, and consequently it measures either A or B; so that it must necessarily measure one of the Numbers A, B, C, or D; but howmany Numbers soever there be, the same Reasoning will evidently hold through them all.

Theorem XVIII.

Of certain Numbers, A, B, C, D, *&c.* if each of them be separately Incommensurable to any Number, N, so is M their $l:\times le$.

Demon. If any Number measures N, M, then some prime Number measures them (*Theor.* VII. *Coroll.* 3.) which therefore (by the last) measures some one of the given Numbers; consequently each of these is not Incommensurable to N, contrary to Supposition.

Theo-

THEOREM XIX.

The greatest Common Measure, m, and the least Common Multiple, M, of two Numbers, A, B, are reciprocally Proportional with them; thus, $m : A :: B : M$.

DEMON. Take $a : b$, least in the Ratio of A, B, then is $Ab = aB = M$; and $A = ma$, $B = mb$. Hence $Ab = amb = M$: But it's manifest that $m : am :: bm : bam$, that is, $m : A :: B : M$.

PROBLEM VI.

Any Number of Ratios being given, to continue them in Integers, in any Order, and in their least Terms.

Rule. This *Problem* may be solved two different Ways.

Method 1. Continue them by *Problem* I. *B.* IV. *Ch.* IV. and then, reduce the Numbers found to their least Terms (by *Corol.* 3. *Theor.* III.)

Observe, If the given Ratios are in their lowest Terms, then being continued in this Manner, the Numbers found, will, in some Cases, be in the least Terms, but not in every Case, as these Examples shew, *viz* $3 : 8$, and $4 : 7$, which are in their lowest Terms, being continued, make this Series; $12 : 32 : 56$, which has a Common Measure, 4, which reduces it to this, $3 : 8 : 14$. Again, These, $2 : 3$, and $4 : 7$, make this Series, $8 : 12 : 21$, which is in its lowest Terms. But if the given Ratios are equal, or the same Ratio to be continued, then this Method is good. But I shall refer the Demonstration till after the second general Method is explain'd.

$A : B,\ C : D$
$E : F : G$
$a : b : c$

Method 2. The given Ratios being in their lowest Terms (or reduced to such.) Then suppose only two Ratios, as $A : B$, and $C : D$; they are continued thus; Find F, the $l{:}\times le$ of $B : C$, (*Prob.* V.) then find E and G, by these Proportions, *viz.* $B : A :: F : E$, and $C : D :: F : G$; then are $E : F : G$ the Numbers sought.

DEMON. That $E : F : G$, continue the given Ratios, is manifest from the Construction: For $A : B :: E : F$, and $C : D :: F : G$. And that they are the least Numbers which continue the given Ratios, is thus proved: Suppose any other Numbers, $a : b : c$, which are in the given Ratios; I say, these are greater Numbers than $E : F : G$; for, $A : B :: a : b$; but $A : B$ are least in their Ratio; therefore B measures b. Again, $C : D :: b : c$; but $C : D$ are least in their Ratio; therefore C measures b; therefore also, F, the $l{:}\times le$ of B, C, must measure b (by *Gen. Corol. Prob.* V.) Again, Since $F : b :: E : a :: G : c$, and F measures b; therefore E and G, do equally measure a and c; consequently a, b, c, are greater than E, F, G, which are therefore the least that continue the given Ratios.

$A : B,\ C : D,\ E : F$
$G : H : I$
$K : L : M : N$
$a : b : c : d$

II. Suppose 3 given Ratios, $A : B$, $C : D$, $E : F$. Find as before, $G : H : I$, the 3 least Numbers that continue the 2 first Ratios; then find M, the $l{:}\times le$ to $I : E$. And lastly, find K, L, N, by these Proportions, $I : M :: H : L :: G : K$, and this $E : M :: F : N$; then are these the Numbers sought, *viz.* $K : L : M : N$.

DEMON. 'Tis plain they continue the given Ratios; and they are the least that do so: For suppose any other Numbers, that also continue the given Ratios, as $a : b : c : d$. I prove that they must be greater than $K : L : M : N$; thus, $A : B :: a : b$; but $A : B$ are least in their Ratio; therefore B measures b. Again, $C : D :: b : c$, and $C : D$ are least in their Ratio; therefore C measures b. Hence again, H, the $l{:}\times le$ of B, C, measures b; but also $H : I :: b : c$; therefore I measures c. Again, $E : F ::$ $c : d$,

c : *d*, and E : F, being in least Terms, E measures *c*; therefore, M, the *l* : ×*le* of I : E, measures *c*, but M : *c* : : K : *a* : : L : *b* : : N : *d*; and because M measures *c*, therefore K, L, N, do equally measure *a*, *b*, *d*: Hence *a*, *b*, *c*, *d*, are greater than K, L, M, N, which are therefore the least that continue the given Ratios.

III. Suppose four Ratios, A : B, C : D, E : F, G : H; continue the first two by the Numbers, I : K : L (K being the *l* : ×*le* of B, C) and continue the first three in the Numbers M : N : O : P, (O being the *l* : ×*le* of L : E) then find T, the *l* : ×*le* of P : G. And lastly, find Q, R, S, U, by these Proportions, *viz.* P : T : : M : Q : : N : R : : O : S, also G : T : : H : U; then Q, R, S, T, U, are the Numbers sought.

A : B, C : D, E : F, G : H
I : K : L
M : N : O : P
Q : R : S : T : U
a : *b* : *c* : *d* : *e*

Demon. They continue the given Ratios plainly; and that they are the least Numbers which do so, I prove, by shewing that any other Numbers, *a*, *b*, *c*, *d*, *e*, which continue the same Ratios, are greater than they: Thus, A : B : : *a* : *b*, and C : D : : *b* : *c*; therefore B and C do both measure *b*; and consequently, K, the *l* : ×*le* of B, C, does measure *b*; and because K : L : : *b* : *c*, therefore L measures *c*. Again, E : F : : *c* : *d*, therefore E measures *c*: And hence again, O, the *l* : ×*le* of L, E, does also measure *c*; and because O : P : : *c* : *d*, and G : H : : *d* : *e*; therefore both P and G measure *d*; and consequently T, the *l* : ×*le* of P, G, does measure *d*: But now T : *d* : : Q : *a* : : R : *b* : : S : *c* : U : *e*; and because T measures *d*, therefore Q, R, S, U, do equally measure *a*, *b*, *c*, *e*. Hence *a*, *b*, *c*, *d*, *e*, are greater than Q, R, S, T, U, which therefore, lastly, are the least that continue the given Ratios.

As I think the Progress of this Rule, and its Reason, *ad Infinitum*, will be clearly perceived after what's explained, I shall carry it no further; but only make this general *Remark*, *viz.* That for every succeeding Series, or every new Ratio added, we begin always with finding the *l* : ×*le* of the Antecedent of the new Ratio, and the last Term of the preceding Series; and making that the last but one of the Series sought, we find the rest by Proportions drawn from this and the Terms of the preceding Series, together with the Ratio added; then the Term which was first found, is a principal *Medium* for demonstrating that the Numbers found continue the given Ratios in the least Terms.

Scholiums.

1*st*. Either of these two Methods are *universal*, whether the given Ratios are different or the same; and are indeed the only Methods that solve this *Problem*, in all Cases; but in that Case where the same Ratio is to be continued, if we take it in its least Terms, and continue it by the *Problem* refered to in the first *Method*, the Series found will be in its least Terms; as I have already said in an Observation after the first *Method*; and which I shall now demonstrate: Thus,

2 : 3
2 : 3
4 : 6 : 9
2 : 3
8 : 12 : 18 : 27
2 : 3
16 : 24 : 36 : 54 : 81
&c.

By the Method of this Operation it's manifest, that the Extremes of the Series are like Powers of the Terms of the given Ratio; and these being Incommensurable (or in their least Terms) their like Powers must be so (*Theor.* XIII. *Cor.*) But the Extremes of a Series being Incommensurable, the Whole must be so.

Or, The Truth of this will also appear from the preceding second *Method*: For if you compare this Operation exactly with the preceding *Rule*, you will find it's the very same Work which would be made by that *Rule*; only in this particular Case it is easier Work, because of its being the same Ratio, and in its least Terms.

Ob-

Obſerve alſo, That the ſame Series is made by the Series of the Powers of the given Terms of the Ratio, multiplied together in a reverſe Order; as has been explained in *Schol.* to *Theor.* VIII. *B.* IV. *Ch.* IV. and which is expreſſed in this general Form:

$$A^n : A^{n-1} \times B : A^{n-2} \times B^2 : \&c. : A \times B^{n-1} : B^n.$$

2*d.* In *B.* IV. *Ch.* IV. *Prob.* I. *Schol.* 2*d.* I have obſerved, That tho' we take all the poſſible Expreſſions of the ſame Ratio, and continue a Series by each of them, yet this would not exhauſt all the Variety of Numbers, in which a Series in the ſame Ratio might appear; and the univerſal Method of finding all that Variety was referred to another Place: And here we plainly have it: Thus,

Raiſe a Series from the leaſt Terms of the given Ratio, to the propoſed Number of Terms; then ſucceſſively multiply this Series by 2, 3, 4, *&c.* taking the Multipliers in the Order of the natural Series, *ad Infinitum*; and you ſhall have hereby the Series in all its poſſible Variety: For it is firſt in its loweſt Terms, and then in all the Multiples of theſe; which neceſſarily exhauſt all the Variety: Becauſe a Series is either in its leaſt Terms, or in the Multiples of theſe (*Theor.* I.) Therefore, if A : B expreſs the leaſt Terms of any Ratio, all the poſſible Variety may be expreſſed in this general Form:

$$A^n x : A^{n-1}Bx : A^{n-2}B^2x : \&c. : AB^{n-1}x : B^n x.$$

Where, according to the infinite Variety of Numbers, in the Order of the natural Series, that x may repreſent, ſo will the Series be different: So x, being 1, the Series is in its leaſt Terms; becauſe 1 does not multiply; but in all other Values of x, the Series is Multiple of the leaſt Terms.

Obſerve alſo, That if x is any Number, which is a Power of the Order n ($n+1$, being the Number of Terms of the Series) then the Series, with that Value of x, is what would ariſe by working, according to the Rule in *Schol.* I. with ſome of the greater Terms of the Ratio, which are always Equimultiples of the leaſt Terms; *Thus*, Let $x=r^n$, then is $A^n x = A^n r^n = \overline{Ar}|^n$ (by *Theor.* I. *B.* III. *Ch.* I.) and $B^n x = B^n r^n = \overline{Br}|^n$, and theſe Extremes are plainly like Powers of $Ar : Br$, Equimultiples of the leaſt Terms of the Ratio A : B. But all the greater Terms of this Ratio are univerſally expreſſed by $Ar : Br$; and the Extremes of a Series rais'd from theſe, are univerſally $\overline{Ar}|^n : \overline{Br}|^n$: Which ſhews that all the Variety that would ariſe by working with all the different greater Terms of the Ratio, is had by making x any Power of the Order n. And laſtly, If x is any Number, which is not a Power of the Order n, then we have all the reſt of the Variety, which cannot ariſe from working according to the preceding Rule; becauſe whatever Terms of the Ratio we work with, as $Ar : Br$, the Extremes will be $\overline{Ar}|^n : \overline{Br}|^n$, ſimilar Powers; but x not being a Power of the Order n, $A^n x : B^n x$ cannot be ſuch Powers (*Cor.* 4. *Theor.* II. *B.* III. *Ch.* I.)

Obſerve, laſtly, That in a Multiple Ratio, or whoſe leſſer Extreme is 1, all the Variety is had by raiſing a Series from all the different Expreſſions of the Ratio; which in this Caſe only has the ſame Effect, as multiplying the leaſt Terms of the Series.

§. II.

§. II. *Relating all to* Geometrical Progressions.

THEOREM XX.

If a Geometrical Series is in its least Terms (*i. e.* all the Terms Incommensurable) the Extremes are such like Powers of the least Terms of the Ratio, whose Index is the Number of Terms less 1. And if the Series is not in its least Terms, the Extremes are Equimultiples of these like Powers of the least Terms of the Ratio.

DEMON. This is plain from the Method of raising a Series in its least Terms, explained in the preceding Problem. (See *Schol.* 1.) For if a Series rais'd from the least Terms of the Ratio, is in its least Terms, and the Extremes are the $n-1$ Powers of the least Terms of the Ratio; also, since two Series in the same Ratio cannot be both in least Terms, and consist of different Numbers; therefore the first Thing propos'd is manifest. Again, if the Series is not in least Terms, it consists of Equimultiples of the least Terms, (see *Schol.* 2. preceding *Problem*) and consequently the Extremes are Equimultiples of the $n-1$, Powers of the least Terms of the common Ratio.

SCHOL. As every Series, not in its least Terms, are Equimultiples of the least; so, according as the common Multiplier is, or is not, a Power of the Order n, the Extremes will be, or will not be, such Powers: And comparing this with the Extremes of the Series, in its least Terms, we may observe, that the Extremes of every Series have one of these three Qualities.

1°. They are Powers of the Order n; but not also Equimultiples of other Numbers, which are such Powers: And this happens only when the Series is in least Terms.

2°. They may be Powers of the Order n, and also Equimultiples of such Powers; which happens only when the least Terms of the Series are multiplied by some Power of the Order n; or, when a Series is raised from any such Terms of the Ratio as are not the least.

3°. They are Equimultiples of Powers of the Order n; but are not such Powers themselves: Which happens when the least Terms of the Ratio (or Series) are multiplied by some Number which is not a Power of the Order n.

COROLLARIES.

1st. If four Numbers are ::l, A : B :: C : D; and if any two of the Comparative Terms, as A, B, or A, C, are Similar Powers of any Order; the other two, C, D, or B, D, are either Similar Powers of the same Order, or they are Equimultiples of such Powers: For if A, B are Powers of the Order n, they admit $n-1$ Means; and so also do C : D; consequently, by this *Theorem*, they are either Powers of the Order n, or Equimultiples of such.

2d. Two Numbers, A, B, that are like Composites of n Factors, are either both Powers of the Order n, or Equimultiples of such; for being like Composites of n Factors, they admit of $n-1$ Means; *i. e.* they are the Extremes of a Geometrical Series of $n+1$ Terms; (*Theor.* IX. *B.* IV. *Ch.* IV.) and consequently, by this *Theorem*, they are either, &c.

THEOREM XXI.

If a Geometrical Series is in its least Terms, or Incommensurable, so also are the Extremes.

DEMON. The Extremes are like Powers of the least Terms of the Ratio; but the least Terms are *Incommensurable* (*Theor.* IV.) and their like Powers are also *Incommensurable* (*Theor.* XIII. *Coroll.*)

COROLL. If the Extremes of a Series are *Commensurable*, the whole Terms are so; for the Whole being *Incommensurable*, so also are the Extremes.

THEOREM XXII.

In every Geometrical Series, whose Ratio is not Multiple, the whole Terms, excluding either of the Extremes, are *Commensurable*.

DEMON. By the Work of the preceding *Problem*, as it is represented in *Schol.* 1. all the Terms of the Series are Multiples of the lesser Term of the Ratio, except the greater Extreme; and all are Multiples of the greater Term of the Ratio, except the lesser Extreme: Wherefore in all Cases, these Terms are Commensurable by that Term of the Ratio.

COROLLARIES.

1*st.* All the Terms of every Geometrical Series, except the lesser Extreme, in some Cases, are Composite Numbers: For if the Ratio is Multiple, the lesser Extreme may be a Prime, and then all the other Terms are Multiples of it: But if the Ratio is not Multiple, all the Terms are Composite Numbers; which are either Powers or Multiples of the Terms of the Ratio. Hence again,

2*d.* No Term of any Geometrical Series, except the lesser Extreme, can be a Prime Number. And hence again,

3*d.* Betwixt two Prime Numbers there cannot be a Geometrical Mean in a whole Number, nor consequently in a mix'd Number; because the Mean is the square Root of the Product of the Extremes; which not having an Integral Root, has none at all (by *Theor.* XIX. *B.* III. *Ch.* I.) And more generally, betwixt two Primes there falls no Number of Geometrical Means; for Integral Means they cannot be, by this *Theorem*; and they cannot be Fractional, as you'll see in *Theor.* XXV.

THEOREM XXIII.

If a Geometrical Series, whose Ratio is not Multiple, is in its least Terms or *Incommensurable*, another Integral Term cannot be added, either increasing or decreasing.

DEMON. The Series being in its least Terms, if we suppose another Integral Term added, then in this increased Series, the whole Terms, excluding this new Extreme added, are not Commensurable; which is contrary to the last *Theorem*.

Or take this other *Demonstration*: Let A, B and L, be the first, second and last Term of a Series, in its least Terms; to which let another Term, x, be added; then is $A : B :: L : x$. But $A : L$ are *Incommensurable*; because the Series from A to L is so (*Theor.* XXI.) therefore A measures B. And hence again, A measures every Term of the Series; the Ratio being in this Case Multiple, *i. e.* every Term being an *aliquot* Part of the next greater, and consequently of every greater: But if A measures L, then A, L are Commensurable, contrary to Supposition.

COROLL. To two Numbers *Incommensurable*, a third in Geometrical Proportion, cannot be an Integer.

SCHOLIUMS.

1°. The Sense of this *Theorem* is the same with this, *viz.* If a Series is in its least Terms (the Ratio not Multiple) the last Term cannot be to any Integral Number in the same Ratio, as the first Term to the second.

2°. Though the Extremes of a Series are Commensurable, and at the same Time also both Composite Numbers; yet it will not follow that another Integer can be added to the Series, as here; $20 : 30 : 45$, the Extremes; 20, 45, are both Composite Numbers,

Numbers, and Commensurable; yet another Term in the same Ratio is not Integral; for it is $47\frac{1}{7}$. But this Truth we may also demonstrate independently of any Particular. *Example.* Thus, In every Series, all the Terms, except the lesser Extreme, are both Commensurable and Composite Numbers: And if the Series is in its least Terms, another Integral Term cannot be added; but if we take the Term next the lesser Extreme, and the greatest Extreme, for the Extremes of a Series, they are both *Commensurable* and *Composite*; yet another Integral Term cannot be added above the greater Extreme; because what is added to this Series, is also added to the Series of which it is a Part, and to which another Integral Term is shewn to be impossible; because its Extremes are Incommensurable.

3°. As to the special Character of a Series, which admits of another Integral Term, it is to be deduced from *Theor.* X. *Thus*, Let A, B, L, be the first, second, and last Terms of a Geometrical Series; if another Term added after L is Integral, let it be called M; then is A : B : : L : M; so that A measures B×L; (for M=BL—A.) Consequently there is no Prime in the Composition of A, but what is found in the Composition of BL (*i. e.* either in B or L) nor any Prime oftner involved in A, than it is in BL: For otherwise A could not measure BL (by *Theor.* X.) and so M would not be Integer.

Thus then we see what are the Conditions of three Integral Numbers, that admit a fourth Proportional, which is also an Integer.

Theorem XXIV.

If there be any one Choice of two Terms in a Series, whereof the lesser measures the greater, then every lesser shall measure every greater: And if there be any two Terms, whereof the lesser does not measure the greater, then none of the lesser shall measure any of the greater.

Demon. 1*st*. Let any Series be thus represented, $A : Ar : : Ar^2 : Ar^3$, *&c.* If any Term is divided by any lesser, the Quote is some Power of r; this is manifest; and if the Quote is an Integer, then r is an Integer; because any Power being Integer, its Root must be so too (*Theor.* XIX. *B.* III. *Ch.* I.) Again, If r is an Integer, all its Powers are Integers; wherefore every lesser Term measures every greater.

2*d*. The second Part is obvious from the preceeding; for if any lesser Term measures any greater, all the lesser would measure all the greater, contrary to Supposition.

Coroll. If A, the lesser Extreme of a Series, measures the second Term, B, it is the greatest common Measure of the whole Series; for it measures all the other Terms, and is its own greatest Measure.

Theorem XXV.

If A, the lesser Extreme of a Series, is a Prime Number, the Ratio of that Series is Multiple; or every lesser Term measures every greater.

Demon. If A does not measure the second Term, B, then being a Prime Number, A, B, are *Incommensurable*; and consequently there cannot be a third Integral Term (by *Coroll: Theor.* XXIII.) contrary to Supposition; and if A measures B, then every lesser Term measures every greater.

Coroll. If the lesser Extreme of a Series is a Prime Number, that Series cannot be in its least Terms; because the lesser Extreme measures the Whole.

Theorem XXVI.

Whatever Number measures the Extremes of a Series, will measure all the middle Terms; or thus, the common Measure of the Extremes is so to the whole Series.

Demon. Let the Series be, $A : B : C$, &c. : : L, and suppose that m measures the Extremes A, L, by these Quotes, a, l; that is, $\frac{A}{m}=a$, and $\frac{L}{m}=l$; then is $A : L : : a : l$; and as many Means as fall betwixt $A : L$, so many fall betwixt $a : l$ in the same Ratio. Let the second Series be a, b, c, &c. l; then is $A : a : : B : b$; but $A=ma$, therefore $B=mb$, hence $\frac{B}{m}=b$; *i. e.* m measures B by b. In the same manner will the Reasoning proceed to the next middle Term C; for, $B : b : : C : c$; but $B=mb$, and therefore $C=mc$, and $\frac{C}{m}=c$; and so of all the rest.

Coroll. Hence we have an easy Rule for finding the greatest common Measure of any Series, *viz.* by finding that of the Extremes.

Schol. This Theorem is true, whether the Series be all Integers or not, and whether m be so or not.

Theorem XXVII.

As many Geometrical Means as fall betwixt any two Numbers, A, L, so many there fall betwixt each of them, and their greatest common Measure.

Demon. 1°. If A, L are the least in their Ratio, then whatever Number of Means is supposed to fall betwixt them, as $n-1$, the whole Series is least in its Ratio, and A, L are Powers of the Order n (*Theor.* XX.) As suppose $A=a^n$, and $L=b^n$; now a^n, b^n being least in the Ratio, 1 is their greatest common Measure, and betwixt $1 : a^n$, also betwixt $1 : b^n$ there fall $n-1$ Means *(Cor.* III. *Probl.* III. *B.* IV. *Ch.* III.)

2°. If A, L are Commensurable, let m be their greatest common Measure, and $A\div m=B$; $L\div m=D$; then are $B : D$ least in the Ratio of $A : L$ (*Theor.* III.) and being in the same Ratio, therefore they admit as many Means *(Theor.* VII. *Book* IV. *Chap.* IV.) But by the last Article, betwixt 1 and B, or D, there fall as many Means as betwixt $B : D$, or $A : L$, as suppose $n-1$. But again, since $A\div m=B$, therefore $m : A : : 1 : B$, and betwixt $1 : B$, there fall $n-1$ Means; consequently there fall as many betwixt $m : A$. And because $L\div m=D$, hence $m : L : : 1 : D$; and betwixt $1 : D$ there fall $n-1$ Means, consequently as many betwixt $m : L$, that is, as many as betwixt $A : L$.

Schol. By the same Reason there will fall as many Means betwixt A or L, and any of their common Measures, as fall betwixt $A : L$ themselves.

Theorem XXVIII.

As many Means as fall betwixt any two Numbers, A, B, so many fall betwixt each of them and their least common Multiple M.

Demon. Let m be the greatest common Measure of A, B; then (by *Theor.* XIX.) $m : A : : B : M$; and (by *Theor.* XXVII.) there fall as many Means betwixt $m : A$ (or $m : B$) as betwixt $A : B$; also (by *Theor.* VII. *Book* IV. *Chap.* IV.) as many betwixt $B : M$ or $A : M$ as betwixt $m : A$ or $m : B$; *that is*, as many as betwixt $A : B$.

Theorem XXIX.

Of a Series in continued Proportion, take the Series of the greatest common Measures, or least common Multiples, to every two adjacent Terms; these are also in one continued Proportion.

$$\begin{array}{l} A : B : C : D : \&c. \\ \quad l \quad m \quad n \\ \quad\; a : b \end{array}$$

Demon. 1°. For the greatest common Measures: Let l be that of A, B, and m of B, C, and n of C, D; also, let $a : b$ be the least Terms of the common Ratio of the first Series. Now, l, m, n do equally measure A, B, C, *viz.* by a; and they also measure B, C, D equally by b; (*Theor.* III.) and, *Reversely*, a measures A, B, C by l, m, n, and b measures B, C, D by l, m, n; hence l, m, n are continuedly in the same Ratio as A, B, C; that is, as $a : b$.

2°. For the least common Multiples: Let l, m, n be the l : ×*les*, then is $l=Ab$, $m=Bb$, $n=Cb$; hence $l : m :: Ab : Bb :: A : B$, and $m : n :: Bb : Cb :: B : C$; therefore $l : m :: m : n$.

Theorem XXX.

If any Numbers, $\div l$, are in their lowest Terms, as A, B, C, D, and L, the least Terms of whose Ratio are $a : b$; whatever Number, m, measures any Term of the Series, it's Commensurable with a or b.

Demon. The Series being in its lowest Terms, and $a : b$ the lowest Terms of the common Ratio, then the Extremes are $A=a^n$ and $L=b^n$; and any middle Term may be expressed $a^{n-r} \times b^r$. (See *Schol.* I. *Probl.* VI.) Now, if m is Incommensurable with a, and b; it's so with any Power of a and b, and with any Product of any Power of the one by any Power of the other, (*Theor.* XII. *Coroll.*) wherefore it cannot measure any Term of the Series, contrary to Supposition.

Theorem XXXI.

If $A : B : C : D$, &c. are $\div l$, and in their lowest Terms, each of them is Incommensurable with the Sum of all the rest.

Demon. Take any one of them, as, B; I say B, and $\overline{A+C+D}$ are Incommensurable; for if they are Commensurable let m measure both; and take $a : b$, the lowest Terms of the common Ratio; then, since m measures B, it is Commensurable to a or b (by the last); suppose to a; and let n measure m, and a, therefore n measures a, B, and $\overline{A+C+D}$ (because n measures m, and m measures B and $\overline{A+C+D}$). But a, the Antecedent of the lowest Terms of the Ratio, measures all the Antecedents of the Series, A, B, C, &c. (these being all Multiples of A, as appears from the Work of *Probl.* VI. as it is in *Schol.* 1*st.*) And since n measures a, by Supposition, therefore it also measures each of these, A, B, C; but it measures also $\overline{A+C+D}$, therefore it measures D (*Axiom* 3.) consequently it measures each of these, A, B, C, D, which therefore are not least in the Ratio, contrary to Supposition.

If instead of a, we take b, the Consequent of the Ratio, the Demonstration will be the same; for then b measures all the Consequents, B, C, D, &c. and consequently n measures them all; and because it measures $\overline{A+C+D}$, therefore it also measures A, *i. e.* it measures each of these, A, B, C, D, &c. contrary to Supposition.

How many Numbers soever you suppose, and which soever of them you take, the Demonstration will still be the same, from a measuring all the Antecedents, and b all the Consequents.

Theorem XXXII.

Of a Series $\div l$, and in their least Terms, any one of them is Incommensurable to the Sum of the whole Series.

Demon.

Demon. Any one of them is Incommensurable to the Sum of all the rest (by the last) and this Sum added to that one (which makes the Sum of the Whole) is Incommensurable to any of the Parts added (*Cor. Ax.* 2d.) *viz.* to that one.

Theorem XXXIII.

If $a : b : c : d :$ &c. $k : l$, $\div l$, and in least Terms, do equally measure A : B : C : D, *&c.* K : L, by m; also if $r : s$ are the least Terms of the Ratio, and neither of them does measure m, then another integral Term cannot be added to the last Series, A : B : C, *&c.* : K : L.

Demon. Since $a : b : c$, &c. do measure A : B : C, *&c.* by m, therefore the last Series is the same as $am : bm : cm$, &c. $km : lm$, and the first Series the same as $r^n : r^{n-1} \times s$, &c. to S^n; then r, s being Incommensurable, so are r, s^n, or $r : l$; and $s : r^n$ or $s : a$. (*Cor. Th.*XIII.) Again, $km : lm : : lm : \frac{l^2m^2}{km} = \frac{l^2m}{k}$, the Term added; which is not Integral; for $\frac{l}{k} = \frac{s}{r}$ hence $\frac{l^2m}{k} = \frac{slm}{r}$, but r is Incommensurable to both s and l, and consequently to $s\,l$; (*Th.*XII.) and also it does not measure m, therefore it does not measure $s\,l\,m$; (*Theor.* VI. *Cor.* 2d.) that is, the Term added is not Integral.

Schol. By the same Method of Demonstration it will appear, that if any two Numbers, $a : b$ are least in their Ratio, and do equally measure other two, A, B, by M, which neither a or b measures, then a third $\div l$ to A : B, is impossible in Integers; for A$=am$ and B$=bm$, and the third $\div l$ is $\frac{b^2m^2}{am} = \frac{b^2m}{a}$. But a being Incommensurable to b, is so to b^2, and it does not measure m, consequently does not measure b^2m.

Theorem XXXIV.

The Extremes of every Geometrical Series, whose Ratio is not Multiple, are like Composite Numbers, whose Index is the Number of Terms less 1: So if the Series has three Terms, the Extremes are composed of two like Factors; if the Series has four Terms, the Extremes are composed of three Factors.

Demon. The Extremes of every Series are either like Powers, whose Index is the Number of Terms less 1, or they are Equimultiples of such like Powers (*Theor.* XX.) If they are Powers, the Truth of this Theorem is manifest; for like Powers are comprehended under the Notion of like Composites: If they are Equimultiples of like Powers, as $xa^n : xb^n$, the same Truth is also manifest; for they may be resolved into $xa \times a \times a$, &c. and $xb \times b \times b$, &c. repeating a in the one, and b in the other equally; then it's plain that $xa : xb : : a : b$; the rest are all a and b.

But now *observe*, That the Similarity of the Composition may be in some Cases, after a Manner different from any of these already represented; in which all the Factors in each Extreme are equal among themselves, as when they are like Powers; or all except one, as in the other Case; for in some Cases, I say, the Similar Composition will be by Factors, which are all different among themselves in each Extreme: And in some Cases, though they are not all different, yet neither will they be all equal, except one. Now, because the preceding Demonstration represents the Extremes of a Series, as composed after two particular Ways, though there are also others; therefore I shall give another Demonstration of the *Theorem*, unlimited to any particular kind of Similiar Composition, and which comprehends them all: And afterwards I shall explain the Limitations of some particular Cases.

Let the Extremes of a Series be A, N, and the Term next the greater Extreme (N) be M. Then, I say,

1st. If

1st. If the *Theorem* be true in every Case where the Number of Terms is n, it is also true when the Number of Terms is $n+1$; which I thus prove:

1°. If from A to N, including both, there are $n+1$ Terms, then from A to N, excluding one of them, there are n Terms; and by Supposition the *Theorem* is true, of n Terms from A to M: Suppose as many Terms, least in their Proportion, as the Series A . . . M; the Extremes of this new Series, are, by Supposition, like Composites; which we may represent thus, *viz.* ab&c. cd&c. supposing as many Factors as $\overline{n-1}$. Dispose these under the other, as in the Margin. Then,

A . . . . . M : N
ab&c. . . . cd&c. :

2°. Since ab&c. : cd&c. are Similar Composites, it is $a : c :: b : d$; and so through all the Factors, comparing them in Order, the least Factor of the one to the least of the other; and so gradually on to the greatest. Again, this new Series being in the least Terms, and in the same Ratio with the other Series, from A to M, every Term in that Series will measure its Correspondent in this equally, and that by their greatest common Measure, suppose m: Therefore $m\times ab$&c.$=$A, and $m\times cd$&c.$=$M.

3°. Compare the second Series from cd&c. to as many of the other, taken from N, towards the Left-hand, and they are also least in the same Ratio with these; because these are Part of the same Series proceeding from A; consequently each Term of the Series, ab&c. . . . . cd&c. measures its Correspondent, in this Part of the first Series, equally, suppose by n; thus, cd&c. measures N by n; hence, $n\times cd$&c.$=$N; so that, for the first and two last Terms of the given Series, we have new Expressions equal to them, which are these standing under them, in the annex'd Scheme on the Margin. Then,

A . . . . M : N
$m\times ab$&c. . . . $m\times cd$&c. : $n\times cd$&c.

4°. Since ab&c. cd&c. or A, M, are Similar Composites, by Supposition, it remains only to be shewn that $m\times ab$&c. and $n\times cd$&c. that is, A and N are also Similar Composites; which is done thus, The Series from $m\times ab$&c. to $n\times cd$&c. is in the continued Ratio of m to n; for the two last Terms, *viz.* $m\times cd$&c. $n\times cd$&c. are plainly so; wherefore the second Term of the given Series being called B, its equivalent Expression in the other Form, is $n\times ab$&c.; but $m\times ab$&c. : $n\times cd$&c. :: $\overline{m\times ab}^n$&c. $\overline{n\times ab}^n$&c. ($=B^n$.) the Index n, being the Number of Terms less 1. (*Theor.* VI. *B.* IV. *Ch.* IV.) Again, because $m\times ab$&c. : $n\times ab$&c. ($=$B) :: $m : n$; therefore $\overline{m\times ab}^n$&c. : $B^n :: m^n : n^n$; (*Cor.* 11. *Theor.* III. *B.* IV. *Ch.* IV.) wherefore also $m\times ab$&c. $n\times cd$&c. :: $m^n : n^n$. Hence again, ab&c. : cd&c. :: $m^{n-1} : n^{n-1}$ ($\overline{n-1}$ being the Number of Factors in ab&c.) but by Supposition, $a : c :: b : d$; so that $\frac{a}{c}=\frac{b}{d}$; and the like being true in the rest of the Factors of ab&c. cd&c. compared, therefore $\frac{a}{c}\times\frac{b}{d}\times$&c.$=\frac{ab\text{\&c.}}{cd\text{\&c.}}$ $=\frac{a^{n-1}}{c^{n-1}}$; wherefore ab&c. : cd&c. :: $a^{n-1} : c^{n-1}$, and ab&c. : cd&c. :: $m^{n-1} : n^{n-1}$. Hence $a^{n-1} : c^{n-1} :: m^{n-1} : n^{n-1}$, and $a : c :: m : n$ (*Cor.* 11. *Theor.* III. *B.* IV. *Ch.* IV.) Wherefore, lastly, mab&c. ncd&c. that is, A, N are Similar Composites of $\overline{n-1}$ Factors, n being the Number of Terms in the Series; which is the first Part of the Demonstration. But,

2d. The Proposition is true of any three Terms, A : B : C in a continued Series, *i. e.* A and C, are like Composites of two Factors; which is thus demonstrated;

Take

$A : B : C$ $a : b$ $c : d$ $ac : cb : bd$	Take $a : b$, the least in the same Ratio, with $A : B$ and $B : C$; they must equally measure A, B, suppose by c; and B, C, suppose by d; that is, $\frac{A}{a} = c$, and $\frac{B}{b} = c$: Also, $\frac{B}{a} = d$, and $\frac{C}{b} = d$; therefore $ac = A$; $cb = B = ad$, and $bd = C$: Wherefore, $ac : bc : bd$, are in the same Ratio continuedly, since they are equal to $A : B : C$; but

it's plain, that $ac : bc :: a : b$, and $bc : bd :: c : d$. Hence $a : b :: c : d$, that is, ac and bd are like Composites of two Factors.

3*d*. Therefore the *Theorem* is true in all Cases; for it is true in Case of three Terms, by *Article* 2d; and it follows from the 1st *Article*, that it's true of four Terms; and from this again it's true of five Terms, and so on for ever.

COROLLARIES.

1*st*. If the Product of two Numbers, A, B, makes a Square Number, these Numbers are like Composites of two Factors; for if $AB = x^2$, then $A : x : B$ is a continued Series, whose Extremes, A, B, are, by this *Theorem*, like Composites of two Factors.

2*d*. If it is $A : B :: C : D$, and if $A : B$ are like Composites of any Number of Factors, C, D are like Composites of the same Number of Factors; because they admit as many Means as A, B, or are the Extremes of an equal Series, with that of which A, B are the Extremes.

SCHOLIUMS.

1*st*. The Reason of limiting the Series to a Ratio, which is not Multiple, is, because if the Ratio were Multiple, the lesser Extreme may be 1, or a prime Number: But as 1 is allowed to be a Power of all Orders, and consequently to be a Composite of 1, as a continual Factor, for it is $1 \times 1 \times 1$, &c. If we also allow 1 to be a Factor in other Cases; then the *Theorem* may be taken without the Limitation.

2*d*. It has been shewn, that the Extremes of every Series are like Powers, or the Equimultiples of like Powers; and in some Cases, that they are both. Those that are like Powers only, and not also Equimultiples of like Powers, have a Similar Composition, by equal Factors only. Those that are not like Powers, have a Similar Composition only by Factors that are not all equal; and which, in some Cases, will be all different; in others not. *Lastly*, Those that are both like Powers and Equimultiples of like Powers, have a Similar Composition, both by equal Factors, and by such as are not all equal. I shall explain these Things a little more particularly.

(1°.) Those that are like Powers only, have a Similar Composition only by equal Factors: The Reason of which, is this; Such a Series is necessarily in its least Terms; for if it is not, the Extremes are either Equimultiples of like Powers, or they are so, and also like Powers; and therefore not like Powers only, contrary to Supposition.

Now, if a Series is in its least Terms, the Extremes are like Powers of the least Terms of the Ratio; the Index being the Number of Terms less 1 (*Theor.* XX.) Thus the least Terms of the Ratio being $a : b$, the Extremes are a^n, b^n; but $a : b$ being Incommensurable, so are a^n, b^n (*Coroll. Theor.* XIII.) so that there is no Prime common to a^n and b^n; for which Reason they can never be resolved into any Number of Similar Factors; because these Factors could be no other than the Primes that compose them, or Products made of these Primes, which cannot be Proportional: For suppose that x, z, are any two Products, made of the Primes of a^n; and y, v, two made of the Primes of b^n; these Products are Incommensurable each to each, *i. e.* x to y, and z to v; because the Primes of a^n are all different from those of b^n; wherefore x, y, z, v, are not Proportional; for if they are, then x, y, being Incommensurable, z, v, must be Commensurable, contrary to what's already shewn.

Hence

Hence we know how to find whether the Extremes of a Series are similary Composite, by equal Factors only, whose Number is the Number of Terms less 1, *viz.* by finding whether they are in their least Terms.

(2°.) Those that are not like Powers, have a Similar Composition only by Factors that are not all equal: This is obvious.

(3°.) If the Extremes are not in their least Terms, and yet are like Powers of the Order n, it's manifest from the two preceding Articles, that they are similarly Composite by Factors all equal; and also by such as are not equal.

But observe, for the two last Cases, that it is not easy in all Examples to determine whether the Extremes admit of a similar Composition, by Factors that are all different, or by Factors that are partly different and partly equal; or whether it may not be both Ways. These Things only I find evident. 1. That for a Series of three Terms, whose Extremes are not Squares, the similar Factors must be all different; because there are but two of them; the Invention of which Factors is easy; thus, a, b, being the least Terms of the Ratio, a^2, b^2, are the Extremes in their least Terms; and therefore in the present Supposition, the Extremes are Equimultiples of these, *viz.* $a^2\times m = a\times am$, and $b^2\times m = b\times bm$. Again, 2. For a Series of four Terms, where the Extremes are not least in their Ratio, and yet are Cubes, these have similar Factors, either all three different, or two of them equal, and the other different from these: The Reason of which, and the Invention of these Factors, you will easily understand, thus; it has been shewn, that such a Case happens only when the Series is raised, after the Manner of *Problem* 1. *B.* IV. *Ch.* IV. from such Terms of the Ratio which are not the least. Suppose then the least Terms of the Ratio, $a : b$, and the Terms from which the Series is raised, to be am, bm (for they must be Equimultiples of the former) then are the Extremes, $\overline{am}^3$, $\overline{bm}^3$; *that is,* $a^3\times m^3$, $b^3\times m^3$, which are resolvable into these Factors, *viz.* $a^3\times m^3 = a\times am\times am^2$, and $b^3\times m^3 = b\times bm\times bm^2$, similiar to the former; or thus, $a^3\times m^3 = a\times a\times am^3$, and $b^3\times{}^3m = b\times b\times bm^3$, similar with the former.

(4°.) In all Cases where the Extremes are like Powers, yet not in their least Terms, and the Number of Terms more than four, they are resolvable into similar Factors that are not all equal, these two Ways, *viz.* 1. Having all the Factors equal, except one, as shewn in the first Demonstration of the *Theorem*: Or, 2. By Factors which are not all equal except one, yet not all different. This last will easily appear, *thus*; The Extremes, according to the Circumstances supposed, are to be expressed $a^n m^n$ and $b^n m^n$, which are resolvable in this Manner, *viz.* $a^n m^n = a\times am^2\times a\times am^2\times$ &c. and $b^n m^n = b\times bm^2\times b\times bm^2\times$ &c. taking so many Factors in this Manner till a and m, in the one, and b, m, in the other, are as oft involved, as the Index n expresses; or they may be variously resolved, as *thus*, $a^n m^n = a\times a\times am\times am^3$ &c. and $b^n m^r = b\times b\times bm\times bm^3$ &c.

Schol. 2*d*. This *Theorem* is a kind of Reverse to *Theor.* IX. *B.* IV. *Ch.* IV.

Theorem XXXV.

If betwixt any two Integers there falls a certain Number of Means, they must necessarily be all Integers.

$$A : \frac{AM}{N} : \frac{AM^2}{N^2} : \frac{AM^3}{N^3}, \text{ \&c.}$$

Demon. Let the first Term of a Series be A, and the Ratio in its least Terms be N : M; then by the common Rules, the Series will be expressed as in the Margin. Now the Extremes being Integers, by Supposition; suppose the last of them to be $\frac{AM^n}{N^n}$; then does N^n measure AM^n (else that Extreme is not Integer) but it cannot measure M^n, since 'tis Incommensurable to it; because N is so to M (*Theor.* XIII. *Coroll.*) wherefore it must measure A (*Theor.* VI.) consequently all the inferior Powers of N measure A: And again,

they

they also measure all the Multiples of A, *i. e.* the Denominators of all the middle Terms, being Powers of N, inferior to N^n, do measure all their Numerators, which are Multiples of A. Therefore, lastly, all these middle Terms are Integers.

Coroll. If in the Progress of a Series, beginning with an Integer, there comes a Fractional Term, simple or mix'd, there can never after that be any more Integers in it; because, if that could happen, then betwixt two Integral Extremes a Fractional middle Term might happen, contrary to what has been demonstrated.

§. III. *Containing a Variety of* Problems, *concerning* Geometrical Progressions, *considered with Regard to their Terms being* Integral *or* Fractional; *whose Solutions depend upon the preceding Doctrine.*

IN a *Geometrical Series* there may be a Variety of Changes from Integers to Fractions [proper or improper] or from these to those; all which depend upon the Relation of the first Term of the Series to the Ratio; and of these, considered also by themselves; as whether the first Term is an Integer or Fraction, and whether the Ratio is Multiple or not. From whence arises a new Set of *Problems*, relating to these Series; which have been referred to this Place, because the Demonstrations depend upon the Composition of Numbers by their Primes. I shall begin with explaining all the various Changes that can be in a Series.

PROBLEM VII.

It is required to shew all the Variety of Changes from Integers to Fractions, and mix'd Numbers, and from these to those, that can possibly be in any Series of Geometrical Proportionals; and to give Rules for the Invention of Series under all the possible Variety.

Solution.

This Complex Problem may be resolved into two Parts, as we consider the Series to increase or decrease: Yet we shall have only one of the Parts to demonstrate; because the Variety of the one is comprehended in the other. I shall first explain the Variety in an increasing Series, and the other will easily be seen in that: But in order to this, there is one general Proposition relating to both Kinds; which being of Use in the particular Parts of the Problem, I shall premise as a

Lemma.

If an increasing Series begins with a Proper Fraction ever so small; if it's a finite or determinate one, and the Ratio also determinate; then, after a certain Number of Terms continued in Fractions, it will increase to a whole or mix'd Number, and that too, greater than any assignable Number.

Again, Let a decreasing Series begin with a Number ever so great; if it's finite, and the Ratio so also; it will decrease to a proper Fraction, and that too, less than any assignable one. The Reason of all which is evident and needs no Demonstration; but if any call for it, they will find it afterwards in *Theor.* I. and II. *Ch.* III.

We proceed to the *Solution* of the *Problem.*

Part I. *For an Increasing Series.*

Case I. *If the first Term of a Series is an Integer, the Varieties are these*;

1°. It may continue in Integers; which necessarily follows from a Multiple Ratio: For it's plain, that the Product of two Integers will be an Integer. That the Series cannot

not continue in Integers, if the Ratio is not Multiple, the next Article will shew,

2°. It may change into a mix'd Number; and there it will continue; which requires and follows from a Ratio not Multiple; as in this *Example*, $4 : 6 : 9 : 13\frac{1}{2} : 20\frac{1}{4}$, &c. the Ratio being $2 : 3$.

$$A : \frac{AM}{N} : \frac{AM^2}{N^2} : \frac{AM^3}{N^3} : \&c.$$

Demon. The lesser Extreme of a Series being A, and the Ratio, in its least Terms, N : M, the Series is expressed as in the Margin.

Now, as the Denominator still increases in every Term in a determined Ratio, *viz.* $1 : N$; so it will become necessarily, after a certain Number of Terms, greater than A (by the preceding *Lemma*) and the first Term wherein that happens (and consequently all above) will be a mix'd Number; let that Term be $\frac{AM^n}{N^n}$. I say, N^n does not measure AM^n; for N, M are Incommensurable; and so are N^n, M^n (by *Coroll. Theor.* XIII.) But if N^n measures AM^n, it will also measure A; (*Theor.* VI.) *i. e.* a greater will measure a lesser, which is absurd: Therefore N^n does not measure AM^n; or $\frac{AM^n}{N^n}$ is a mix'd Number: For the same Reason all the Terms above are mix'd.

But again, *Observe*, that there may be a mix'd Number before the Denominator is greater than A; as in this *Exam.* $6 : \frac{6\times 3}{2} = 9 : \frac{6\times 9}{4} = 13\frac{1}{2}$: And lest it be supposed, that after a mix'd Number, which comes into the Series before N^n is greater than A, there may come an Integer; we ought to demonstrate universally, that after a mix'd Number, there can never be an Integer; and this you have already seen, in *Cor. Theor.* XXXV. Where you also see, that from an Integer there can be no more Changes but into a mix'd Number.

Or, the whole Demonstration may be made more simply; *thus*, It's evident we can chuse a Number, N, which is Incommensurable to A; and being so also to M, no Power of N can measure the Product of A into any Power of M; because these are Incommensurable (*Theor.* X.) Hence none of the Terms of the Series, after A, will be Integers.

Case II. *The first Term being a mix'd Number, the Varieties are,*

$$\frac{A}{B} : \frac{AM}{B} : \frac{AM^2}{B} : \frac{AM^3}{B}, \&c.$$

1°. It may continue in mix'd Numbers for ever; which is necessarily effected, either, 1. By a Multiple Ratio, $1 : M$, provided M be such, as to be Incommensurable with B: Or if B has any Prime different from any of the Primes of M, or any the same Primes, oftner involved; for then, by what's said in the last Case, B can never measure any of the Numerators. 2. By a Ratio not Multiple, as $N : M$, provided N has in its Composition any Prime which is not in A; or any the same Prime oftner involved; or if B has any Prime which is not in M: For in any of these Circumstances, none of the Denominators can ever measure its Numerator.

$$\frac{A}{B} : \frac{AM}{BN} : \frac{AM^2}{BN^2} : \frac{AM^3}{BN^3} : \&c.$$

Exam. 1. $3\frac{1}{2} : 10\frac{1}{2} : 31\frac{1}{2} : 94\frac{1}{2}$; the Ratio being $1 : 3$.

Exam. 2. $\frac{8}{3} : \frac{40}{12} : \frac{200}{48} : \frac{1000}{192}$; the Ratio being $4 : 5$.

 2°. It

$\frac{A}{B} : \frac{AM}{B} : \frac{AM^2}{B} : \&c.$

2°. It may paſs from a mix'd Number into an Integer, and ſo continue for ever; which can be effected only by a Multiple Ratio, 1 : M, ſuch, that M have in its Compoſition all the Primes that compoſe B. I ſhall firſt prove that ſuch a Ratio will bring the Series to an Integer; which is thus evident: M having all the Primes that are in B; if they are alſo as oft involved, then it's plain, that B meaſures M; and conſequently $\frac{AM}{B}$ is an Integer. Again, However oftner the ſame Primes are involved in B than in M, yet being more and more involved in the higher Powers of M, there muſt be a Power in which they are all oftner involved than in B; and hence it's plain, that there will at laſt be a Term in the Series in which B meaſures the Power of M in the Numerator, therefore that Term muſt be Integral.

Again, after an Integer comes into the Series, it muſt continue for ever in Integers, becauſe the Ratio is Multiple.

In the next Place, I ſhall demonſtrate, that no other Conditions will produce this Variety: And, (1.) A Ratio not Multiple, though it could bring the Series to an Integer, yet it could not continue ſo, as has been proved in *Caſe* I. *Art.* 2. (2.) If M has not in it all the Primes of B, it cannot meaſure B (*Theor.* X.) and hence alſo B will never meaſure any Power of M, becauſe M^n has no other Primes than M, (*Cor.* 5. *Theor.* X.) And ſince B is Incommenſurable to A, therefore (by *Cor.* 2. *Theor.* VI.) it cannot meaſure AM^n, that is, none of the Terms can ever be Integers.

Exam. $3\frac{1}{8} : 14\frac{7}{12} : 112\frac{1}{2} : 675 :$ *&c.* the Ratio 1 : 7

Obſerve, Examples of this Variety may be invented more ſimply, thus, *viz.* Take any Integer, and any Multiple Ratio, whoſe greater Term is leſs than the aſſumed Integer, and yet is not an *aliquot* Part of it; by this you'll find at leaſt one mixt Term below that Integer; from which again the Series will proceed upwards in the manner propoſed. But the other Method ſhews the fundamental Reaſon of the Caſe from the Compoſition of Numbers; and is of Uſe in ſome of the following Problems.

(3°.) It may paſs from a mixt Number into an Integer, and from that again into a mix'd Number; and ſo continue; which can be effected only by a Ratio not Multiple, as N : M; ſuch, that M do contain in its Compoſition all the different Primes of B, and A all thoſe of N, which muſt alſo be as oft at leaſt involved in A as in N. The Reaſon of which is this; if the Ratio were Multiple, then either the Series would never paſs into an Integer, or if it did, it would continue ſo (by *Caſe* II.) Again, if there be any Prime in B, which is not in M; or in N, which is not in A, it is manifeſt (from *Theor.* X.) that none of the Diviſors, as BN^n will ever meaſure the Dividend AM^n. Alſo, the Primes of N muſt not only be all involved in A, but they muſt be at leaſt as oft involved in A as in N; for, if any of them is oftner in N, it will be much more ſo in the higher Powers of N; and hence (by *Theor.* X.) none of the Denominators can ever meaſure its Numerator, *i. e.* none of the Terms will ever be an Integer. Theſe are the general Conditions: But yet more particularly, if any of the Primes of M are oftner involved in B; then conceive, that one of them whoſe Index in B exceeds its Index in N, by the greateſt Difference; alſo conceive the leaſt Number, which multiplying the Index of that Prime in M will produce a Number not leſs than its Index in B; then muſt N be ſuch, that taking the Prime, whoſe Index in N and A differ leaſt, and multiplying its Index in N by the ſame Multiplier, the Product ſhall not be greater than its Index in A. The Reaſon of which is alſo plain, for otherwiſe, before we come to a Term in which

$\frac{A}{B} : \frac{AM}{BN} : \frac{AM^2}{BN^2} : \frac{AM^3}{BN^3} :$ *&c.*

which the Power of M contains all the Primes of B as oft involved, which is necessary to make an Integer, the like Power of N in the Denominator will contain all the Primes of A as oft involved; and therefore in all the Terms above they will be oftner involved in N, and consequently the Denominator will never measure the Numerator, *i.e.* there will never be an Integer in the Series.

These Conditions are plain enough to shew the Invention of such a Series; whereof take this Example, $\frac{56}{45} : \frac{840}{90} : 70 : 525 : \frac{7875}{2}$: the Ratio being $2 : 15$; and that you may compare it with the preceding Conditions, the Composition of the first Term, represented by $\frac{A}{B}$, is this, $\frac{56}{45} = \frac{2\times2\times2\times7}{3\times3\times5}$, and the Ratio, $\frac{M}{N}$ is $\frac{15}{2} = \frac{3\times5}{2}$.

These are all the Changes from a mixt Number, because when a mixt Number follows Integers, the Series can never pass again into Integers; as has been shewn in *Art.* 2. *Case* I.

Case III. *The first Term being a proper Fraction, the Varieties are these.*

[1°.] It may pass into a mixt Number; and continue so; which is effected by a Ratio Multiple or not Multiple, qualified as in *Variety* I. *Case* II; for $\frac{A}{B}$ may be either a proper or improper Fraction; and since with the Qualifications refer'd to there can never be an Integer in the Series, and by the premised Lemma it must encrease either to a whole or mixt Number; therefore it will pass into a mixt Number, and there continue.

Exam. 1. $\frac{2}{13} : \frac{6}{13} : \frac{18}{13}$, &c. the Ratio $1 : 3$:

Exam. 2. $\frac{3}{29} : \frac{15}{58} : \frac{75}{116} : \frac{375}{232}$, &c. Ratio $2 : 5$.

[2°.] It may pass into a whole Number, and there continue; which requires a Multiple Ratio, qualified as in *Var.* 2. *Case* II. with this further, that supposing M^n the least Power of M which B measures, the Products of all the inferior Powers of M multiplied into A, must be less than B; else it's plain, that it will pass first into a mixt Number.

Examples are easily invented by taking M equal to B, or to any Multiple of it; or taking any Integer A, and a Multiple Ratio $1 : M$, such, that M is greater than A; for then continuing the Series downwards, it will fall first into a proper Fraction, from which it proceeds upwards in the Manner proposed.

Exam. $\frac{2}{9} : \frac{2}{3} : 2 : 6 : 18$, &c.

[3°.] It may pass first into a mixt Number, and then into an Integer, and there continue; which requires a Multiple Ratio qualified as in *Var.* 2. *Case* II. with this further, that supposing M^n to be the least Power of M which B measures, some of the next inferior Powers multiplied by A shall produce a Number greater than B; which is easily invented; or may be had by taking any *Example* of *Var.* 2. *Case* II. and continuing it downwards; for it must first pass into a proper Fraction, since there cannot be an Integer both above and below a mixt Number in the same Series.

Exam. $675 : 112\frac{1}{2} : 14\frac{7}{12} : 3\frac{1}{8} : \frac{25}{56}$, &c. Ratio $7 : 1$.

[4°.] It may pass first into a whole Number, and then into a mixt, and so it must continue, by Virtue of *Theor.* XXXV; and this is effected by a Ratio not Multiple,

ple, qualified as in *V.* 3. *Case* II. with this further, that supposing $\frac{M^n}{N^n}$ the least Power of $\frac{M}{N}$, which multiplied by $\frac{A}{B}$ makes an Integer, all the inferior Powers produce a proper Fraction. The Invention of Examples is easy, by assuming any Integer A, and a Ratio not Multiple, N : M, such, that NA be less than M; for then $\frac{AN}{M}$ is a proper Fraction, and $\frac{AN}{M} : A : \frac{AM}{N} : \frac{AM^2}{N^2}$, &c. represents the Series required.

Exam. $\frac{12}{17} : 4 : \frac{68}{3} :$ &c. Ratio 3 : 17.

[5°.] It may pass first into a mixt Number, then into an Integer, and lastly, into a mixt Number; which requires a Ratio not Multiple, qualified as in *V.* 3. *Case* II. with this further, that supposing $\frac{M^n}{N^n}$ the least Power of $\frac{M}{N}$, which multiplied into $\frac{A}{B}$ produces an Integer, some of the next inferior Powers shall produce a mixt Number; Examples of which are invented by taking an Example of *V.* 3. *Case* II. which being continued downwards must fall into a proper Fraction, but never into a whole Number, because there is a whole Number above the mixt.

Exam. $\frac{112}{675} : \frac{56}{45} : \frac{840}{90} : 70 : 525 : \frac{7875}{2}$, &c. Ratio 2 : 15.

Part II. *For a decreasing Series.*

The Varieties here are included in the former, and therefore I need only to name them, thus,

Case I. *The first Term being an Integer, then*

1°. It may pass first into a proper Fraction, and so continue.
2°. It may pass first into a mixt Number, and then into a proper Fraction.

Case II. *The first Term being a mixt Number.*

1°. It may pass first into a proper Fraction and continue so.
2°. It may pass first into a whole Number, and then into a proper Fraction.
3°. It may pass first into a whole Number, then into a mixt, and lastly, into a proper Fraction.

Case III. *The first Term a proper Fraction.*

It can only continue in Fractions.

Schol. In the following Problems, a given Ratio, or the same Ratio, do always comprehend a Ratio with its reciprocal; either of which is to be supposed, as we take a Series to encrease or decrease, and we suppose a Ratio always in its least Terms.

PROBLEM VIII.

Any whole Number being given, with any Ratio, to find how many Integral Terms can possibly be joined in the same continued Series with A, taking it either encreasing or decreasing, or both Ways, in that given Ratio.

Solution.

We must consider this Problem in two Parts, according as the given Ratio is Multiple, or not.

Case

Case I. The given Ratio Multiple. Then it's plain that the Series proceeds encreasing *ad Infinitum* in Integers; but is limited decreasing. Thus; let the given whole Number be A, and the Ratio M : 1; then if M is greater than A, there can be no Integral Term added; if M=A, then there will only be one which will be 1; but if M is less than A, then the Number of Integral Terms that can be added is equal to the Index of the highest Power of M which measures A; and if M does not measure A, there can no Integer be added; all which is manifest in this general Expression of the Series, $A : \frac{A}{M} : \frac{A}{M^2} : \frac{A}{M^3} :$ &c. But this Solution requiring, that every Term of the Series be actually raised, we may solve it otherwise; *thus*, resolve A and M into all their Primes; and if there is any Prime in M, which is not in A, or any the same Prime oftner involved, then M does not measure A, and so there will never be an Integral Term after A: But if A contains all the Primes of M, and all of them at least as oft involved; take that Prime whose Index in A exceeds its Index in M by the least Difference; also seek the greatest Number, which multiplying the Index of that Prime in M, makes a Number not greater than its Index in A, that Number is the Number sought. The Reason of which is obvious, for these Terms will always be whole Numbers, till some one at least of the Primes of A is oftner involved in the Power of M, which is the Denominator; and it's evident that this will happen first to that Prime, the Index of whose Involution in M comes short of its Index in A by the least Difference; and since any Power of M, as M^n, is the Product of the like Powers of all the Primes of M, the Truth of the Rule is manifest.

Exam. Let 72 be the given Number, and the Ratio 4 : 1; then taking the Primes of 72 and 4, it is $72=2\times2\times2\times3\times3$, and $4=2\times2$. Now there being but one Prime in 4, whose Index is 2, and the Index of the same Prime in 73 being 3, I find that 1 is the greatest Number, which multiplying 2, the Index of 2 in 4, produces a Number not greater than 3, the Index of 2 in 72; therefore there can be but 1 Integral Term added, as here, $72 : 18 : 4\frac{1}{2} :$ &c.

Case II. The Ratio not Multiple. It's plain that the Number of Integral Terms is limited, both encreasing (by *Part* I. *Case* I. of the preceding *Probl.*) and decreasing (by the preceding *Lemma*;) and to find the Number, take it first encreasing,

$$\frac{AN^3}{M^3} : \frac{AN^2}{M^2} : \frac{AN}{M} : A : \frac{AM}{N} : \frac{AM^2}{N^2} : \frac{AM^3}{N^3} : \text{\&c.}$$

and suppose the given Number A, and the Ratio in its least Terms to be N : M; the Index of the greatest Power of N, which measures A, is the Number sought. The Reason of which is this; suppose the last Integral Term to be $\frac{AM^n}{N^n}$, so that N^n measures AM^n, but it does not measure M^n, they being Prime to one another, because N and M are so; therefore, by (*Theor.* VI.) N^n does measure A; but it is also the greatest Power of N that does so; for if a greater does, as N^{n+1}, then that will also measure AM^{n+1}; and so $\frac{AM^n}{N^n}$ is not the last Integral Term, contrary to Supposition. *Lastly*, Because the Index of the Powers of N shew their Distance after A, therefore the Rule is true.

2°. If the Series is to be taken decreasing, then the Index of the greatest Power of M, which measures A, is the Number sought; for the same Reason as was explained in the preceding Article.

Observe, That this Case may be solved the same Way as *Case* I. by comparing the Primes of A with those of N for the encreasing Series, and with those of M for the decreasing.

3°.

3°. If it's proposed to find the greatest Number of Integral Terms that can be joined in the same Series with A in the Ratio N : M, taken both encreasing and decreasing; then find separately how many can be added encreasing in the Ratio N : M, and how many decreasing in the Ratio M : N; the Sum is plainly the Number sought.

Exam. Let the given Number be $72=2\times2\times2\times3\times3$, and the Ratio be $2 : 3$; then is the greatest Number of Integral Terms that can be joined with it, 5, *viz.* 3 above and 2 below, making this Series, $21\frac{1}{3} : 32 : 48 : 72 : 108 : 162 : 243 : 364\frac{1}{2}$. For 8 the third Power of 2 is the greatest Power of it that measures 72, and 9 the second Power of 3 is its greatest Power that measures 72.

Schol. We may make the preceding Problem yet more general and unlimited, by supposing no particular given Ratio; but proposing to find the greatest Number of Integral Terms that can possibly be joined in the same Series with a given Integer in any Ratio whatever; the Answer to which is an infinite Number, because with a Multiple Ratio the Series goes on encreasing *ad infinitum*; but if we take these Limitations, *viz.* 1°. A Ratio Multiple and a decreasing Series; then though no particular Ratio is given, the Problem may be solved. *Thus*; resolve the given Number, A, into its Primes, the Index of that one which is oftnest involved is the Number sought, and that Prime it self is a Ratio which will make the Series. *Example*, If $72=2\times2\times2\times3\times3$ is the given Number, then you can join in the same Series decreasing, only three Integral Terms, 3 being the Index of 2 in 72, and the only Ratio that can effect this is $2 : 1$, which makes this Series $72 : 36 : 18 : 9 : 4\frac{1}{2}$. *Observe also*, That if there are more than one of the Primes of A involved to the same Power, which is the highest of any concerned in it, then any of these Primes, or the Product of any two or more of them, may be made the Ratio; so if $216=2\times2\times2\times3\times3\times3$ is the given Number, the Answer is also 3; and the Series may be made in any of these Ratios, and in these only, *viz.* $2 : 1$. or $3 : 1$: or $6 : 1$, making these different Series $216 : 108 : 54 : 27$: or $216 : 72 : 24 : 8$. Or, *Lastly*, $216 : 36 : 6 : 1$. Thus we learn, not only how to solve the Problem, but to find also all the Variety of Ratios that can possibly solve it. The Reason of which is obvious.

Again, 2°. We shall suppose the Problem limited to a Ratio not Multiple, without any determined or given one; then is the Solution made thus, resolve A into its Primes, and take these two, the Sum of the Indexes of whose Involution in A, is the greatest, that Sum is the Number sought; and these Primes are the Terms of a Ratio, in which the Series may be made; and if several of these Sums which are the greatest, are equal among themselves, then the Primes to which they correspond, or the Products of any two or more of them, make all the different Ratios in which the Series can be made. *Exam.* Let the given Number be $3024=2\times2\times2\times2\times3\times3\times3\times7$, then is the Number sought 7, the Sum of the Indexes of 2 and 3, *viz.* 4 and 3, which is greater than the Sum of any other two Indexes; and therefore the Ratio is $2 : 3$, which in this Example is the only one that can make a Series of 7 Integers joined to 3024; which Series is this, $896 : 1344 : 2016 : 3024 : 4536 : 6804 : 10206 : 15309$.

3°. If it's proposed only to find the greatest Number of Terms that can be joined, increasing or decreasing, separately considered; then take the Index of the Prime, which is oftnest involved in A, and that is the Number sought for an encreasing Series; that Prime, or any other as oft involved, or the Product of any two or more of these, being the Antecedent or lesser Term of the Ratio, which can produce the Series, to which we may take any other greater Number, which is Prime to it for a Consequent. Then for a decreasing Series we must chuse that Prime whose Index is the greatest, that has another Number lesser than it self, and Prime to it. So in the preceding Example, 3024, the greatest Number encreasing is 4, the Index of 2 in 3024; and in this Case there is an infinite Choice of Ratios, because we may take any

any Number greater, and Prime to 2, for a Consequent; but for a decreasing Series, the Number is only 3, the Index of 3 in 3024, and there is but one Ratio, *viz.* 3 : 2, which will produce the Series, because there is but one Number, *viz.* 2, which is lesser than 3, and Prime to it. Take this other *Example*, $9261=7\times7\times7\times3\times3\times3$. To this can be joined at most 3 Terms encreasing, the Index of 7 or 3 being 3; and the Series may be produced by any Ratio, whose Antecedent is 7 or 3, or 21. Again, decreasing there can at most be added 3, which also can be effected by any Ratio whose Antecedent is 3 or 7, or 21; which makes a limited Number of different Ratios, because there is a limited Number of Terms lesser than these Antecedents and Prime to them, which are these, 3 : 2, 7 : 5, 7 : 3, 7 : 2; and if 21 is made the Antecedent, the Consequents are these, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 19, 20. But the absolutely greatest Number of integral Terms that can stand in the same Series with 9261 is 6, whereof 3 will be greater, and 3 lesser, in the Ratio 3 : 7, which is the only one that will make the Series.

PROBLEM IX.

A whole Number being given, to find a Ratio in which it is possible to join a given Number of integral Terms, and no more, in the same Series with the given whole Number.

Solut. If we take the given Number for the greater Extreme of a Series, then the Problem being possible in the Circumstances of the given Numbers, may be solved in some Cases by a Multiple Ratio only; in others by a Ratio not Multiple and in others by both: But if the lesser Extreme is given, it requires a Ratio not Multiple; which it does also, if we consider the given Number as one of the middle Terms. *Thus,*

$$A : \frac{A}{M} : \frac{A}{M^2} : \text{\&c.}$$

1°. Let us take the given Number, A, for the greater Extreme of a Series, then resolve it into its Primes; and if the Index of any of its Primes is precisely equal to the given Number of Terms, that Prime, or the Product of any two or more such, or any such Product multiplied again into any of the Primes which have a greater Index, is the greater Term of a Multiple Ratio which solves the Problem. And if there is any Number less than it, and Prime to it, that being made Consequent, makes a Ratio not Multiple, which also solves the Problem. But if none of the Primes of A has an Index exactly equal to the given Number of Terms, the Problem is impossible, making the given Number the greater Extreme of the Series.

Exam. Given 567, and the Number of Terms 4, then the Ratio sought is 3 : 1; for $567=3\times3\times3\times3\times7$, and there is not another Ratio that will answer; but take $189=3\times3\times3\times7$, and the Problem is impossible.

2°. The given Number being the lesser Extreme, find the Antecedent of the Ratio sought the same way, as in the preceding Case; and for a Consequent, any Number greater, and Prime to it: So 567 being the lesser Extreme, and the Number of Terms 4, the Ratio sought must take 3 for the Antecedent, and any Number greater, and Prime to it, for the Consequent.

3°. If the given Number is considered as one of the middle Terms indefinitely, then take these two Primes of A, the Sum of whose Indexes is equal to the given Number of Terms, and these are the Terms of a Ratio, which solves the Problem; and of the Terms to be added, there will be as many greater than the given Number, as the Index of the lesser one of these two Primes, and as many decreasing as the Index of the greater.

Exam. Given $72=2\times2\times2\times3\times3$, and the Number of Terms 5, the Ratio is 2 : 3, the Number of Terms greater being 3, and the lesser being 2; and in this Case there is no other Variety in the Solution. But suppose $3528=2\times2\times2\times3\times3\times7\times7$, and

 the

the Number of Terms 5, it may be solved by this Ratio 2 : 3 or 2 : 7, with 3 Terms greater and 2 lesser; and if the Number of Terms is 4, then it is solved by this Ratio 3 : 7 with 2 Terms greater, and 2 lesser, or with this Ratio 6 ($=2\times3$) : 7 with 2 greater and 2 lesser. *Observe*, If the Sum of no two of the Indexes is exactly equal to the given Number of Terms, then the given Number cannot stand as a middle Term in the proposed Circumstances.

The Reason of all this is plain, especially if it be compared with the preceding Problem.

Schol. If the proposed Number of integral Terms is *o*, That is, to find a Ratio with which it's impossible to join any integral Term to a given Integer, we have no more to do, but, take any two Numbers which contain a Prime different from any in, A, or any the same Prime oftner involved, and make these the Terms of your Ratio; for then it's plain, that in such a Ratio you can join no integral Term, either greater or lesser, because none of these Numbers can measure A.

Again, if the Ratio is required to be such, that the Number of integral Terms to be added is infinite, then take any Multiple Ratio.

PROBLEM X.

To find a whole Number, with which a given Number, and no more, of integral Terms can be joined in the same Series in a given Ratio.

Solut. 1°. If the Ratio is Multiple, the Problem is plainly impossible; for to any Integer an infinite Number of integral Terms may be added in any Multiple Ratio, yet if we limit it so as to consider the Number sought as the greater Extreme of a Series, it is solved thus; if the greater Term of the given Ratio has any of its Primes involved to an Index precisely equal to the given Number of Terms, then any Number which is a Multiple of that Power of that Prime, is the Number sought; but if none of its Primes is so oft involved, then is the Problem also impossible.

2°. If the Ratio, is not Multiple, raise its Terms to the Power whose Index is the given Number of Terms less 1; and these are the Extremes of a Series in that Ratio, having the given Number of Terms, and to which it is impossible to add another in Integers (*Theor.* XXIII.) And these Powers, or any of the middle Terms of the Series, whereof they are the Extremes, which are easily found (vid. *Probl.* VI. *Schol.* I.) are Numbers which solve the Problem, and are indeed the only Numbers that do so. And *observe*, if it's proposed to find the Number sought, so that it shall be any one of the Extremes, or any one of the middle Terms, it's easily done by what's shewn.

Exam. The Ratio 2 : 3, and Number of Terms 6; I take the 5th Power of 2 : 3, *viz.* $2\times2\times2\times2\times2=32$, and $3\times3\times3\times3\times3=243$, and these are Extremes of the Series; the middle Terms being composed of the Powers of these, according to this Form $A^5 : A^4B : A^3B^2 : A^2B^3 : AB^4 : B^5$.

Scholiums.

I. If the proposed Number of Terms is *o*, then with a Multiple Ratio the Problem is impossible, if the given Number is to be made the lesser Extreme; but if we make it the greater, then we may take any Number lesser than the Antecedent of the Ratio, or greater, provided it has not all the different Primes of the given Antecedent, or has some of them less involved.

Again, with a Ratio not Multiple, we have no more to do but take any Number which has not in its Composition all the Primes of either of the Terms of the Ratio, or has any one of each of them less involved.

II.

II. If there is no particular Ratio given, nor any Limitation, *i. e.* if we demand an Integer, with which can be join'd only a certain Number of integral Terms in any Ratio, then the Problem is impossible, because in a Multiple Ratio there may be an infinite Number of Terms encreasing; but suppose these general Limitations,

1°. That the Number sought be the greater Extreme of a Series; then it may be found, thus; of any prime Number take that Power whose Index is the given Number of Terms; or take any Multiple of such a Power, provided it has no Prime involved to a higher Power; and to that Number can be joined the given Number of integral Terms decreasing, in any Ratio whose Antecedent that Prime is.

Exam. Let the Number of Terms be 3, then 8 is the Number sought, and the Ratio 2 : 1. Or $8\times5=40$, and the Ratio 2 : 1; but 16 is not such a Number, because it admits of 4 Terms in the Ratio 2 : 1. Again, $27=3\times3\times3$ is a Number answering the Problem, and the Ratio is either 3 : 1, or 3 : 2, making these Series 27 : 9 : 3 : 1, or 27 : 18 : 12 : 8.

2°. Suppose the Limitation is to any Ratio not Multiple; then take, as before, that Power of some Prime, whose Index is the given Number of Terms, and that is the Number sought; the said Prime being the Antecedent of a Ratio which solves the Problem, encreasing, any greater Number and Prime to it being made the Consequent. Also, if there is any Number lesser than it, but greater than 1, that being made Consequent will be a Ratio which will solve it decreasing. *Lastly*, If you take any two Primes, and involve them each to such a Power, that the Sum of their Indexes be equal to the given Number of Terms, then the Product of these Powers is a Number to which the given Number of Terms, and no more, can be added, taking part of them greater, and part lesser; which may also be distributed in any Proportion; for whatever Index the lesser Prime is raised to, that will express the Number of Terms greater than the Number sought, and the Index of the other is the Number lesser, these two Primes being the Terms of a Ratio that answers the Problem; and in this Case the given Number of Terms is the greatest Number of integral Terms absolutely that can be joined with the Number sought.

Exam. Given Number of Terms 5. I take $3\times3\times3\times3\times3=243$, which is a Number to which can be joined 5 integral Terms, and no more increasing, the Ratio of which must have 3 for its Antecedent, and any Number greater, and Prime to it, for a Consequent as 4; and decreasing it can also admit 5 Terms, but limited to this one Ratio not Multiple, *viz.* 3 : 2, because 2 is the only lesser Number prime to 3. Again, 108 ($=3\times3\times3\times2\times2$) can have joined with it only 5 integral Terms in the Ratio 2 : 3, whereof 2 will be greater and 3 lesser, as here, 32 : 48 : 72 : 108 : 162 : 243.

III. If it be proposed to find an Integer, to which it is impossible to join any Integer in the same Series, in any Ratio, the Problem is impossible; because to any Integer can be joined at least another Integer.

PROBLEM XI.

Any Fraction, proper or improper, $\frac{A}{B}$, being given, to find how many fractional Terms can possibly be joined with it in the same Series.

Solution.

If there is no further Limitation of this Problem, the Answer is an infinite Number; for if the Fraction is proper it admits an infinite Number of fractional Terms, decreasing in the same Series in any Ratio whatever; and if it's an improper Fraction, then also an infinite Number of fractional Terms can be joined encreasing or decreasing, or both Ways; for the Series being continued both Ways, there can't be one Integer

above and another below; (by *Theor.* XXXV.) therefore the Series will continue in Fractions one or both Ways.

But as the Number sought may be limited on one Side (though it can't on both) we shall enquire upon what Side it is limited, and to what Number of Terms. Thus,

Case I. *The given Ratio Multiple, as* 1 : M, *Then,*

(1°.) The Series will continue for ever in Fractions decreasing: If the given Fraction is proper, this is manifest; and if it's improper, you see the same Truth from this Consideration, *viz.* That if ever there comes an Integer into the Series, from that reversely it will continue upwards to a mixt Number (from whence it proceeded downwards) which is impossible in a Multiple Ratio, because every Term is the Product of two Integers. Therefore there can no Integer come into the Series. But,

(2°.) Take the Series encreasing from the given Fraction, and the Number of fractional Terms that can be added, will in some Cases be finite, and in some infinite, which the following Rule will discover, whether the given Fraction is proper or improper; *Thus*, resolve M and B both into their Primes;

$\frac{A}{B} : \frac{AM}{B} : \frac{AM^2}{B} : \&c.$

and if there is any Prime in B which is not in M, then the Number of Terms required is infinite; for B can never measure any Power of M, and being Prime to A, it cannot measure any of the Numerators (*Cor.* 2. *Theor.* VI.) Again, if all the different Primes of B are found in M, and as oft at least involved; then is B an *aliquot* Part of M, and hence $\frac{AM}{B}$ is an Integer, therefore there can no fractional Term be added: *Lastly*, If all the Primes of B are in M, and some of them also oftner involved in B than in M, then take all of these whose Indexes in B and M have the same Difference, and that, the greatest of any other of them; and chusing that one of these, whose Index is greatest, find the greatest Number, which multiplying its Index in M, will make a Product less than its Index in B; that Number is the Number sought; for it's the Index of the greatest Power of M which B does not measure, since B measures none of the Powers of M till you come to one which has all the Primes of B, at least as oft involved; but any Power of M is the Product of the like Powers of all its Factors (as follows from *Theor.* I. *Book* III. *Chap.* I.) Hence the Truth of the Rule is clear.

Exam. 1. $\frac{2}{15}$ or $\frac{37}{15}$, and the Ratio 1 : 7; there can never be any integral Term, because all the Primes of 15, which are 3, 5, are different from 7.

Exam. 2. $\frac{2}{3}$ or $\frac{7}{3}$, and the Ratio 1 : 6. There can be no fractional Term added, because 3 is an *aliquot* Part of 6, whereby the very next Term will be an Integer.

Exam. 3. $\frac{7}{24}$ or $\frac{75}{24}$, and the Ratio 1 : 6. Here $24 = 2\times2\times2\times3$ and $6 = 2\times3$; then the Index of 2 in 24 is 3, and in 6 it is 1, and the greatest Number, which multiplying 1 will produce a Number less than 3 is 2; therefore there can be but 2 fractional Terms joined with the given one; as here, $\frac{7}{24} : \frac{42}{24} : \frac{252}{24} : \frac{1512}{24} = 63$: And here, $\frac{75}{24} : \frac{450}{24} : \frac{2700}{24} : \frac{16200}{24} = 675$.

Case

Case II. *The given Ratio not Multiple, as* N : M, *which we shall again subdivide according as the given Fraction is proper or improper.*

(1°.) If the given Fraction is proper, it's already shewn, that the Series must continue for ever in Fractions decreasing, but will not in all Cases do so encreasing; and the Number sought may be found thus; resolve A, B, N, M, into their Primes, and if there is any Prime in N which is not in A, or the same Prime oftner involved; or again, any Prime in B which is not in M, then the Series will continue for ever in Fractions; because the Denominators can never measure any of the Numerators.

$$\frac{A}{B} : \frac{AM}{BN} : \frac{AM^2}{BN^2} : \&c.$$

But if none of these Circumstances happen, *i. e.* If there is no Prime in B which is not in M, nor in N which is not in A, nor any the same Prime oftner involved in N than in A, then if all the Primes of B are at least equally involved in M, the very first Term after $\frac{A}{B}$ is an Integer. But if any one or more Primes of B are not so oft involved in M; take all of those whose Indexes in B and M have the same Difference, and that too, the greatest of any other of them, and chuse that one of these whose Index is greatest; then find n, the greatest Number which multiplying its Index in M, produces a Number less than its Index in B, and that is the greatest Number of fractional Terms; supposing there can be an Integer; the Reason is easy: And to know if there can be an Integer do this; take all the Primes whose Indexes in N and A have the same Difference, and that, the least of any other, then chuse that one whose Index is greatest; and if the Product of its Index in N, multiplied by $n+1$, produces a Number not exceeding its Index in A, then the $n+1$ Term after $\frac{A}{B}$ is an Integer, and n is the Number sought; but if it's greater, the Series will continue for ever in Fractions.

Exam. 1. $\frac{A}{B}=\frac{21}{55}=\frac{3\times7}{5\times11}$ and $\frac{M}{N}=\frac{65}{22}=\frac{5\times13}{2\times11}$ or $\frac{M}{N}=\frac{98}{45}=\frac{2\times7\times7}{3\times3\times5}$. Here the Series will continue for ever in Fractions.

Exam. 2. $\frac{A}{B}=\frac{42}{55}=\frac{3\times7\times2}{5\times11}$ and $\frac{M}{N}=\frac{275}{21}=\frac{5\times5\times11}{3\times7}$. Here the very first Term after $\frac{A}{B}$ is an Integer, *viz.* $\frac{42}{55}\times\frac{275}{21}$ or $\frac{3\times7\times2}{5\times11}\times\frac{5\times5\times11}{3\times7}=10$, by taking away common Factors, out of the Numerators and Denominators.

Exam. 3. $\frac{A}{B}=\frac{80}{567}=\frac{2\times2\times2\times2\times5}{3\times3\times3\times3\times7}$ and $\frac{M}{N}=\frac{105}{2}=\frac{3\times5\times7}{2}$. Here there will be 3 fractional Terms after $\frac{A}{B}$, which I find thus; 3 is the Prime in B, whose Index exceeds its Index in M by the greatest Difference, and no other have the same Difference; also 3 is the greatest Number which multiplying 1, its Index in M, makes a Product 3 less than its Index in B, which is 4. Again, as there is but one Prime in N; *viz.* 2, whose Index is only 1, this multiplied by 4 $(=n+1)$ produces 4, a Number not exceeding 4, the Index of 2 in A; therefore 3 is the Number sought. But if we suppose A to be $40=2\times2\times2\times5$, then because the Product 4 exceeds 3, the Index of 2 in A, therefore the Series must continue for ever in Fractions, because the $n+1$ Term after $\frac{A}{B}$ is the first that could possibly be an Integer, upon the Consideration of the Primes of B compared with the same in M. But then this is hindered by the Consideration of the

the Primes of N, compared with the same in A. For at the $n+1$ Term, there is a Prime in N, whose Index is greater than its Index in A; and consequently cannot measure it, nor can it ever after; because its Index in N is still increasing.

(2°.) If the given Fraction is improper, the Number of fractional Terms in the Series after $\frac{A}{B}$, either increasing or decreasing, is found the same Way as in the last Article. And observe, that if the Number is limited increasing, then it's certainly infinite decreasing; but if it's found infinite, increasing, it may be either finite or infinite, decreasing.

SCHOLIUMS.

1st. In proposing to find how many fractional Terms can be join'd to a given Fraction, we have not distinguish'd betwixt proper and improper Fractions; but that I shall here consider. And,

1°. It's plain, that from an improper Fraction, all the fractional Terms, increasing, must be improper; but decreasing, they may be partly proper and partly improper; and if there stands an Integer below the given improper Fraction, all the intermediate Terms are improper Fractions.

2°. From a given proper Fraction, all the fractional Terms decreasing, are proper; but increasing, they may be partly proper, partly improper; and, in some Cases, all improper.

Hence there are but two Cases concerning the distinct Number of proper and improper Fractions, wherein a Rule is wanted: The first is to determine how many improper Fractions can be joined, decreasing, to a given one in any given Ratio. The second is to find how many proper Fractions can be joined, increasing, to a given one in any Ratio. Now if the Ratio is not Multiple, in either of these Cases, I know of no Rule but actually raising the Series; but if the Ratio is Multiple, they can be solved otherwise: Thus,

Question 1. To find how many improper Fractions can be added, decreasing below a given one, $\frac{A}{B}$, in a given Multiple Ratio, $1:M$.

Rule. Take the integral Part of the given Number; which Integer call q. Then take the greatest Power of M, which does not exceed q; the Index of that Power is the Number sought.

DEMON. The Index of M, expresses the Distance of each

$\frac{A}{B} : \frac{A}{BM} : \frac{A}{BM^2} : \&c.$ | Term from $\frac{A}{B}$; and if M^n is the greatest Power of M, not exceeding q, then is $\frac{A}{BM^n}$ the last improper Fraction: For, if any of the Powers of M, in the Divisors, exceeds q, it must be, at least, equal to $q+1$; and as soon as that happens, we have a proper Fraction; because, if $\frac{A}{B}=q$, with a Remainder, therefore qB is less than A, and $\overline{q+1}\times B$, greater than A: So that $\frac{A}{\overline{q+1}\times B}$ is necessarily a proper Fraction, and $\frac{A}{qB}$ an improper: Therefore, if M^n is the greatest Power of M, not exceeding q, then is $\frac{A}{BM^n}$ the last improper Fraction; for a greater Power, as M^{n+1}, must be, at least, equal to $q+1$; and since $\frac{A}{\overline{q+1}\times B}$ is a proper

 Fraction,

Fraction, therefore so is $\frac{A}{BM^{n+1}}$, and consequently $\frac{A}{BM^n}$ is the last improper one; and n the Number of Terms sought.

Exam. $\frac{A}{B}=\frac{65}{7}=9\frac{2}{7}$ and $M=2$, whose third Power, 8, is the greatest, not exceeding 9; therefore 3 is the greatest Number of improper Fractions that can stand in the same Series below $\frac{65}{7}$, in the Ratio 2 : 1, as here $\frac{65}{7}:\frac{65}{14}:\frac{65}{28}:\frac{65}{56}:\frac{65}{112}$.

Question 2. To find how many proper Fractions can be join'd, increasing in the same Series, above a given one, $\frac{A}{B}$, in a given Multiple Ratio 1 : M.

Rule. Take the reciprocal improper Fraction, $\frac{B}{A}$, and find, as in the last Question, how many improper Fractions can be join'd to it, decreasing; and you have the Number sought: For if $\frac{B}{AM^n}$ is the last improper Fraction, then taking the whole Series reciprocally, $\frac{AM^n}{B}$ is the last proper Fraction after $\frac{A}{B}$, increasing.

2*d*. We may consider the *Problem* unlimited to any particular given Ratio; only in general, suppose the Ratio is Multiple or not Multiple; then,

1°. Suppose an increasing Series such, that it comes at last to an Integer, and a Multiple Ratio; (though there is no particular one given) then find the Prime of B, whose Index is the greatest, that Index less 1 is the Number sought; and if the Product of that Prime, multiplied into all the other Primes of B, is made the Consequent of a Multiple Ratio, by that you may find an Example of the Series: The Reason of the Rule is evident. But again,

2°. If the Problem is limited to a Ratio not Multiple, the Series increasing as before; then take the Prime of B, which has the greatest Index, provided also that there be a Prime in A, which has an equal or greater Index; the Index of that Prime in B, less 1, is the Number sought: And you may find an Example of the Series by a Ratio whose Antecedent is that Prime of A, and its Consequent the Product of all the Primes of B; provided it be a Number greater than the Antecedent; else, take some Multiple of it greater than the Antecedent. Hence, if the Index of the Prime, referred to in B, is 1, there can be no improper Fraction joined with the given Number.

Example for both these Articles. $\frac{A}{B}=\frac{405}{56}=\frac{3\times3\times3\times3\times5}{2\times2\times2\times7}$. Here 2 is the Prime in B, whose Index, 3, is the greatest; and therefore 2 is the Number sought, in the first Article, and 1 : 14 is a Ratio agreeing to this Solution.

Again, If the Ratio is supposed not Multiple, 2 is also the Number sought; because, according to the Rule, there is a Prime in A, whose Index is greater than 3: And if you take for a Ratio, 3 : 14, it will agree to the Solution.

But suppose $\frac{A}{B}=\frac{105}{26}=\frac{3\times5\times7}{2\times13}$; if there comes an Integer into the Series, it must be the very first Term after $\frac{A}{B}$; because there is no Index either in A or B, greater than 1; for no Ratio Multiple can answer the Supposition of the Series coming at last to an Integer, except this, 1 : 26 $(=2\times13,)$ and it's plain, that the very first

Term

Term added will be 105. Again, No Ratio, not Multiple, can answer, but one of these in which the Antecedent is, 3. or 5. or 7. or 3×5. or 3×7. or 5×7. or $3\times5\times7$. and the Consequent 2×13, or some Multiple of this that is greater than the Antecedent: In all which Cases it's manifest, that the first Term added will be an Integer.

3°. Suppose a given improper Fraction, and that a Series ought to proceed from it, decreasing, in a Multiple Ratio: To find the greatest Number of fractional Terms, improper, that can be join'd: From a Table of Powers seek the greatest Power (*i. e.* a Power of the greatest Index) which is a Number not greater than q, the integral Part of $\frac{A}{B}$; and of whose Root the next greater Power is greater than q; the Index of the former Power is the Number sought; and that Root is the Antecedent of a Multiple Ratio, which will make a Series, having the Number of fractional Terms found, after $\frac{A}{B}$.

Exam. $\frac{A}{B}=\frac{65}{7}=9\frac{2}{7}$. The Number sought is 3; for 8, the third Power of 2, is the greatest Power, which does not exceed 9; and the next Power of whose Root (*viz.* 16.) does exceed 9: And by the Ratio, 2 : 1, we have the Series, $\frac{65}{7}:\frac{65}{14}:\frac{65}{28}:\frac{65}{56}$.

PROBLEM XII.

A Fraction, $\frac{A}{B}$, being given for the lesser Extreme of a Series, to find a Ratio, in which a given Number, n, of fractional Terms, and no more, can possibly be in the same Series above the given one.

Solution. 1°. Suppose we are to find a Multiple Ratio; then find the Prime that's oftnest involved in B; if its Index does not exceed n, then the Problem is impossible: For M, the Consequent of any Ratio, 1 : M, which will bring the Series to a whole Number, must contain all the Primes of B; therefore it contains the Root, at least, of that Prime which is most involved in B; but unless the Index of that Involution be greater than n, M^n will be equal to, or greater than B; and consequently the nth Term after the given Extreme, is a whole Number; or, perhaps, there may be one before this. Again, though the Index of some of the Primes of B exceeds n, yet the Problem may be impossible; for to make it possible, the Index of some of these Primes must be such, that there be some inferior Power, whose Index, multiplied by n, makes a Product less than its Index in B; but multiplied by $n+1$, makes a Number not less than that Index; and then the Product of that inferior Power of any such Prime in B, multiplied continually into such Powers of all the other Primes of B, as that their Indexes, multiplied by $n+1$, makes a Number not less than their Indexes in B, may be taken for the Consequent of the Ratio sought. The Reason of all this is manifest, by considering that an Integer can never come into the Series, till the Power of M, in the Numerator, contains the several Primes of B, involved, at least, as oft: And as soon as that happens, we have a whole Number.

$$\frac{A}{B}:\frac{AM}{B}:\frac{AM^2}{B}:\frac{AM^3}{B}:\&c.$$

Exam. 1. $\frac{A}{B}=\frac{36}{175}=\frac{2\times2\times3\times3}{5\times5\times7}$, and $n=3$. The Problem is impossible; because there is no Index of a Prime in B, which exceeds 3.

Exam.

Exam. 2. $\frac{A}{B} = \frac{25}{3456} = \frac{5\times5}{3\times3\times3\times2\times2\times2\times2\times2\times2\times2}$, and $n=3$; then is the Problem possible; because the Index of 2 in B, is 7; and if we take the second Power of 2 (*viz.* 4) the Index of this, multiplied by 3, is 6; and multiplied by $3+1$, or 4, makes 8; therefore the Ratio sought is 1 : 12; and the Series is $\frac{25}{3456} : \frac{300}{3456} : \frac{3600}{3456} : \frac{43200}{3456} : \frac{518400}{3456} = 150$. And the Reason of the Rule will appear in this Example, by setting it down in this Manner, *viz.*

$$\frac{25}{3^2\times2^7} : \frac{25\times3\times2^2}{3^3\times2^7} : \frac{25\times3^2\times2^4}{3^3\times2^7} : \frac{25\times3^3\times2^6}{3^3\times2^7} : \frac{25\times3^4\times2^8}{3^3\times2^7} = 25\times3\times2.$$

Case II. *Suppose a Ratio not Multiple, as* N : M, *is to be found.*

The Problem is possible only when there is some Prime in B, whose Index is greater than n, in the Manner described in the preceding Case; and when at the same Time there is some Prime in A, whose Index is greater than n, so much that there be some inferior Power of it, whose Index multiplied by $n+1$, shall produce a Number not exceeding its Index in A. Then for the Consequent of the Ratio, take such a Number as was directed in the first Case; and for the Antecedent, take any one, or the Product of any two or more of such inferior Powers of the Primes in A, whose Indexes are as above described. And observe, that if the Consequent, found by the Direction of the first Case, is a Number less than any of the Antecedents, found by the Direction of this Rule, then multiply it by some other Prime Number, which is not in B; so that the Product be greater than the Antecedent.

The *Reason* of all this Rule is to be found from the same Consideration as that of the last Case.

Exam. $\frac{A}{B} = \frac{135}{56} = \frac{5\times3\times3\times3}{7\times2\times2\times2}$, and $n=2$. The Problem is possible; because the Index of 2 in B, is 3, and $2\times1=2$ (1 being the Index of the first Power of 2) and $2+1$, or $3\times1=3$. Then again, the Index of 3 in A is 3, and $3\times1=3$ (1 being the Index of the first Power of 3) and for the Ratio sought, we have 3 : 14 ($=2\times7$) as in this Series, $\frac{135}{56} : \frac{1890}{168} : \frac{26460}{504} : \frac{370440}{1512} = 245$; which we may also represent in the following Manner; whereby the Reason of the Rule will clearly appear,

$$\frac{5\times3^3}{7\times2^3} : \frac{5\times3^3\times2\times7}{7\times2^3\times3} : \frac{5\times3^3\times2^2\times7^2}{7\times2^3\times3^2} : \frac{5\times3^3\times2^3\times7^3}{7\times2^3\times3^3} = 5\times7^2.$$

SCHOLIUMS.

1st. If it's proposed to find a Ratio, in which no Fractional Term can be added; take any Number, which is a Multiple of B, for the Consequent, and any *aliquot* Part of A, or A it self, for the Antecedent. The Reason is plain.

Or, if it's proposed that all the Terms be Fractional *ad Infinitum*; we have no more to do, but take $\frac{M}{N}$, such, that N contain some Prime which is not in A, or the same Prime oftner involved; or take M, such, that it contain not all the different Primes of B.

2*d*. The greatest limited Number of Fractional Terms that can possibly be joined in the same Series with a Fraction, must necessarily lie all on one Side; and not partly above and partly below; because Fractions cannot lie betwixt Integers in the same Series: And as the greatest Number may lie either above or below, in different Cases; the Reason why I have limited the Problem to an increasing Series, is plainly because I know of no determinate Rule that will reach to all Cases for a decreasing Series: The Reason of which seems to be this, That it does not depend only upon the Consideration of like or unlike Primes, and their different Involutions, but also upon their particular Quantities and Proportions; yet we have these two Particulars to observe:

(1°.) That as from a proper Fraction a Series decreases in proper Fractions, for ever, in whatever Ratio: So,

(2.) The given Fraction being improper, if the Ratio is required to be Multiple, the Problem is impossible; the Reason of which has been explained in *Probl.* XI. *Case* I. But if it's only required to find such a Multiple Ratio, in which a given Number of improper Fractions, and no more, can be joined to a given improper Fraction, $\frac{A}{B}$, decreasing; this we can find, *thus*, Take q, the integral Part of the given mix'd Number, and seek a prime Number, M, different from any of the Primes of A; and such also, that it's n Power (n being the given Number of Terms) be the greatest of its Powers, which does not exceed q; and that Prime is the Antecedent of a Ratio, which solves the Question: For it has been shewn, that $\frac{A}{BM^n}$ is the last improper Fraction; and that it is not an Integer, also that none of the added Terms are so is plain; because M is a Prime different from any of these in A.

PROBLEM XIII.

To find a Fraction with which may be joined in the same Series, increasing, a given Number, and no more fractional Terms, in a given Ratio.

SOLUTION.

Case I. *Suppose a Ratio Multiple*, 1 : M, *and the given Number of Terms*, n.

$$\frac{A}{B} : \frac{AM}{B} : \frac{AM^2}{B} : \&c.$$

Resolve M into all its Primes; and for the Denominator sought, take any Number, B, composed of some or all of the same Primes, and no other than are in M: But let some one of them, at least, be oftner involved in B than in M; so much, that taking all of these, whose Indexes in B and M have the same Difference, and that too, the greatest of any other of them; and chusing any one of these, whose Index is the greatest; the Product of its Index in M, multiplied by n, shall be less than its Index in B; but multiplied by $n+1$, it shall not be less.

Then for a Numerator, take A, any Number whatever Prime to B; and you have the Fraction sought; which is proper or improper, as A is less or greater than B.

Exam. M=72, and n=4. The Primes of M are $2\times2\times2\times3\times3$: The Index of 3 in 72 is 2; which multiplied by 4 gives 8; and multiplied by $4+1$, or 5, gives 10: Therefore I take a Number, for B, which has 3 involved in it to the 9th Power; because 9 is greater than 8, but not greater than 10: Such a Number is the 9th Power of 3 it self, *viz.* 19683; which we may also multiply by any other of the Primes of M, as 2, or any such Power of any of these Primes, whose Index is either not less than its Index in M, or which does not want of it the Difference betwixt the Indexes of the first assumed Primes in B and M. The rest of the Work is obvious.

Case

Case II. *Suppose the Ratio not Multiple, as* N : M.

Find the Denominator, B, the same Way as before, from the Primes of M: Then for a Numerator, take A, any Number composed of all the Primes of N (and any other you please, provided they are none of those of B) but let them be so much oftner involved, that taking all of these whose Indexes in N and A have the same Difference, and that, the least of any other of them; and chusing that one of them whose Index is greatest, the Product of its Index in N by $n+1$, shall not exceed its Index in A.

The Invention of the Numbers sought, in these Cases, has no Difficulty; and the *Reason* is contained in what has been explained in the preceding Problems, especially *Probl.* XI. And 'tis to be observed too, that this Problem is always possible, *i. e.* with any Ratio, and any Number of Terms.

SCHOLIUMS.

1st. Suppose the Number required ought to be such, that the very first Term after it is a whole Number; then for a Multiple Ratio, 1 : M, take for B any *aliquot* Part of M, or M it self; and for A, any Number prime to B. Again, for a Ratio not Multiple, as N : M, take B, as before; and for A take N, or any Multiple of it, so that $\frac{A}{B}$ be not a whole Number.

If the Number of Terms to be added is infinite, take B, such, that it have some Prime, which is not in M.

2d. If no particular Ratio is given, *i. e.* If it's required to find such a Fraction, that a given Number is the greatest Number of Fractional Terms that can possibly be joined with it in the same Series increasing, in any Ratio whatever; then the Problem is impossible: For no such Fraction can be found; because with every Multiple Ratio, an infinite Number of Fractional Terms can be join'd in the same Series with any Fraction; and it may be so too, with many Ratios not Multiple: But if we suppose the Series limited to this Condition, that it shall not continue for ever in Fractions; then,

(1°.) Suppose a Multiple Ratio: Take for B, any Number whereof that Prime which is oftnest involved shall have $n+1$ for its Index; and for A, take any Number Prime to B: And to find an Example of the Series, take for M, (the Consequent of the Ratio) the Product of the Roots of all the Primes in B, or that Product multiplied by any other Prime.

(2°.) Suppose a Ratio not Multiple; take B, as before; and for A, let it be a Number prime to B, and such that all its Primes be involved to the $n+1$ Power: Then the Invention of a Ratio, answering the Problem, is this; Take the Consequent M, as before; and for the Antecedent N, take any of the Primes of A, or the Product of any two or more of them, provided it be less than the Consequent.

3d. What has been said in *Schol. 2d.* of the last Problem, may be repeated here, for the Reason why this Problem is limited to an increasing Series: Yet concerning decreasing Series, we have to add these Particulars, *viz.*

(1°.) Suppose the given Ratio is Multiple; then for a decreasing Series the Problem is impossible: For the Number of Fractional Terms will be infinite after any Fraction.

(2°.) Suppose we ought to find a mix'd Number, with which may be join'd a given Number, n, of mix'd Terms, and no more, decreasing in a given Multiple Ratio, M : 1; it may be found, *thus*; Take the n Power of M, and then any Number, q, not exceeding M^n. *Lastly*, Take any mix'd Number, $\frac{A}{B}$, whose integral Part is q;

but take Care alſo that A be ſuch that there be ſome Prime in M, which is not in A, or the ſame Prime oftner involved.

(3°.) If the mix'd Number in the laſt Article is required to be invented in its leaſt Terms; take for B any Number at Pleaſure, which is not a Multiple of 2 (*i. e.* any odd Number) and for A, multiply B by the n Power of 2; to the Product add 1 [or more generally add any Number, whereby the Sum may be Prime to B; and at the ſame Time a Number leſſer than the Product of B by the $n+1$ Power of 2] and make this Sum the Numerator: And if 2 is made the Ratio, 'tis evident there will be n mix'd Terms and no more. Again, with a greater Ratio than 2, it's certain there will be fewer mix'd Terms; for the Denominator will ſooner exceed the Numerator: Hence 'tis plain that $\frac{A}{B}$ is ſuch a mix'd Number as was required. Or ſee the Reaſon of it, *thus*; The Number found, according to this Rule, is thus expreſſed, $\frac{B\times 2^n+1}{B}$; to which n Number of Terms being joined in the Ratio, 2 : 1, the laſt of them is $\frac{B\times 2^n+1}{B\times 2^n}$, which is an improper Fraction; and the very next Term would be a proper one, *viz.* $\frac{B\times 2^n+1}{B\times 2^{n+1}}$; where 2^{n+1} is manifeſtly greater than 2^n+1.

§. IV. *Of Numbers* odd *and* even.

THEOREMS I, II, III, IV.

IN *Addition of Numbers* theſe Things are true;

1ſt. The Sum of two or more even Numbers, is an even Number; for 2 meaſures each of the Parts, therefore it meaſures the Whole (*Ax.* 1.) *Exam.* $4+6+8=18$.

2d. The Sum of two odd Numbers is an even Number; for 1 taken from each of them, leaves an even Number, but $1+1=2$: So that the Sum is, the Sum of two even Numbers and 2 added. *Exam.* $5+7=12=4+6+2$.

3d. The Sum of an even and an odd Number, is an odd Number; for 2 cannot meaſure both the Parts, ſince it cannot meaſure an odd Number; (*Cor.* 1. *Ax.* 3.) therefore it cannot meaſure the Whole (*Ax.* 1.) Or thus, 1 taken from the odd Number, leaves an even; ſo that the Sum is the Sum of 2 even Numbers, (which is an even) and 1, which makes an odd Number (by *Cor.* 4. *Defin.* 8.)

4th. If more than 2 Numbers are added, which are all odd Numbers, or partly even partly odd; the Sum is even or odd, according as the Number of odd Parts is even or odd; which follows eaſily from the former Articles.

Exam. $3+5+7=15$, and $3+5+13+19=40$. Alſo,
$3+4+6=13$, and $3+5+4=12$.

THEOREMS V, VI, VII.

In *Subſtraction* theſe Truths are evident, being the Reverſe of the former, *viz.*

5th. The Difference of 2 even Numbers, is an even Number: So $8-4=4$.

6th. The Difference of 2 odd Numbers, is an even Number: So $9-5=4$.

7th. The Difference of an even and an odd Number, is an odd Number: So $14-5=9$, and $19-8=11$.

THEOREMS VIII, IX.

In *Multiplication* theſe Truths are evident from the firſt *Theorem*; becauſe Multiplication is only a repeated Addition.

8th.

8th. The Product of two even Numbers, or of one odd and even Number, is an even Number; for it's only the Repetition of an even Number, or a Sum of even Numbers: So $4\times8=32$, and $6\times7=42$.

9th. Two odd Numbers produce an odd Number; for it's the Sum of an odd Number of odd Parts: So $3\times7=21$.

Coroll. The Powers of an even Number are all even; and of an odd Number are all odd. Also the Product of more than 2 odd or even Numbers, is odd or even; and the Product of several Factors partly odd, partly even, is even.

Theorems X, XI, XII.

In *Division*, these Truths follow from the last; because this is the Reverse of that.

10th. An even Number measures an even Number by an even, or an odd, in different Cases: So $12\div4=3$, and $12\div6=2$; the Reverse of which is, that the Measure of an even Number may be odd or even.

11th. An odd Number measures an even, only by an even: So $24\div3=8$.

12th. An odd Number measures an odd, only by an odd: So $35\div7=5$.

COROLLARIES.

1st. The Roots of odd or even Numbers are all odd or even.

2d. The Number which an odd Number measures, may be either odd or even; which is plain from the second and third Articles: But the Number which an even Number measures, must be even; else the Product of two even Numbers, or of an even and odd (*viz.* the Quote and Divisor) would be odd, contrary to *Theor.* 8; which last Part we may also express thus, *viz.* An even Number cannot measure an odd: Or also thus, There is no even Number in the Composition of an odd; and so, lastly, An odd Number only can measure an odd.

Theorem XIII.

There is no Number whatever (excluding 1) that will measure all odd Numbers, because an infinite Number of those are prime Numbers; but all even Numbers have a common Measure, *viz.* 2, from the Definition.

Theorem XIV.

Two odd Numbers, that differ by 2 (*i. e.* every two adjacent Terms in the Series of odd Numbers) are Incommensurable; for dividing them by the Rule for finding their greatest common Measure, the first Remainder is 2; and the next will be 1; which is the greatest common Measure.

Theorem XV.

If an odd Number measures an even, it will also measure the Half of the even Number. *Exam.* 3 measures both 12 and 6.

A)	B	(C
3	12	4

Demon. If A, an odd Number, measures B, an even, let the Quote be C, it is an even Number (*Theor.* XI.) which will back again measure B by A; but B and C, being both even, are measurable by 2. Also $C : B :: \frac{C}{2} : \frac{B}{2}$. Now since C measures B by A, therefore $\frac{C}{2}$ measures $\frac{B}{2}$ by A; and back again, A measures $\frac{B}{2}$ by $\frac{C}{2}$: Otherwise thus, any even Number may be expressed 2N; and if A measures 2N by C; then is $AC=2N$, and $A : N :: 2 : c$. But 2 measures C, which is an even Number; therefore A measures N, the Half of 2N.

Theo-

Theorem XVI.

If an odd Number, A, be Incommenſurable with any Number, B; it will be ſo alſo with the Double of B, or 2B.

	B
m	A . d
	2B

Demon. If A and 2B are Commenſurable, ſuppoſe m their common Meaſure; it muſt be an odd Number, becauſe it meaſures an odd Number (for no even can meaſure an odd, *Cor.* 2. *Theor.* XII.) let the Quote be d, it is alſo an odd Number (*Theor.* XII.) Again, 2B is an even Number; and becauſe m, an odd Number, meaſures it, therefore it meaſures its Half, B (*Theor.* XV.) conſequently, A, B, are not Incommenſurable, contrary to Suppoſition.

Coroll. For the ſame Reaſon, A will be Incommenſurable with any Product of B, multiplied into ſome odd Number.

Theorem XVII.

All even Numbers are either ſome Power of 2, or ſome of thoſe multiplied into ſome odd Number.

2) A
2) B
2) C
2) D
E

Demon. All the Powers of 2 are even Numbers (*Cor.* after *Theor.* IX.) but an even Number, which is not any Power of 2, is the Product of ſuch a Power by ſome odd Number: Suppoſe any even Number, A, divide it by 2, and let the Quote be B; divide this again by 2, and let the Quote be C; and ſo on, as in the Margin: As long as 2 meaſures the ſucceeding Quotes, the laſt Quote will be either 1, or ſome odd Number; for as long as any of thoſe Quotes is an even Number, it's again meaſurable by 2; and ſo that is not the laſt Quote. If it's 1, then A is ſome Power of 2; for it's the Product of all theſe Diviſors; and if it's not 1, it muſt therefore be ſome odd Number; conſequently A is compoſed of thoſe Factors, *viz.* that odd Number which is the laſt Quote, and 2 as oft involved as the Number of Diviſions, *i. e.* ſuch a Power of 2, whoſe Index is the Number of Diviſions: Thus, if B is an odd Number, then is $A=2B$; if C is an odd Number, $A=2\times2\times C$, and ſo on.

Or, the whole Demonſtration may be made thus; No even Number is a Prime but 2; and all other Primes are odd, by Definitions; therefore, let any even Number, A, be reduced to its component Primes, ſome one, or more of thoſe, muſt be 2; elſe it cannot be an even Number: Then either there is no other Prime amongſt theſe but the Number 2; ſo that A is a Power of 2; or if there are alſo other Primes, they muſt be odd Numbers; and if there are more than one ſuch, their Product is an odd Number (*Theor.* IX) Conſequently A is the Product of ſome Power of 2, by ſome odd Number.

Theorem XVIII.

All even Numbers, above 2, have ſome one of theſe Qualities, *viz.* they are either,

1*ſt*. Evenly even only, *i. e.* They are not alſo oddly even; and ſuch are all the ſuperior Powers of 2, and none other.

2*d*. Oddly even only, *i. e.* They are not alſo evenly even; and ſuch are all the Doubles of every odd Number, or the Products by 2, and none other.

3*d*. Both evenly even, and oddly even; and ſuch are the Products of any odd Number, by any Power of 2 (above 2) and none other.

Demon. It's evident that all even Numbers, above 2, muſt have ſome one of the three Qualities mentioned; and that the ſame Number can have but one of them. What is to be demonſtrated, is therefore this; That the ſeveral Claſſes of Numbers deſcrib'd, have the Qualities aſſigned to them; and that none other have theſe Qualities.

1°. The

1°. The superior Powers of 2 are evenly even; this is manifest: Then again, they are not oddly even; for 2 being a Prime, no Number can measure any of its Powers, but some other of its Powers (*Cor. 6. Theor.* X. *Ch.* 1.) which are all even Numbers; therefore no odd Number can measure them, *i. e.* they are not oddly even.

2°. The Multiple of any odd Number A by 2, or 2A, is odly even by the Definition; but it's not evenly even; for if we suppose it is, let $2A=BC$ (two even Numbers) then $2 : B :: C : A$, but 2 measures the even Number, B; and therefore so must the even Number, C, measure the odd Number, A; which is impossible (*Theor.* XII. *Cor.* 2.)

3°. Any superior Power of 2, multiplied into an odd Number, produces a Number which is both evenly even and oddly even: It is oddly even by the Definition; but it's also evenly even; for let the odd Number concerned in it, be called o, and the supposed Power of 2 be m; then is m equal to the next Power below, multiplied by 2: Call that mix'd Power, p, so is $2p=m$, consequently $o\times m=2\times p\times o$; but po is an even Number (*viz.* the Multiple of an even Number, p) therefore $2\times po$ ($=mo$) is evenly even.

4°. There can be no even Number of any of the Qualities mentioned, but these described; for it's shewn, that none of the several Classes can be of any other of these Qualities, but that assigned to it; and these three Classes comprehend all even Numbers, by (*Theor.* XVII.) Since they are either Powers of 2, or some of those multiplied by some odd Number; the first Class comprehends all the superior Powers of 2; the second comprehends 2 and its Products by all odd Numbers; the third comprehends the Product of all the superior Powers of 2 by all odd Numbers.

From the Nature and Manner of producing the Species of even Numbers, here explained, the following Consequences will easily appear.

COROLLARIES.

1*st*. Every Number, which is evenly even, has an even Half, or is measurable by 4; for it's either some superior Power of 2, as 4 . 8 . 16 . &c. each of which is measurable by 4; or it's the Multiple of such a Power by some odd Number.

2*d*. The Product of two Numbers, evenly even only, is evenly even only; being the Product of two Powers of 2; which is also some Power of 2 from the Nature of Powers.

3*d*. A Number evenly even only, multiplied by any Number oddly even only, or both oddly and evenly even, produces a Number which is both oddly and evenly even; because there is in the Composition of the Product some odd Number, and also some higher Power of 2.

4*th*. A Number oddly even only, multiplied by any even Number whatever, produces a Number both oddly and evenly even; but multiplied by an odd Number, produces a Number oddly even only; the Reason of both which is manifest.

5*th*. A Number both oddly and evenly even, multiplied by any Number whatever, produces a Number both oddly and evenly even.

6*th*. An evenly even only, can be measured by none but another such, or the Root 2; because it is the Power of a Prime Number (2) which can be measured by none but some of the inferior Powers of that Prime; wherefore an evenly even only, being measured by another such, quotes another such, or 2.

7*th*. A Number both oddly and evenly even, may be measured by any Kind of Number whatever.

8*th*. A Number oddly even only, can be measured by no Number, but 2 or some odd Number: Whence again,

9*th*. The

9th. The Number which is meaſured by an evenly even only, is either evenly even only, or both oddly and evenly even; and the Number which one both oddly and evenly even, or one oddly even only, meaſures, is alſo both oddly and evenly even.

10th. If you take the Series of even Numbers, 2. 4. 6. 8. 10. 12. 14 *&c.* and beginning at 6, take every other Term, *i. e.* paſſing one take the next, *viz.* 6. 10. 14 *&c.* you have the Series of Numbers oddly even only, becauſe the Series of even Numbers are the Doubles of the ſeveral Terms of the natural Progreſſion 1, 2, 3 *&c.* whereof every other Term, beginning at 3, make the Series of odd Numbers, and the Doubles of thoſe is the Series of Numbers oddly even only; ſo that 6 being 2×3, every other Term after this makes the Series oddly even only.

Theorem XIX.

1	3.	5.	7.	9.	11, *&c.*
2	6.	10.	14.	18.	22, *&c.*
4	12.	20.	28.	36.	44, *&c.*
8	24.	40.	56.	72.	88, *&c.*
16	48.	80.	112.	144.	176, *&c.*

Take the Series of odd Numbers from 1, then the Series of their Doubles, then the Doubles of this Series, and ſo on; as the firſt Line is the Progreſſion of odd Numbers, ſo the ſecond Line is the Series of Numbers oddly even only, and the firſt Column on the left Hand (excluding 1.2) is the Series of Numbers evenly even only: The other Columns below the ſecond Line containing the Numbers that are both oddly and evenly even.

The Truth of all this is evident from the Definitions of theſe Kind of Numbers, and the Conſtruction of this Table; upon which this alſo is remarkable, *viz.* That each Line (taken from left to right) is an Arithmetical Progreſſion, whoſe common Difference is double the firſt Term; the Reaſon of which will be plain from theſe Conſiderations, *viz.* 1°. Becauſe it is ſo in the firſt Line or Series of odd Numbers. 2°. By the Conſtruction of the Table the firſt Term of every Line is double the firſt Term of the preceding. 3°. The Product of any Arithmetical Progreſſion is an Arithmetical Progreſſion, whoſe Difference is the Product of the former Difference by the common Multiplier. Again, each Column from top to bottom is a Geometrical Progreſſion in the Ratio of 1 : 2, as is plain from the Conſtruction. So that the whole Syſtem of Numbers that are both oddly and evenly even, proceed either from the ſeveral Numbers which (excluding 2) are oddly even only, by multiplying each of theſe continually by 2, making ſo many Geometrical Progreſſions; or from the Numbers evenly even only, by adding to theſe continually their Doubles, making ſo many Arithmetical Progreſſions.

Theorem XX.

The Sum of Numbers that are all oddly even only, may be oddly even only, or evenly even only, or both; *Thus*, if the Number of Terms added is odd, the Sum is oddly even only; but if it's even, the Sum is evenly even only, or both oddly and evenly even.

Exam. $6+10+14=30$, oddly even only,
$6+10 = 16$, evenly even only,
$6+10+14+18=48$, both oddly and evenly even.

Demon. Let $a, b, c, d,$ &c. be any odd Numbers, then are $2a, 2b, 2c, 2d,$ &c. Numbers oddly even only; and their Sum $2a, +2b+2c, + 2d$ &c.$=2\times\overline{a+b+c+d}$ &c. is oddly even only, if the Number of Terms added is odd; for an odd Number of odd Numbers makes together an odd Number, and an odd Number doubled makes a Number oddly even only. Again, if the Number of Terms added is even, then their Sum is even, and may be repreſented by 2 A, and therefore its double is 4A, or 2×2A, which, it's plain, cannot be a Number oddly even only; and therefore muſt be either

either evenly even only, or both oddly even and evenly even. That in some Cases it will be the one way, in some the other, the preceding Examples shew; and you'll learn afterwards how to invent Examples of each Kind.

Theorem XXI.

The Sum of any Number of Numbers that are all evenly even only, is both oddly and evenly even. *Exam.* $4+8+16+32=60=2\times30=6\times10$.

Demon. The given Numbers may be represented thus, 2^n, 2^{n+r}, 2^{n+s} 2^{n+t}; and in the first Place, because $2^{n+r}=2^n\times2^r$, therefore $2^n+2^{n+r}=2^n+2^n\times2^r=2^n\times\underline{2^r+1}$. But 2^r+1 is odd, therefore $2^n\times\underline{2^r+1}$ is oddly and evenly even. This Sum we may now represent by $2^n\times o$ (o being the odd Number $=2^r+1$) then is $2^n\times o+2^{n+s}=2^n\times\underline{2^s+o}$; which for the same Reason as before is both oddly and evenly even. Call this Sum again $2^n\times o$, and the next Sum is $2^n\times o+2^{n+t}=2^n\times\underline{2^t+o}$, which is both oddly and evenly even; and so on.

Theorem XXII.

The Sum of any Number of Terms, all both oddly and evenly even, is either evenly even only, or both oddly and evenly even; and particularly, if the Number of Terms is 2, the Sum is both oddly and evenly even.

Demon. The Sum cannot be oddly even only, because each of the Terms has an even half, or is measurable by 4; whence the Sum is also measurable by 4, and consequently it is either evenly even only, or both oddly and evenly even (of which you'll find Examples and Rules how to invent them afterwards). Again, if there are but 2 Terms, the Sum is both oddly and evenly even: For every Number, both oddly and evenly even, is the Product of an odd Number, by some superior Power of 2; wherefore let a, o represent two odd Numbers, and 2^n, 2^{n+r}, two Powers of 2, then will $2^n\times o$, $2^{n+r}\times a$, represent any two Numbers both oddly and evenly even; but their Sum is $2^n\times o+2^{n+r}\times a=2^n\times\underline{o+2^r\times a}$; and 2^r being even, $2^r\times a$ is even also, and $o+2^r\times a$ is odd; consequently $\overline{2^n\times o+2^r\times a}$ (or $2^n\times o+2^{n+r}\times a$) is both oddly and evenly even, being the Product of an odd Number by a superior Power of 2.

Theorem XXIII.

The Sum of any Number of Terms all evenly even only, and any Number all both oddly and evenly even is either evenly even only, or both oddly and evenly even; and particularly, one Number evenly even only, being added to another both oddly and evenly even, the Sum is both oddly and evenly even.

Demon. The Sum cannot be oddly even only, because each Part is measurable by 4, therefore it is either evenly even only, or both oddly and evenly even. Again, if there is but one of each, the Sum is both oddly and evenly even; for the 2 supposed Numbers may be expressed thus, 2^n, and $2^{n+r}\times o$, whose Sum is $=2^n+2^{n+r}\times o=2^n\times\underline{1+2^r\times o}$; but 2^r is even, and o is odd, therefore $2^r\times o$ is even, and therefore $1+2^r\times o$ is odd, and hence $2^n\times\underline{1+2^r\times o}$ is both oddly and evenly even; or the 2 Numbers may be represented thus, 2^{n+r}; $2^n\times o$, whose Sum is $2^{n+r}+2^n\times o=2^n\times\underline{2^r+o}$; but 2^r+o is odd, and 2^n even, hence $2^n\times\underline{2^r+o}$ is both oddly and evenly even.

Theorem XXIV.

Any Number of Terms oddly even only, with any Number evenly even only, or both oddly and evenly even, make a Sum either oddly even only, or both oddly and evenly even; particularly, 1st. Any odd Number (among which reckon 1) of Terms,

all of them oddly even only, added to one or any Number, all evenly even only, or both oddly and evenly even, makes the Sum oddly even only.

2°. Any even Number of Terms all oddly even only, added to one or any Number of Terms all evenly even only, or both oddly and evenly even, make a Sum both oddly and evenly even.

DEMON. For Article first, which must be subdivided thus,

(1°.) Suppose one Number, oddly even only, added to one either evenly even only, or both oddly and evenly even, their Sum is oddly even only; for let o be any odd Number, and e even, then $2\times o$ represents any Number oddly even only; and if e is a Power of 2, then $2\times e$ represents any Number evenly even only; else $2\times e$ is a Number both oddly and evenly even; but it's plain, that $2\times o+2\times e=2\times\overline{o+e}$, and $o+e$ is odd, hence $2\times o+2\times e$ is oddly even only.

(2°.) Suppose more generally any odd Number (including 1) of Terms all oddly even only, added to any Number, evenly even only, or both oddly and evenly even, the total Sum is oddly even only; for the Sum of the Numbers oddly even only is oddly even only (*Theor.* XX.) and the Sum of these that are evenly even only is both oddly and evenly even (*Theor.* XXI.) which two Sums make the total Sum oddly even only (by what's last demonstrated.) But if the other Part consists of Numbers, both oddly and evenly even, then their Sum is either evenly even only, or both oddly and evenly even (*Theor.* XXII.) either of which added to the former, which is oddly even only, the Total is oddly even only (*Case* I.)

For Article second, the Sum of an even Number of Terms all oddly even, is either evenly even only, or both oddly and evenly even (*Theor.* XX.) then the Sum of any Number of Terms, all evenly even only, is both oddly and evenly even (*Theor.*XXI.) Also the Sum of any Number of Terms both oddly and evenly even, is either evenly even only, or both oddly and evenly even (*Theor.* XXII.) wherefore it's plain, that what we have to consider in this Article is this, *viz.* What kind of a Sum is that of 2 Numbers, both of them evenly even only, or both of them oddly and also evenly even; or the one evenly even only, and the other both oddly and evenly even (for of these Kinds are the Sums of the two Classes of Numbers added) in all which three Cases the Sum is both oddly and evenly even, by *Theor.* XXI, XXII, and XXIII, the last part of which shews the Truth of the last Case.

THEOREM XXV.

If there are three Numbers in Arithmetical Progression, whereof the middle Term is evenly even only, and one of the Extremes oddly even only, the other Extreme is also oddly even only.

DEMON. Let the three Terms be $2\times o$, 2^n, N, the first being oddly even only (for o is an odd Number) and the second being evenly even only, or some Power of 2, then is $2\times o+N=2\times 2^n=2^{n+1}$; but since 2^{n+1} is even, so must $2\times o+N$; and because also $2\times o$ is even, so is the Remainder N. Let it be supposed that $N=2a$, then is $2^{n+1}=2\times o+2a=2\times\overline{o+a}$, but a is an odd Number, for else $o+a$ will be odd (*viz.* the Sum of an odd Number, and even Number) and then $2\times\overline{o+a}$ is oddly even only, *i. e.* 2^{n+1} $(=2\times\overline{o+a})$ is oddly even only, which is impossible, for 2^{n+1} is a Power of 2, or evenly even only; wherefore a must be odd, and consequently N $(=2a)$ is oddly even only.

PROBLEM I.

To find a proposed even Number of Numbers, which are all oddly even only, and whose Sum is evenly even only.

Rule

Rule 1. If the proposed Number is 2, take any Number oddly even only (as A in *Exam.* 1st.) Also any Number evenly even only and which is greater than the former Number (as B) then take a third in Arithmetical Progession to A, B, as, C; and A, C, are the Numbers sought.

Exam. 1.

A,	B,	C,
14,	16,	18,

Demon. By *Theor.* XXV, C is oddly even only; then $A+C=2B$, which is a Number evenly even only, *viz.* some Power of 2, because B is such.

2°. If the Number is 4, take 2 Numbers oddly even only, as A, B, *Example* second; also any Power of 2, as C, which is a greater Number than B; then take D : E, as much greater than C, as B A are lesser; and A, B, D, E, are the Numbers sought.

Exam. 2.

A :	B :	C :	D :	E
6 :	*10 :*	*16 :*	*22 :*	*26*

Demon. D, E, are oddly even only, by *Theor.* XXV. and $B+D=2C$, also $A+E=2C$, therefore $A+B+D+E=4C$, which is a Power of 2, because both 4 and C are so.

3°. Let the proposed Number be any even Number, above 4; find first four of the Numbers sought, as in the last Case; then take the next Power of 2 above C, as G; and below it take a Number, as F, oddly even only, and which is different from any of the preceding, and another as far above it as H; then take in the next Power of 2, as K, and take below it a Number oddly even only, and another as far above; and so on, till you have as many Numbers as are required.

A :	B :	C :	D :	E :	F	G	H	I	K	L
6 :	*10*	:	*22 :*	*26 :*	*30*	:	*34 :*	*62*	:	*66*
		16				*32*			*64*	

Demon. For the first 4 we have the Demonstration already: Then for the next 2; $F+H=2G$; But $2C=G$, therefore $4C$ $(=A+B+D+E)$ is $=2G$, consequently $A+B+D+E+F+H=4G$, which is a Power of 2. The Reason goes on the same Way to the next Two, and so for ever.

Schol. If the proposed Number is it self some Power of 2, we may work thus; take any Number of Terms all oddly even only, which is equal to the Half of the Number proposed, then take a Power of 2 greater than the greatest of them, and as many Terms above it, at the same Distance as the former half are below it; thus, to find 8 Terms, I first take 4, as A, B, C, D, then a Power of 2, as E, greater than

A,	B,	C,	D,	E,	F,	G,	H,	I
6 ·	*10 ·*	*14 ·*	*18 ·*	*32 ·*	*46 ·*	*50 ·*	*54 ·*	*58*

D; and lastly, F, G, H, I, as much greater than E, as D, C, B, A, are lesser. The Reason is plain, for here E multiplied by the Number proposed is the Sum of the other Numbers found; but E, and the Number proposed, being both Powers of 2, so is the Product or Sum.

PROBLEM II.

To find an even Number of Terms oddly even only, whose Sum is both oddly and evenly even.

Rule 1°. If the given even Number is greater then 2, then take as many Terms as half the Number proposed out of the Series of Powers of 2, beginning at any Power above 4; then take a Number oddly even only below each of these, and another as far above it; and you have the Numbers sought.

Exam. To find 6 ſuch Numbers; they are A, C, D, F, G, I.

A	.	C	.	D		F	.	G		I
6	.	10	.	14	.	18	.	30	.	34
	8		.		16		..		32	
	B				E				H	

DEMON. $A+C=2B$. Again, $D+F=2E$, and $G+I=2H$, therefore $A+C+D+F+G+I=2\times\overline{B+E+H}$; but B. E. H. being evenly even only, their Sum $B+E+H$ is both oddly and evenly even (by *Theor.* XXI.) and ſo alſo is $2\times\overline{B+E+H}$ (*Cor.* 5. *Theor.* XVIII.) and how many Terms ſoever you thus find, the Reaſon is plainly the ſame; alſo the Reaſon why you muſt begin above 4 is, becauſe there is not a Number oddly even only below 4.

2°. If the given Number is 2, take any Number oddly even only, as A; then a Number greater, as B, which is evenly even only; and a Number oddly even only, C, as far above B as A is below it; then laſtly, take D, the Number oddly even only, which is the next above C; and A, D, are the Numbers ſought.

Example.

A	.	B		C		D
6	.	.	.	10	.	14
		8				

DEMON. $A+C=2B$, alſo $C+4=D$, therefore $A+D=A+C+4=2B+4$; ſuppoſe next, that $B=2x$, and then $2B+4=4x+4=4\times\overline{x+1}$; but x is an even Number (ſince B is at leaſt 4) therefore $x+1$ is odd, and therefore $4\times\overline{x+1}$ is both oddly and evenly even.

PROBLEM III.

To find a propoſed Number of Terms which are both oddly and evenly even, and whoſe Sum is evenly even only.

Rule 1°. If the propoſed Number is even, take as many Numbers oddly even only, and whoſe Sum is evenly even only, by *Probl.* II. multiply each of them by ſome Number evenly even only, and you have the Numbers ſought.

Exam. To find 4 ſuch Numbers; they are E, F, G, H.

	A	B	C	D	
Oddly even only	10+	14+	18+	22= 64,	the 6th Power of 2.
Multiplier			4		
	40+	56+	72+	88=256,	the 8th Power of 2.
	E	F	G	H	

DEMON. Numbers oddly even only, as A, B, C, D, being multiplied by ſome Power of 2, produce Numbers both oddly and evenly even; but $\overline{A+B+C+D}=$ ſome Power of 2; therefore $4\times\overline{A+B+C+D}$ is alſo ſome Power of 2. Alſo $4\times\overline{A+B+C+D}=E+F+G+H$, which is therefore ſome Power of 2, or a Number evenly even only.

2°. If the propoſed Number is odd, take the next leſſer Number which is even, and find as many Terms both oddly and evenly even, and whoſe Sum is evenly even only (by *Caſe* I.) to this Sum add the Number evenly even only, which is the *next greater*, and this laſt Sum is the remaining Term ſought.

Exam. To find 3 Numbers; find 12 and 20, Numbers both oddly and evenly even, whoſe Sum is 32, evenly even only; to this I add 64, the next greater evenly even, the Sum 96 is the remaining Number ſought; for $12+20+96=128=$ the 7th Power of 2.

DEMON.

Demon. Let A, B, C, D, *&c.* be any Numbers both oddly and evenly even, and whose Sum S is evenly even only, then the next greater Number evenly even only is $2S$. Also their Sum $S+2S=3S=4S-S$, to which add the preceding Numbers found, *viz.* $A+B+C+D$ *&c.* or their Sum which is S, the total Sum is $4S-S+S=4S$, which is evenly even only, because 4 and S are so.

PROBLEM IV.

To find a Number of Terms, all of them both oddly and evenly even, and whose Sum is both oddly and evenly even.

Rule 1°. If the Number of Terms is even, take (by *Probl.* II.) as many Terms (as the Number proposed) which are all oddly even only, and whose Sum is both oddly and evenly even; multiply them by 2, or by any Number evenly even only, and you have the Numbers sought.

Exam. first, to find 4 Numbers.

Oddly even only,	$6+10+14+18=48$
Multiplier	2
	$12+20+28+36=96$

Numbers sought both oddly and evenly even.

Demon. The Products are Numbers both oddly and evenly even (*Theor.* XVIII. *Cor.* 4.) and the Sum of the Numbers multiplied being both oddly and evenly even, its Product by the same Multiplier is both oddly and evenly even (*Cor.* 5. *Theor.* XVIII.) and is also equal to the Sum of the Numbers formerly produced.

2°. If the Number of Terms is odd, take as many oddly even only, their Sum is always oddly even only; multiply them by 2, or some Number evenly even only, you have the Numbers sought.

Example second.

	$A+B+C=S$
Oddly even only	$6+10+14=30$
Multiplier	2
Numbers sought	$12+20+28=60$

All both oddly and evenly even.

Demon. Any odd Number of Terms, A, B, C, *&c.* all oddly even only, have a Sum S oddly even only (*Theor.* XX.) and these, or their Sum being multiplied by 2, or any Power of it, produce Numbers both oddly and evenly even. (*Cor.* 5. *Theor* XVIII.) Also the Sum of these Products is the Product of the Sum of the former, *viz.* $A+B+C$, *&c.* by the same Power of 2, which we have already said is both oddly and evenly even.

PROBLEM V.

To find any Number of Terms, all both oddly and evenly even, with any Number of Terms evenly even only, whose Sum all together is evenly even only.

Rule. Find the Number of Terms proposed both oddly and evenly even, and whose Sum is evenly even only (by *Probl.* III.) Take that Sum as the least of the Terms sought evenly even only; and take the rest of them immediately adjacent to that, and greater, in the Order of the Series of Numbers evenly even only.

Exam. To find 6 Numbers, whereof 3 are both oddly and evenly even, and 3 of them evenly even only; and whose Sum is evenly even only.

Both oddly and evenly even, $12+20+96=128$ $(=2^7)$
Numbers evenly even only, $128+256+512$
Sum$=1024=2^{10}$

Demon. By the Rule of Geometrical Progressions, the Sum of a Progression, whose Ratio is 2 (*i. e.* the Sum of any Number of immediately adjacent Powers of 2) is equal to the Difference betwixt double of the greatest Extreme, (which is equal to the next greater

greater Term in the Progression) and the lesser Extreme (for the Sum of any Geometrical Progression is thus expressed $\frac{rl-a}{r-1}$, but r being 2 the Sum is $2l-a$) therefore if that lesser Extreme be added to the Sum, this Sum is equal to the next greater Term in the Progression; for $2l-a+a=2l$. Now let any Number of Terms both oddly and evenly even, and whose Sum is evenly even only, be $A+B+C+D+$&c. $=M$; and let as many Terms evenly even only, be M, N, O, P, &c. the Sum of these last is, $2P-M$, to which if we add the Sum of the preceding Numbers, which is M, the Total is $2P$, the next greater Number evenly even only.

PROBLEM VI.

To find Numbers, as in the last, whose Sum is both oddly and evenly even.

Rule. Find the proposed Number of Terms, both oddly and evenly even, whose Sum is both oddly and evenly even (by *Probl.* IV.) then take as many Terms evenly even only, any where out of the Series of the Powers of 2.

Exam. To find 3 Terms of each Kind.

	A	B	C	S
Both oddly and evenly even	12+	20 +	28=	60
Evenly even only	64+	128+	256=	448
	M	N	O	R
Sum both oddly and evenly even	508=2×254=4×127			

Demon. Let any Number of Terms both oddly and evenly even, and whose Sum is both oddly and evenly even, be A, B, C, &c. and their Sum S; then take the proposed Number of Terms, all Powers of 2, any where out of that Series, and call them M, N, O, &c. and their Sum R; this Sum is both oddly and evenly even (*Theor.* XXI.) and the total Sum is therefore $S+R$, both Parts of which being both oddly and evenly even, their Sum is so also (by *Theor.* XXII.)

Theorem XXVI.

Betwixt two Numbers, both even, or both odd (whereof one of them may be 1) there is at least one Arithmetical Mean in Integers.

Demon. The Sum of two even, or two odd Numbers, is an even Number, and consequently is measurable by 2, but the half Sum of the Extremes is the Arithmetical Mean; therefore

Exam. Betwixt 4 and 6, there is one Mean, 5; and betwixt 5 and 7 there is one Mean, 6.

Theorem XXVII.

Betwixt an even Number, and an odd (which may be 1) there are at least 2 Arithmetical Means in Integers, or there are none at all; nor can there possibly be any odd Number of Means.

Demon. The Sum of an even and odd Number is odd, therefore they do not admit of one Arithmetical Mean, because the Sum being odd is not measurable by 2, consequently there must be 2 Means at least if there are any. Hence again, there cannot be an odd Number of Means; for then there would be one odd Mean, contrary to what's last shewn.

Theorem XXVIII.

If a Geometrical Progression is in its lowest Terms, they are either all odd Numbers, or all even, except one of the Extremes, which must be odd.

Demon.

DEMON. Let $A : B$ be the lowest Term of the Ratio of any Geometrical Progression; then will the Series be thus represented, $A^n : BA^{n-1} : B^2A^{n-2} : B^3A^{n-3} :$ &c. $AB^{n-1} : B^n$ (*Probl.* VI. *Schol.* 1. *Ch.* I.) Now A and B are either both odd, or one odd and the other even; for if they are both even they are not in the lowest Terms: But all the Powers of odd Numbers are odd, and of even Numbers are even; and the Product of an odd Number by an odd, is odd, and of an even by an odd, is even: Whence the *Theorem* is manifest.

THEOREM XXIX.

If an even Number is a Square, it has an even Half, or is measurable by 4; but if an odd Number is a Square, then being divided by 4, it leaves a Remainder of 1; or 1 taken from an odd Square, leaves a Multiple of 4.

DEMON. 1°. Any even Root may be expressed $2n$; and its Square will be $4n^2$, *viz.* a Multiple of 4; which shews the first Part. Again,

2°. Any odd Root may be expressed $2n+1$, and its Square will be $4n^2+4n+1$, *viz.* $4\times\underline{n^2+n}+1$; which is plainly a Multiple of 4, and 1 remaining over.

COROLLARIES.

1*st*. The Sum of any Number of even Squares is measurable by 4 (or has an even Half.)

2*d*. The Sum of 2 or 3 odd Squares, divided by 4, leaves a Remainder of 2 or 3. *Universally*, if the Number of odd Squares added, is a Multiple of 4 (as 4 . 8 . 12 . 16 . 20 &c.) the Sum is measurable by 4; otherwise there will always be a Remainder; particularly if that Number is the Sum of a Multiple of 4, and 1, or 2, or 3; the Remainder will be accordingly, 1, or 2, or 3.

3*d*. The Sum of an even and odd Square, divided by 4, leaves a Remainder of 1; and *universally*, if any Number of even Squares is added to any Number of odd Squares, the Remainder will be the same that would happen with the Sum of the odd Squares; because the Sum of the even Squares leaves no Remainder.

4*th*. The Sum of any two integral Squares, being divided by 4, cannot leave a Remainder of 3; for if they are both even, the Remainder is *0*; since each of these Squares is measurable by 4, by this *Theorem*; and consequently their Sum is so. If the one is even, and the other odd, the Sum will leave a Remainder of 1 (*Coroll.* 3.) *Lastly*, If both are odd, the Remainder will be 2 (*Coroll.* 2.)

THEOREM XXX.

The Terms of an Arithmetical Progression are either all even or all odd; or they are alternately even and odd; *i. e.* the 1st, 3d, 5th, *&c.* Terms are all even or odd; and the 2d, 4th, 6th, *&c.* all odd or even. Also the Sum of the Whole is odd or even, according as the Number of odd Terms is odd or even; but if all the Terms are even the Sum is even.

DEMON. This depends all upon the lesser Extreme, and the Difference compared with *Theorem* I. Thus,

1°. If the lesser Extreme and Difference are both even, so must the whole Series be; because even Numbers are still added to even. So if the lesser Extreme is 2, and the Difference 4, the Series is 2 . 6 . 10 . 14, &c.

2°. If the 1st Term is odd, and the Difference even, all the Terms are odd; because they are each the Sum of an even and odd Number. So the 1st Term being 3, and the Difference 4, the Series is 3 . 7 . 11 . 15, &c.

3°. If the 1st Term is even and the Difference odd, or if both are odd, the Terms are alternately odd and even; because an odd and even makes the Sum odd; and two odds make an even.

4°. That

4°. That the Sum of the Series will be odd or even, according as the Number of odd Terms is odd or even, is alſo evident from the ſame Principles; for the Sum of every two odd Numbers is even: So that if the Number of odd Terms is even, the Sum of them, and conſequently of the whole Series, is even; but if it's odd, the Sum is odd; which added to the even Sum of the even Terms, makes the total Sum odd. If all are even, the Sum is manifeſty even.

Theorem XXXI.

Take the odd Series, 1 . 3 . 5 . 7 . 9 . &c. The Sum of any even Number of Terms of this Series, taken in the continued Order of the Series, and beginning at any Term, is a Number both oddly and evenly even (*i. e.* it has an even Half; or is meaſurable by 4.) For *Exam.* $5+7=12=3\times4$, and $5+7+\ \ +11=32=8\times4$.

Demon. 1°. The Sum of any two adjacent Terms is meaſurable by 4; for it is equal to the Double of that Term of the natural Series which lies betwixt them, and is the Arithmetical Mean; but that Mean is an even Number; and double of an even Number is evidently meaſurable by 4; or is an even Number with an even Half.

2°. Since this is true of any two adjacent Terms, it muſt be true of any even Number of adjacent Terms; becauſe theſe being diſtributed into [illegible] 2's, the Sum of each 2 is meaſurable by 4; conſequently the Sum of the [illegible] meaſurable by 4.

Theorem XXXII.

If out of the odd Series, 1 . 3 . 5 . 7 . 9 . &c. be taken in the continued Order of the Series, any odd Number of Terms, beginning at any Term, the Sum of them is an odd Number; whoſe Place, in the ſame Series, has this conſtant and regular Connection with the Number of Terms, and the Place of the leſſer Extreme of the Terms added, *viz.* that, if you take the Product of that Number of Terms, by the Place of the leſſer Extreme; then again, Take the Half of the Square of the Number of Terms leſs 1; add this Half Square to the former Product; the Sum is the Place of the Sum of the Terms added.

Thus, if there are 3 Terms added, and the Place of the leaſt be n, the Place of the Sum is $3n+2$. If there are 5 Terms added, it is $5n+8$; and ſo on, as in this Table.

$3n+2$ $(=2\times1\ )$
$5n+8$ $(=2\times4\ )$
$7n+18$ $(=2\times9\ )$
$9n+32$ $(=2\times16)$
&c.

Demon. 1°. That the Sum is an odd Number is already proved in *Theorem* I.

2°. That the Place of the Sum is according to the *Theorem*, is deduced from the Rules concerning Arithmetical Progreſſions; Thus,

Call the Place of the leſſer Extreme of the Series added, n; then that Term it ſelf is $2n-1$ [from the Nature of the Series; for it is $1+\overline{n-1}\times2=1+2n-2=2n-1$.] Let the Number of Terms added be a; the greateſt Extreme added muſt be $2n-1+2\times\overline{a-1}=2n+2a-3$ [for $2n-1$ is the leſſer Extreme, 2 the common Difference, and a the Number of Terms] then the Sum of the Extremes is $\overline{2n-1}+\overline{2n+2a-3}=4n+2a-4$; and the total Sum is $\overline{4n+2a-4}\times\frac{a}{2}=2an+aa-2a$. Now this being a Term of the odd Series, 1 . 3 . 5 . &c. ſuppoſe its Place to be N; then that Term of the odd Series, whoſe Place is N, is it ſelf $2N-1$ (as above ſhewn for the Place n;) ſo that $2N-1=2an+aa-2a$: Add 1 to both, and then divide by 2, and it is $N=\frac{2an+aa-2a+1}{2}=an+\frac{aa-2a+1}{2}$. But $\overline{a-1}^2=a^2-2a+1$, and the Half of this is $\frac{aa-2a+1}{2}$; whence the Rule is evidently demonſtrated.

Schol.

SCHOL. The half Squares of the Number of Terms less 1, are the Products of the Series of Square Numbers, 1 . 4 . 9 &c. multiplied by 2, as I have marked them in the Table; and that it will go on so for ever, will be plain, thus; Let a be any odd Number, the next greater odd Number is $a+1$: Take 1 from each of them, the Remainders are $a-1$, $a+1$; whose Squares are a^2-2a+1, a^2+2a+1; whose Halfs are $\frac{aa-2a+1}{2}$, $\frac{aa+2a+1}{2}$: But if the first of these is the Double of any square Number, the other must be double of the next greater Square; for suppose $\frac{aa-2a+1}{2}=2\times b^2$, then is $aa-2a+1=4\times b^2=\overline{2b}^2$; hence $a-1=2b$. Add 2 to each, and it is $a+1=2b+2=2\times\overline{b+1}$; therefore $\overline{a+1}^2$, or $a^2+2a+1=4\times\overline{b+1}^2$, and $\frac{aa+2a+1}{2}=2\times\overline{b+1}^2$.

§. V. *Of Numbers,* Perfect, Abundant, *and* Deficient.

THEOREM XXXIV.

IF the Geometrical Progression, 1 : 2 : 4 : 8, &c. is carried on till the Sum be a prime Number; and that Sum be multiplied by the last Term of the Series, the Product shall be a *perfect* Number; thus, $1+2=3$, a Prime, and $3\times2=6$, a perfect Number. Again, $1+2+4=7$ and $7\times4=28$, a perfect Number; for its *aliquot* Parts are $1+2+4+7+14=28$.

DEMON. Let $1+2+4+8+$&c. $+2^n=S$, a prime Number; then is $S\times2^n$ a perfect Number: For,

1°. If from S we raise a Series in the Ratio 1 : 2, having as many Terms as the preceding; the last Term of it will be $S\times2^n$, as is evident from the Method of raising the Series.

1, 2, 4, 8, &c. 2^n	
S, 2S, 4S, 8S, &c. $2^n\times S$	

2°. It's evident from the Composition of these Numbers, that all the Terms of both these Series, from 1 and S, are *aliquot* Parts of $S\times2^n$; for the first Series after 1, are all Powers of the same Root, 2; which therefore measure 2^n, and consequently $2^n\times S$; and the second Series being only the Multiples of the first by S, therefore each of them also measures $2^n\times S$.

3°. By *Probl.* IV. *B.* IV. *Ch.* III. The Sum of all the Terms of a Geometrical Progression, excluding the greatest Extreme, is the Quote of the Difference of the Extremes, divided by the Ratio less 1; but the Ratio here being 2, therefore $S+2S+4S+8S+$&c. $=2^n\times S-S$, and $1+2+4+8$ &c. $+2^n=S$; also $2^n\times S-S+S=2^n\times S$; therefore $1+2+4+8$ &c. $+2^n+S+2S+4S+8S$ &c. $=2^n\times S$.

4°. It being proved that $2^n\times S$ is the Sum of all the other Numbers in these two Series, and that each of these are *aliquot* Parts of it; it remains to shew that no other Number can be an *aliquot* Part of it. Thus, Every other Number must necessarily have in its Composition some other Prime than 2 or S, or some higher Power of one or both of these than is in $2^n\times S$; but by *Theor.* X. no such Number can measure $2^n\times S$, or can be an *aliquot* Part of it.

SCHOL. In this *Theorem* there is a certain Way of finding as many perfect Numbers, as the Number of Cases in which the Sum of the Series, 1 . 2 . 4 . 8 &c. can be a prime Number; in which *observe*, that there is no more to do, but from every Term of the Series, as it goes on, to take 1; the Difference is the Sum of all the preceding Terms; and if it's a Prime, then being multiplied into the preceding Term, it gives a perfect Number. That there are some perfect Numbers found this Way, is certain;

 for

for such a Number is $6=\overline{1+2}\,(=3)\times 2$; also $28=\overline{1+2+4}\,(=7)\times 4$; and $496=\overline{1+2+4+8+16}\,(=31)\times 16$; and $8128=\overline{1+2+4+8+16+32+64}\,(=127)\times 64$. And $33550336=\overline{1+2+4+8+16+32+64+128+256+512+1024+2048+4096}\,(=8191)\times 4096$. Besides these there are but a few more found. Dr. *Harris* says there are but ten perfect Numbers betwixt 1 and 1,000,000,000,000, but does not express them. And *Tacquet* observes, that the Reason why more are not found, is, That in the Progression, 1 . 2 . 4 . &c. the Intervals of the Numbers, which lessened by an Unit, become Primes, are very great; and when the Numbers are great, the finding whether they are Primes or not is a vast Labour. There is one Thing more I would observe here, that it has not yet been demonstrated, as far as I know, that there can be no perfect Numbers but what may be found by this *Theorem*, *i. e.* that every perfect Number is the Product of two Numbers, whereof one is a Prime, and the Sum of a certain Number of Terms of the Series, 1 . 2 . 4 . &c. the other, the last of these Terms. Again, it wants also to be demonstrated, that the Number of perfect Numbers is infinite.

THEOREM XXXV.

Every Prime Number is a deficient Number.

DEMON. A Prime has no *aliquot* Part but 1; which is less than any prime Number.

THEOREM XXXVI.

Every Power of 2 is a deficient Number.

DEMON. No Number can measure or be an *aliquot* Part of any Power of 2, but 1, or the inferior Powers of 2 (*Coroll.* 6. *Theor.* X.) But any Power of 2, is more by 1 than the Sum of all the inferior Powers and 1 (by the Rule for summing the Geometrical Progression, 1 . 2 . 4 . &c.) and therefore is a deficient Number.

THEOREM XXXVII.

Every Number is *abundant*, which is measured by a perfect or an abundant Number: Or thus, A perfect or abundant Number can measure no Number but an Abundant.

p	n
a	r
b	s
&c.	&c.
1	u

DEMON. Let p, a Number perfect or abundant, measure n; and let a, b, &c. 1, be the *aliquot* Parts of p; and take r, s, &c. u, in the same Ratios to n, as a, b, &c. 1 are to p; they will all be Integers; for since $p:a::n:r$, then $p:n::a:r$; but p measures n, therefore a measures r, which consequently must be Integer; and so of the rest. Now because $p:n::a:r::b:s:$ &c. $::1:u$, therefore $p:n::a+b$ &c.$+1:r+s$ &c.$+u$; but $a+b$ &c.$+1$, is either equal to, or greater than p, as this is a perfect or abundant Number: And hence, $r+s$ &c.$+u$ (which are all *aliquot* Parts of n, because $a+b$ &c. are so of p) is equal to, or greater than n; if greater, the *Theorem* is demonstrated; if equal, then 1, which is an *aliquot* Part of n, different from any of these, r, s, &c. u, being added to them, makes the Sum greater than n; which is therefore abundant.

COROLLARIES.

1st. A perfect or deficient Number can be measured only by a Deficient; because what is measured by a Perfect or Abundant is Abundant.

2d. An abundant Number may be measured by any Number; for the Multiple of any abundant or perfect Number is an abundant Number; and what is measured by a perfect Number is measured by all the *aliquot* Parts of it, which are all deficient Numbers (by the 1st *Coroll.*) Hence again:

3d. A

3*d*. A deficient Number meaſures any kind of Number.

SCHOL. As in *Theorem* XXXIV. we have a Rule for finding perfect Numbers, ſo from this *Theorem* we have a Rule for finding abundant Numbers; for ſuch are all the Multiples of any perfect Number. And from *Coroll.* 1. we have a Rule for finding deficient Numbers; all the *aliquot* Parts (except 1) of any perfect Number being ſuch.

Exam. 1. 6 is perfect, and 18 is its Multiple, whoſe *aliquot* Parts are $1+2+3+6+9=21$.

Exam. 2. 28 is Perfect, and 14 its Half, whoſe *aliquot* Parts are $1+2+7=10$.

THEOREM XXXVIII.

If any Number, A, multiplying a Prime, p, produces a perfect Number, N; the ſame A multiplying another Number M, which is leſs than p, and which does not meaſure A, produces an abundant Number, O.

Exam. $4\times7=28$, a perfect Number, and $4\times6=24$, an abundant Number, whoſe *aliquot* Parts are $1+2+3+4+6+8+12=36$.

	N	O	
A	p	M	
a	r	m	x
b	s	n	
&c.	&c.	&c.	
1			

DEMON. Let a, b, &c. 1, be all the *aliquot* Parts of A; then becauſe $N=A\times p$, a, b, &c. will meaſure N; and becauſe p is a Prime, N has no other Meaſure except A, a, b, &c, 1, and the Products of theſe by p (as you ſee in *Probl.* IV. §. I.) But it's alſo plain, that N being divided by a, b, &c. the Quotes are alſo *aliquot* Parts of N, and therefore muſt be the ſame Numbers as the Products of p by a, b, &c. though not anſwering in the ſame Order, *i. e.* if $\frac{N}{a}=r$, this is not the ſame as ap; but as they muſt neceſſarily be the ſame Numbers, however the Correſpondence be, let us ſuppoſe $\frac{N}{a}=r$, $\frac{N}{b}=s$, &c. Then again, ſince $O=AM$, therefore a, b, &c. which meaſure A, do alſo meaſure O; let $\frac{O}{a}=m$, $\frac{O}{b}=n$, &c. wherefore $N : O :: p : M :: r : m :: s : n$, &c. and compoundly, $N : O :: \overline{p+r+s}$, &c. : $\overline{M+m+n}$, &c. But $\overline{A+a+b \text{ \&c.}+1+p+r+s}$ &c. $=N$; ſo that $\overline{A+a+b \text{ \&c.}+1}$ is the Remainder, after $\overline{p+r+s}$ &c. is taken out of N; let x be the Remainder after $\overline{M+m+n}$ &c. is taken out of O (which muſt be greater than that Sum, ſince N is greater than $\overline{p+r+s}$ &c.) then is $N : O :: \overline{A+a+b}$, &c. $+1 : x$; but M being leſs than p, O is leſs than N; and conſequently x is leſs than $\overline{A+a+b}$ &c. $+1$; alſo $\overline{M+m+n}$ &c. $+x=O$; therefore $\overline{M+m+n}$, &c. $+\overline{A+a+b}$, &c. $+1$ (each of which meaſures O) is greater than O. And ſince, laſtly, M does not meaſure A, therefore M, m, n, &c. A, a, b, &c. 1, are all different *aliquot* Parts of O, which is therefore Abundant.

THEOREM XXXIX.

If a Number, A, multiplied into another, B, produces either a perfect or abundant Number; then if A is multiplied into any Multiple of B, the Product is Abundant.

Exam. $2\times3=6$, a perfect Number, $2\times5=10$, and $3\times10=30$, an abundant Number, whoſe *aliquot* Parts are $1+2+3+5+6+10+15=42$.

DEMON. Let M be a Multiple of B; then is $B : M :: AB : AM$. And becauſe B meaſures M, ſo does AB meaſure AM; but AB is, by Suppoſition, *Perfect* or *Abundant*; therefore (by *Theor.* XXXVII.) AM is *Abundant*.

CHAP. II.

Of Figurate Numbers.

§ 1. *Definitions.*

I. NUMBERS are called *Figurate* from Geometrical Figures, which they are capable of reprefenting in a certain Manner, by a particular Difpofition of their Units (as fhall be prefently explained;) which is a Part of the antient *Pythagorean* Speculations about Numbers and Geometrical Figures; from the Comparifon of which they found fuch Likeneffes and Correfpondencies, whence they pretended to difcover many Myfteries and Secrets of Nature. Our Bufinefs here is to confider thefe Numbers as a Subject purely Arithmetical, and upon the Principles of Numbers only to explain their Connections and Properties; yet it being neceffary to have Names for Things, and fimple Names being more convenient than long Defcriptions; and the Geometrical Names (defcribed below) being ftill in ufe, we fhall retain them, and explain the Reafon and Meaning of them, for their fake who have Acquaintance enough with Geometry to underftand it, or Imagination to conceive it by the following Explications; for others, they muft take them as mere Names, by which thefe Numbers are defigned and diftinguifhed.

II. Take any Arithmetical Progreffion, beginning with 1, and whofe common Difference is any integral Number; then take the Sums of thefe Series continually from the beginning; and again, the Sums of thefe Sums, and fo on for ever. Thefe feveral Series of Sums are called in general Figurate Numbers, but more particularly, the firft Sums are called plain Figurates, and alfo Polygons; the fecond Sums are called folid Figurates, and alfo Pyramids; the third Sums are called fecond Pyramidals, and fo on. But again,

III. Polygons are diftinguifhed thus, If the common Difference in the Series $\div l$, whence they proceed, or whofe Sums they are, is 1. as 1 . 2 . 3 . &c. the Sums 1 . 3 . 6 . &c. are called Triangles. If the Difference is 2, as 1 . 3 . 5 . &c. the Sums 1 . 4 . 9 . &c. are called *Quadrangles*, and particularly Squares. If the Difference is 3, as 1 . 4 . 7 . &c. the Sums 1 . 5 . 12 . &c. are called *Quinquangles* or *Pentagons*, and fo on; the Name of the Polygon expreffing a Figure of a Number of Angles, which is 2 more than the common Difference of the Series $\div l$. In the fame Manner,

IV. Pyramids, and all the following Sums, are diftinguifhed by the Polygon whence they proceed; and thus we have Triangular Pyramids, Square Pyramids, &c. alfo Triangular and Square, fecond Pyramidals, third Pyramidals, and fo on.

V. Since the Pyramidals do all proceed from Polygons, they may alfo be called Polygonal Numbers; and the whole Order of Sums be more conveniently diftinguifhed, by calling them Polygonals of the firft, fecond, &c. Order: Thus, the firft Sums or Polygons, are Polygonals of the firft Order; the fecond Sums, or Pyramids, are Polygonals of the fecond Order. And again, for the feveral Orders proceeding from different Series $\div l$, they are to be diftinguifhed by the Name of the Polygon, which is particularly applied to the firft Order, and fo on. Thus all the Order of Sums proceeding from the Series 1 . 2 . 3 . &c. are Triangulars of the firft or fecond, &c. Order. Thefe from the Series 1 . 3 . 5 . &c. are Polygonals of the fquare Kind, and fo on. *Obferve again*, That inftead of thefe Names Triangular, &c. it will be fometimes more convenient

venient to diſtinguiſh them by firſt Species, ſecond Species, *&c.* and then the Arithmetical Denominations of firſt, ſecond, *&c.* being the ſame Numbers as the common Differences of the Series ÷ *l*, theſe are clearly marked by this Denomination; and thus as all the different Series of Sums come under the general Name of Polygonal Numbers, ſo theſe from different Series ÷ *l* are diſtinguiſhed by different *Species*, and the different Series of Sums proceeding from the ſame Series ÷ *l* are diſtinguiſhed by different Orders. But in the laſt Place obſerve, that we ſhall ſometimes uſe the ſimple Name Polygon; or alſo particularly, Triangle, Square, *&c.* when we ſpeak of the firſt Sums, or firſt Order of *Polygonals*; alſo the ſimple Name *Pyramid* for the ſecond Sums, or Sums of Polygons.

I ſhall now repreſent all theſe Series in diſtinct Tables, according to their Species and Orders; and then explain the Reaſon of the particular Names.

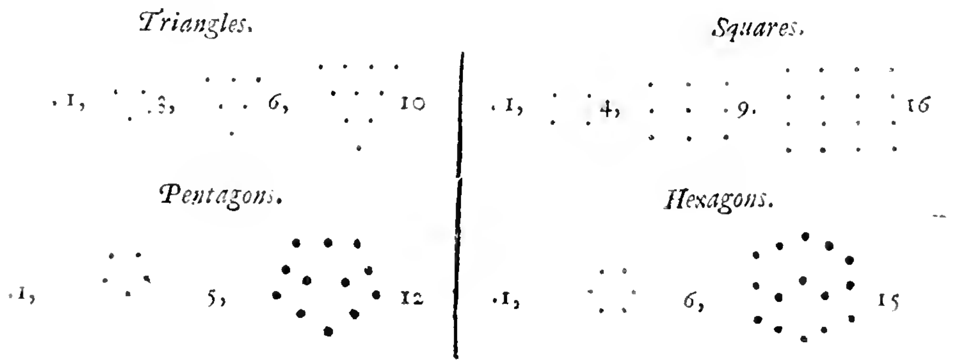
Table of Polygonal *Numbers.*

Series ÷ *l*		*Polygons*, or *Polygonals*, *1ſt Order.*		*Pyramids*, *Polyg. 2d Order.*		*2d Pyramidals* *Polyg. 3d Order.*
1. 2. 3. 4	*1ſt Sums.*	Triangles, 1. 3. 6. 10	*2d Sums*	1. 4. 10. 20	*3d Sums*	1. 5. 15. 35
1. 3. 5. 7		Squares, 1. 4. 9. 16		1. 5. 14. 30		1 6. 20. 50
1. 4. 7. 10 *&c.*		Pentagons, 1. 5. 12. 22 *&c.*		1. 6. 18. 40 *&c.*		1. 7. 25. 65 *&c.*
1. 5. 9. 13		Hexagons, 1. 6. 15. 28		1. 7. 22. 50		1. 8. 30. 80
1. 6. 11. 16		Heptagons, 1. 7. 18. 34		1. 8. 26. 60		1. 9. 35. 95
1. 7. 13. 19		Octogons, 1. 8. 21. 40		1. 9. 30. 70		1. 10. 40. 110
&c.		*&c.*		*&c.*		*&c.*

The Reaſon of the Names.

A Number is called a *Polygon*, from the Repreſentation of a plain Figure having many Angles, and ſuch too as is Regular, or has equal Angles, and equal Sides. Thus *Triangles* repreſent Equiangular *Triangles*, Squares, equal angled *Quadrangles*, and ſo on. Which Repreſentation you ſee in the following Schemes; wherein 1 is of all Species, becauſe every Thing is an Unit of its Kind.

Polygons.

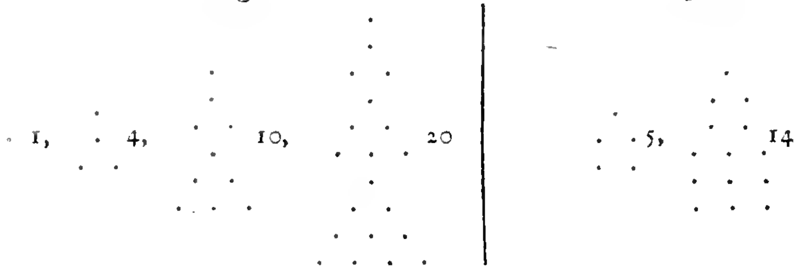


Such is the Diſpoſition of the Units of theſe Numbers, from whence they are called *Triangles*, &c. and ſo will the Repreſentation go on as it is here begun, both as to the

the Continuation of theſe here repreſented, and all the other Species. But as I meddle no further with theſe Speculations, ſo I ſhall ſay no more for the Demonſtration of it; only this one Thing I muſt here obſerve, That the Sums of the Series 1 . 3 . 5 *&c.* *viz.* 1 . 4 . 9 *&c.* are not only ſquare Numbers, but they are the Series of Squares of the natural Progreſſion, 1 . 2 . 3 *&c.* So far as the Series is here carried you ſee the Truth of the Obſervation; and that it muſt continue ſo for ever may be eaſily perceived from the Conſideration of the Numbers, and the Manner of diſpoſing their Units. But I ſhall not leave the Demonſtration of it merely upon this; in another Place I ſhall propoſe and demonſtrate it diſtinctly by it ſelf; and till then, conſider theſe Numbers only as the Sums of the Series 1 . 3 . 5 *&c.*

Pyramids.

Triangular. *Square.*



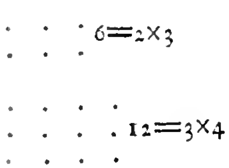
By conceiving the Planes of each of the Polygons which compoſe a Pyramid to be placed parallel over one another, and diſpoſed, with reſpect to the Situation of their Angles and Diſtances, ſo that the reſpective Angles of each Polygon be in a right Line with one another, and with the vertical Point or Unit; this does in a Manner repreſent a *Pyramid*, and hence the Name.

The other Orders of Pyramidals have no ſuch Repreſentation, and are mere Combinations of the preceding, called Pyramidals only for a Diſtinction from the Pyramids whence they proceed.

VI. The Place of any Term in any Series of *Polygonals*, which is the Number of Places from the beginning to that Term, is called the *Root* or *Side* of that Polygonal; becauſe in the Polygon, repreſented it is the Number of Points or Units that makes the Side of the Figure; ſo 10 is the 4th Term of the Triangles, and 20 the 4th Term of the Triangular Pyramids; wherefore 4 is called their Root or Side; or we may as well call it the Place of any Term.

VII. Polygonals that ſtand in the ſame Places of their reſpective Series, are called *Collaterals* (*i.e.* having the ſame Side.)

VIII. The Product of any two Numbers is called alſo a plain Figurate Number; and is particularly a Quadrangle, becauſe it can repreſent ſuch a Figure; and the two Factors are called the Sides of the Figure, as in the annex'd Examples. And obſerve, that though Squares are Quadrangles, yet becauſe every Quadrangle is not a Square, therefore they may be diſtinguiſhed by applying the general Name Quadrangle to all the Species excepting Squares. But the Difference will be better marked by diſtinguiſhing them into *Oblongs* and *Squares*. Yet again, obſerve, that the Name Oblong is more particularly applied to that kind wherein



wherein the Sides differ by 1, which are the only Oblongs we consider here, because of their Connection with the Figurates above described; the whole Series of which Oblongs is made out by taking the natural Series 1 . 2 . 3, *&c.* and multiplying each Term into the next, as here,

	1	2	3	4	5	6	*&c.*
Oblongs	2	6	12	20	30		*&c.*

IX. The Product of any 3 Numbers is called also a solid Figurate Number, and particularly a Prism; and yet more particularly it's a Quadrangular Prism; for the Product of two Numbers is a Quadrangle, and the Product of this by the remaining Factor makes a Prism; because by taking any Quadrangle (or other plain Figure) a certain Number of Times, and conceiving them all placed parallel to one another at equal Distances, and so situated, that their respective Angles are in a right Line, they do in a manner represent what in Geometry is called a Prism. But again, in this Doctrine of Figurates, if we take the Product of any of the above described Polygons multiplied by its Side, that is called a Prism (though some of them are not composed of 3 Factors; and such as are so, yet are not considered in that Manner.)

These Prisms are also distinguished by the Polygon whence they proceed. Again, taking the Sums of these Series of Prisms, and the Sums of those Sums, and so on, we have new Series, which may be called in general *Prismatick Numbers*, to be distinguished the same Way as *Polygonals*, by different Orders and Species, as in the following Tables.

Again, *Prisms* being multiplied by their Sides, produce a new Kind of *Prismaticks*; and these again multiplied by their Sides, produce another Kind, and so on; all which we may distinguish by the Names of different Degrees, calling the Products of Polygons by their Sides, *Prismaticks* of the first Degree; the Products of these again by their Sides, *Prismaticks* of the second Degree, and so on. Observe also, That these several Degrees of Prismaticks are the Products of their Polygons by such a Power of their Sides as expresses that Degree; for a being any Polygon, and n its Side, the Prismaticks of the several Degrees proceeding from this Polygon are $a\times n$. $a\times n\times n.=a\times nn$. $a\times nn\times n=a\times nnn$. &c. Again, the Sums of Prismaticks of any Degree make also different Orders of Prismaticks of that Degree. *Lastly*, By the simple Name of Prisms always understand the first Degree, or Product of Polygons by their Sides.

Polygons.	
Triang.	1.3. 6.10
Square	1.4. 9.16 &c.
Pentag.	1.5.12 22
Hexa.	1.6.15.28
	&c.

Prisms, or *Prismaticks* of the 1*st Degree* and 1*st Order.*		*Prismaticks*, 1*st Degree*, 2*d Order.*
Triang.	1. 6. 18. 40	1. 7. 25. 65
Cubes,	1. 8. 27. 64 &c.	1. 9. 36.100 &c.
Pent.	1.10. 36. 88	1.11. 47.135
Hex.	1.12. 45.112	1.13. 92.204
	&c.	*&c.*

2*d Degree*, 1*st Order*,		2*d Degree*, 2*d Order.*
Triang.	1.12. 54. 60	1.13. 67.227
4*th Powers*	1.16. 81.256 &c.	1.17. 98.354 &c.
Pent.	1.20.108.352	1.21.129.481
Hex.	1.24.135.448	1.25.160.608
	&c.	*&c.*

Schol. We have observed already, That the Polygons of the second Species are the Series of Squares of the Progression 1 . 2 . 3 *&c.* (which shall be demonstrated afterwards.) And now from this and the Construction of Prismaticks, it follows, that these of the second Species, and 1st Order of all the Degrees successively, are the several Series of the superior Powers of the same Progression 1 . 2 . 3, *&c.* Thus, those of the first Degree are Cubes or third Powers; and universally, those of the n Degree are $n+2$ Powers. Therefore the whole Doctrine of Powers and Roots may be considered as a Part of this of Figurate Numbers; but as the calling them Figurates proceeds from a Consideration which is not properly Arithmetical; so the Order and Connection of Things in Arithmetick required that this Part concerning Powers and Roots, which is the most useful and necessary, should be particularly handled in another Place, as it is in *Book* III. and several Properties of these and other Composite Numbers (which are all Figurates) you have in *Chap.* I. of this Book.

We proceed now to explain the Properties of such Figurates as have not been yet handled, and some remaining Properties of those that have been in part considered already.

§ 2. *Of* Polygonal *Numbers.*

Theorem I.

Every Number is a Polygonal of every Species, and also of every Order whose Denominations are less than it by 2, or by any greater Number.

Exam. 5 is a Polygonal of the Third, and of all the preceding Species and Orders.

Demon. The first Term in every Species and Order being 1; the second Terms in the several Orders of the same Species, and of the same Order in all the different Species, are, by the Construction, in Arithmetical Progression, with the common Difference 1. Again, the second Term of the first Species in every Order is more by 2 than the Denomination of that Order, and is the least Number, except 1, of all the other Polygonals of that Order; comparing these Things, the Truth proposed is manifest.

Lemma.

Let any Number of different Series, $a. b. c. d.$ &c. $e. f. g. h.$ &c. as in the Margin, be such, that each collateral Column, as $d, h, m. q.$ &c. is an Arithmetical Progression. Also, let A . B . C . D *&c.* be the Sums of the former, thus, a=A. $a+b$=B. and so on. Then are the Collaterals of this last Table also $\div l$, and their common Difference is the Sum of the Differences of all the collateral Columns of the first Table backwards, from that which is in the same Place with any given Column of the second. Thus, let $e-a=x$. $f-b=y$. $g-c=z$, and $h-d=v$, whose Sum call S; then is D . H . M . Q in the common Difference S, or H—D=S $(=x+y+z+v.)$

a	b	c	d		A	B	C	D
e	f	g	h		E	F	G	H
i	k	l	m		I	K	L	M
n	o	p	q		N	O	P	Q

Demon. $e+f+g+h=a+b+c+d+$S; for S$=\overline{e-a}+\overline{f-b}+\overline{g-c}+\overline{h-d}$; also $i+k+l+m=e+f+g+h+$S, and so of the other Series; then because D$=a+b+c+d$, H$=e+f+g+h$ M$=i+k+l+m$. Q$=n+o+p+q$, therefore 'tis evident that H=D+S. M=H+S. Q=M+S, and so on: The Reason is the same, how large soever the Tables are, and which soever Column we chuse.

THEOREM II.

The Collaterals of the ſeveral Series $\div l$ (whoſe Sums make Polygons) are alſo $\div l$, whoſe common Difference is the preceding Term of the firſt Line; ſo 4 . 7 . 10 . 13 have the common Difference 3.

Again, the Collaterals of the ſeveral Species of any Order of Polygonals, are $\div l$, whoſe Difference is the preceding Term of the firſt Line or Triangular Species; ſo in the firſt Order, 10 . 16 . 22 . 28. differ by 6.

DEMON. 1°. For the Collaterals of the Series $\div l$, the firſt and leaſt Term of each Series is 1; and calling the Difference in any Series d, the n^{th} Term is $1+\overline{n-1}\times d$ (by the Rules of Progreſſions $\div l$.) But in the ſeveral Series the Differences are gradually 1 . 2 . 3 &c. wherefore in the Expreſſion $1+\overline{n-1}\times d$, d being ſucceſſively 1 . 2 . 3 &c. it follows that the Collaterals expreſſed univerſally $1+\overline{n-1}\times d$, are theſe $1+\overline{n-1}$: $1+\overline{n-1}\times 2$: $1+\overline{n-1}\times 3$ &c. which is a Series in the Difference $n-1$, which is the Term of the firſt Series preceding n, the Place of the Collaterals.

2°. For the Collaterals of the ſeveral Orders of Polygonals, the Theorem follows plainly from the preceding Lemma: For the Polygonals of the 1ſt Order proceed from the Series $\div l$, whoſe Collaterals are $\div l$, therefore, by the Lemma, theſe laſt are alſo $\div l$, and their Differences are the Sums of the Differences in the correſponding, and all the preceding Columns of the other: Alſo theſe other Differences being the preceding Term of the firſt Line, their Sum is the preceding Term of the firſt or Triangular Species of the firſt Order of Polygonals, or of the ſimple Polygons. For the ſame Reaſon, the Thing propoſed is true in the ſecond, and all the following Orders of Polygonals.

COROL. A Polygonal Number of any Order and Species is equal to the Sum of the Collateral Polygonal of any preceding Species of the ſame Order, and the Product of the Diſtance of theſe two Species (*i. e.* the Number of the Species, leſs 1, from the one Species to the others) multiplied by the preceding Polygonal of the firſt Species. This is manifeſt, becauſe it is nothing but the Rule for expreſſing the greateſt Term of a Series $\div l$, by means of the leſſer Term, the Number of Terms, and the common Difference. Thus, for Example, in the firſt Order, 28 is in the 4th Place of the 4th Species, and 28= the Sum of 16 (the 4th Term of the 2d Species) +12 (the Product of 6, the preceding Triangular, and 2, the Diſtance of the 4th and 2d Species.)

General SCHOLIUM.

In order to find the Sum of any Series of Polygonals, or any Term of any Series of Polygonals, it is plain that we want only a Rule for finding any Sum, or Term of any Series of the 1ſt or Triangular Species; becauſe thereby we can find any Term of the Collaterals of the Order given, by Coroll. preceding. *Again obſerve*, That any Polygonal of the firſt Order being the Sum of an Arithmetical Progreſſion, we know how to find any of theſe by the Rules of Progreſſions, and what we want is a Rule for the other Orders; but there is one general Rule which comprehends them all; in order to the Inveſtigation of which, and to make it the more ſimple and eaſy, we muſt conſider the natural Progreſſion 1 . 2 . 3 . &c. (from which the Triangulars proceed) as the Sums of a Series of Units 1 . 1 . 1 . 1 . &c. for the Sums of theſe continually from the beginning are 1 . 2 . 3 . 4 . &c.

Let us then begin with the Series of Units, and take the Series of their Sums, and the Sum of theſe Sums, and ſo on; and thus we ſhall have, from the moſt ſimple Original, the whole Orders of Polygonals of the Triangular Kind; and though, properly

ſpeaking, the Sums of the Series 1 . 2 . 3 . *&c.* are the firſt Order of Triangulars, yet it is convenient that we diſtinguiſh all the Series of the following Table by different Orders, calling the Series 1 . 1 . 1 . *&c.* the firſt Order, and 1 . 2 . 3 . *&c.* the ſecond; for they may be alſo call'd Triangular Numbers, becauſe what are more properly ſo proceed from them. It is true indeed, that by this Means the numbering of the Orders is different from the Method already laid down, but that will cauſe no Difficulty, becauſe this Way of numbering the Triangulars is uſed only with relation to the Rule we are now inveſtigating, and has this conſtant Connexion with the other, that the Number of the Order in this Method is always 2 more than that in the other Method; and beſides, by adding two Words we can ſave all Ambiguity; thus, when we ſpeak of any Order of Triangulars, for Example, the fourth Order, ſay the fourth Order from Units, and then all is clear; and if that is not added, you are to underſtand the Order number'd from the Series of Triangles (or Sums of the Series 1 . 2 . 3 . *&c.*) as in all the other Species the Orders are conſtantly number'd from their Polygons.

Table of Triangular Numbers from a Series of Units.

Orders									
1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9
3	1	3	6	10	15	21	28	36	45
4	1	4	10	20	35	56	84	120	165
5	1	5	15	35	70	126	210	330	495
6	1	6	21	56	126	252	462	792	1287
7	1	7	28	84	210	462	924	1716	3003

From the Conſtruction of this Table I make this uſeful *Obſervation.*

Every Term of every Order is equal to the Sum of the Collateral Term of the preceding Order, and the preceding Term of the ſame Order. Thus, 35 (the 5th Term of the 4th Order) is $=15$ (the Collateral or 5th Term of the preceding or 3d Order) $+20$ (the preceding or 4th Term of the ſame 4th Order.) The univerſal Truth of which Obſervation is manifeſt from the Conſtruction.

THEOREM III.

The Series of Numbers of any Order of the Triangulars, deduced from a Series of Units, is the ſame Series as the Series of Collaterals, the Number of whoſe Place from the beginning is equal to the Number of the Order of the other. Thus the Collaterals in the 6th Place are the ſame as the Triangulars of the 6th Order, *viz.* 1 . 6 . 21 . *&c.*

DEMON. The Truth of this Theorem appears in the preceding Table, ſo far as it is carried: and the Conſtruction of the Table attentively conſider'd will make the Univerſality of it plain. But to remove all Difficulty, I ſhall prove it thus,

1°. Every Term of any Collateral Column is equal to the Sum of all the Terms of the preceding Column, from the ſame Order upwards. So in the Collaterals of the 4th Place, the Term $35=15+10+6+3+1$, the preceding Column. And the Univerſality of this is manifeſt from the Obſervation made above upon the Conſtruction of the Table; for the 1ſt Term in every Column is the ſame, *viz.* 1; then the 2d Term is the Sum of the firſt Term of the ſame Column, and the 2d Term of the preceding Column (by that Obſervation) *i. e.* the Sum of the 1ſt and 2d Terms of the preceding Column; the 3d Term is the Sum of the preceding, or 2d Term of the ſame Column (*viz.* the Sum of the 1ſt and 2d Terms of the preceding Column) and the correſponding or 3d Term of the preceding Column; and ſo on.

2°. From what is laſt ſhewn it is manifeſt, that the ſeveral Collateral Columns are alſo the Sums of Numbers taken continually from a Series of Units, which is the 1ſt Column;

lumn; and thence it appears plainly, that the perpendicular Columns at any Diſtance from the 1ſt Column of Units, muſt be *ad infinitum* the ſame as the tranſverſe Lines or Series of Numbers, at equal Diſtance from the Series of Units, which is the firſt Line.

Coroll. The Term in any Place of any Order of Triangulars (in the preceding Table) is equal to that Term whoſe Place is the Number of the Order of the former, and of ſuch an Order whoſe Number is the Place of the other. Thus, the 3d Term of the 5th Order is equal to the 5th Term of the 3d Order; and univerſally, the a Term of the b Order, is the ſame as the b Term of the a Order.

PROBLEM I.

To find the Triangular Number in any given Place, of any given Order (from Units.)

Rule. Let the Order be call'd a, and the Place b (or contrarily the Order b and the Place a) then take $n=a+b-2$, and carry on this Series $1 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$ &c. to $\frac{b}{a-1}$; the continual Product of all theſe Factors is the Number ſought.

Exam. To find the 5th ($=b$) Term of the 7th ($=a$) Order; then is $a+b-2$ ($=n$) $=7+5-2=10$; and the Number ſought is $1 \times \frac{10}{1} \times \frac{9}{2} \times \frac{8}{3} \times \frac{7}{4} \times \frac{6}{5} \times \frac{5}{6} = 210$.

Demon. 1°. The firſt Thing in order to the Demonſtration of this Rule, is to obſerve, That it is the very ſame Thing in effect as the Rule for finding the Coefficient of the $\bar{a}$ Term of a Binomial Power (or Power of a Binomial Root) whoſe Index is n, or $\overline{a+b-2}$, for which ſee *Book* III. *Chap.* II. So that what remains to be prov'd is this Correſpondence of Coefficients and Triangulars, *viz.* that the b Triangular (or Triangular in the b Place) of the $\bar{a}$ Order, is the ſame as the $\bar{a}$ Coefficient (or Coefficient of the $\bar{a}$ Term) of the $n=a+b-2$ Power of a Binomial Root. And to ſhew this let us,

2°. Compare the Table of Coefficients (*Book* III. *Chap.* II.) with this Table of Triangulars, and it's manifeſt they are the very ſame Numbers, only diſpoſed in another Manner. For it is plain, they are the ſame Numbers taken in their perpendicular Columns, as being produc'd the ſame Way from the Column of Units by continual Addition, *i. e.* what are there called ſimilar Coefficients are the ſame Numbers as what are here call'd Collaterals, being taken at equal Diſtance from the Beginning or Column of Units: The Difference being this, that in the Table of Triangulars the firſt Terms of every perpendicular Column ſtand in one Line, and ſo do the 2d Terms, and ſo on; but in the Table of Coefficients the firſt Term of the 2d Column ſtands in a Line with the 2d of the firſt Column, and ſo on; whence it's plain, that Coefficients in different Places, and different Powers, are the ſame Numbers under a different Name, with Triangulars in different Places and different Orders; and for their mutual Correſpondence let us conſider what is ſhewn, *Book* III. *Chap.* II viz.

3°. The a Coefficient of the n Power is equal to the $n-a+2$ Coefficient of the ſame Power, reckoning from either Extreme; alſo that the $n-a+2$ Coefficient of the n Power is the ſame as the $n-a+2$ Term of the ſimilar Coefficients in the a Place of different Powers, *i. e.* (by what is before ſhewn) the Triangular in the a Place of the $n-a+2$ Order (for the different Places in the Column of ſimilar Coefficients anſwer to the different Orders of Triangulars; and different Places of the Coefficients of the ſame Powers anſwer to different Places in the ſame Order of Triangulars.) Now ſup-

 poſe

pose $n-a+2=b$, then is $n=a+b-2$; wherefore the a Coefficient, and the b Coefficient of the n Power are the same. Also the b Coefficient of the n Power is the b Term of the similar Coefficients in the a Place of different Powers, equal to the a Triangular of the b Order: But the a, or also the b Coefficient of the n Power is $1 \times \frac{n}{1} \times \frac{n-1}{2}$, &c. so $\frac{n-a+2=b}{a-1}$, which must therefore be the a Triangular of the b Order (or b Triangular of the a Order) as it is, according to the Rule, which is therefore good.

Another Rule *for solving the preceding* Problem.

For the sake of a particular Use of it, I give you here another Rule for the preceding *Problem*; which is this,

Let a express the given Order of Triangulars, and n the Place of the Term sought; then the continual Product of these Factors, $1 \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$, *&c.* to $\frac{n+a-2}{a-1}$ is the Number sought.

Exam. To find the 5th Term of the 7th Order; it is, $210 = 1 \times \frac{5}{1} \times \frac{6}{2} \times \frac{7}{3} \times \frac{8}{4} \times \frac{9}{5} \times \frac{10}{6}$.

DEMON. In the Demonstration of the former Rule it is shewn, that the b Term of the a Order, is the a Coefficient of the $n=a+b-2$ Power; and if instead of b we put n, then the n Term of the a Order is the a Coefficient of the $\overline{a+n-2}$ Power; which by the Rule of Coefficients is, $1 \times \frac{a+n-2}{1} \times \frac{\overline{a+n-2}-1}{2} \times \frac{\overline{a+n-2}-2}{3}$, *&c.* to $\frac{n}{a-1}$, which is the same as the preceding Rule for Triangulars (*i. e.* for the n Term of the a Order;) but this Series is the same in effect as the other Series $1 \times \frac{n}{1} \times \frac{n+1}{2}$, *&c.* $\frac{n+a-2}{a-1}$; for it is manifest that the Denominators are the same, and the Numerators also, only in a reverse Order (which makes no change in the Product;) for both the Series have the same Number of Terms, as the Series of Denominators does clearly shew; and for the Numerators, the first and last of them are the same Numbers in both, only the first in the one is the last in the other; and since their Progression is by a continual Difference of 1, it follows plainly, that they must be the same Numbers only in a reverse Order; consequently this Rule is good, since it is the same (only in a different Form) with the former, which is demonstrated to be good.

But observe, That this last Rule may also be demonstrated independently of the other, from the immediate Consideration of the Triangular Numbers, without any Comparison of them with Coefficients. *Thus,*

The n Term of the a Order is the a Term of the n Order (*Cor. Theor.* III.) and therefore it is the same Thing to which of these we apply the Rule; but for the present Demonstration we must take it, the a Term of the n Order; and then, I say, if the Rule is good in one Case, or for one Order of Triangulars, as the n Order, it will therefore be good in the next Case, or the $n+1$ Order, and consequently it is good in all superior Cases; and to prove this, first consider the annex'd Series, wherein, because 1 does not multiply, therefore I have omitted it in every Term as useless, except in the first Term, which is it self 1.

Tri-

	1st.	2d.	3d.	4th.	
Triangulars of the Order n, according to the Rule,	$1.$	$\frac{n}{1}.$	$\frac{n}{1}\times\frac{n+1}{2}.$	$\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}.$	&c.
The Equivalents to these; (the Numerators being only in a different Order)	$1.$	$\frac{n}{1}.$	$\frac{n+1}{1}\times\frac{n}{2}.$	$\frac{n+1}{1}\times\frac{n+2}{2}\times\frac{n}{3}.$	&c.
Triangulars of the Order $n+1$ according to the same Rule.	$1.$	$\frac{n+1}{1}.$	$\frac{n+1}{1}\times\frac{n+2}{2}.$	$\frac{n+1}{1}\times\frac{n+2}{2}\times\frac{n+3}{3}.$	&c.

Now 'tis plain, that the Terms of the last Series are the Triangulars of the Order $n+1$, according to the Rule; and that they are truly the Numbers sought, upon Supposition that those of the former Series are the Triangulars of the Order n, I thus shew. By the Observation made upon the Construction of the Table of Triangulars, every Term of any Order is the Sum of the Collateral Term of the preceding, and the preceding Term of the given Order; but comparing the 2d and 3d Series (*i. e.* the Series for the Order n, and that for the Order $n+1$) it's plain that the last is compos'd according to that Property now mention'd; thus $1=1$, the first Terms; $1+\frac{n}{1}=\frac{n+1}{1}$ (*i. e.* the 2d Term of the Order $n+1$ equal to the 2d Term of the Order n, and 1st Term of the Order $n+1$;) $\frac{n+1}{1}\times\frac{n}{2}+\frac{n+1}{1}$ $\left(=\frac{n+1}{1}\times\overline{\frac{n}{2}+1}\right)=\frac{n+1}{1}\times\frac{n+2}{2}$. Again, $\frac{n+1}{1}\times\frac{n+2}{2}\times\frac{n}{3}+\frac{n+1}{1}\times\frac{n+2}{2}$ $\left(=\frac{n+1}{1}\times\frac{n+2}{2}\times\overline{\frac{n}{3}+1}\right)=\frac{n+1}{1}\times\frac{n+2}{2}\times\frac{n+3}{3}$, and so on; for it's manifest, that according as these two Series proceed, they must always have the same Connection, *viz.* that any Term of the last is the Sum of the Collateral Term of the former, and the preceding Term of the same last Series. Wherefore if the former is the Series of Triangulars of the Order n, the last must be that of the Order $n+1$.

But the Rule is true when apply'd to the first Order or Series of Units; for here $n=1$; and hence it's plain that the Numerator and Denominator in every Factor are equal, and therefore they are each equal to 1; hence every Term of the Series is 1.

Lastly, The Rule being good for the first Order, it must therefore, by what was first proved, be good for the second Order, and so for the third, and all the following for ever.

Observe, Lest any body should think the universal Connexion of the two Series, for the Order n and $n+1$, any thing obscure, I shall make this universal Demonstration of it, *viz.* That any Term of the last is the Sum of the Collateral Term of the former, and the preceding Term of the same last. Thus, any Term of the Series for the Order $n+1$ may be expressed $\frac{n+1}{1}\times\frac{n+2}{2}$, &c. to $\frac{n+a}{a}$ inclusive; and the preceding Term is therefore $\frac{n+1}{1}\times\frac{n+2}{2}$, &c. to $\frac{n+a}{a}$ exclusive; also the Collateral Term of the Series for the Order n is $\frac{n+1}{1}\times\frac{n+2}{2}$, &c. to $\frac{n}{a}$, inclusive: Now the Sum of

$n+1$

$\frac{n+1}{1}\times\frac{n+2}{2}$ &c. and $\frac{n+1}{1}\times\frac{n+2}{2}$ &c. $\times\frac{n}{a}$ is $=\frac{n+1}{1}\times\frac{n+2}{2}$ &c. $\times\overline{1+\frac{n}{a}}=\frac{n+1}{1}\times$ $\frac{n+2}{2}$ &c. $\frac{n+a}{a}$, the Thing to be proved.

Schol. As this 2d Rule has been demonstrated independently of the Rule for Coefficients; so the Rule of Coefficients may be also demonstrated, by Means of this Rule, for Triangulars, with the Correspondence betwixt the two, as above explain'd. Thus, the n Term of the a Order of Triangulars, being $1\times\frac{n}{1}\times\frac{n+1}{2}$ &c. $\times\frac{n+a-2}{a-1}$; let us only invert the Order of the Numerators, which does not alter the Value of the Total Product, and it is $1\times\frac{n+a-2}{1}\times\frac{n+a-2-1}{2}$ &c. $\times\frac{n}{a-1}$. Again, Take m $=n+a-2$, and the Series is $1\times\frac{m}{1}\times\frac{m-1}{2}$ &c. $\times\frac{m-a+2}{a-1}$, which is the Rule for the a Coefficient of the m Power, as it ought to be; since it is shewn that the n Term of the a Order of Triangulars, is the a Coefficient of the $n+a-2$ Power, *i. e.* of the m Power.

Hence we have also a new Rule for Coefficients, which is this; Let n, or a, be the Place of the Coefficient, and $a+n-2$, the Index of the Power; then is $1\times\frac{n}{1}\times\frac{n+1}{2}$ &c. $\times\frac{n+a-2}{a-1}$, the Coefficient; for this is the a Term of the n Order of Triangulars; which is equal to the n Coefficient of the $a+n-2$ Power, as already shewn. Also the n Coefficient of the $a+n-2$ Power, is equal to the a Coefficient of the $a+n-2$ Power; for if you call $a+n-2=b$, then the n Coefficent of the b Power is also the $b-n+2$ Coefficient of the b Power (as has been shewn) *that is*, the $a+n-2-n+2$ $(=a)$ Coefficient of the $a+n-2$ Power.

In the last Place we shall set before us, in one View, these two Rules, as they relate both to Coefficients and Triangulars.

$1\times\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}$ &c. $\times\frac{n-a+2=b}{a-1}=$ $\left\{\begin{array}{l} a\text{, or also the}\\ n-a+2\ (=b) \end{array}\right\}$ Coefficient of the n $(=a+b-2)$ Power. a Triangular of the b Order, or b Triangular of the a Order.

$1\times\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}$ &c. $\times\frac{n+a-2=b}{a-1}=$ $\left\{\right.$ n or a Coefficient of the b $(=n+a-2)$ Power. n or a Triangular of the a or n Order.

PROBLEM II.

To find the Polygonal Number in any Place, of any Order, and of any Species.

Rule. Find, by the last Problem, the Polygonal of the given, and also of the preceding Place, of the given Order of the first or triangular Species. Take the Number of the given Species, less 1; by which multiply the Polygonal found of the preceding Place; to the Product add the Polygonal found of the given Place; the Sum is the Number sought. But here *observe*, That in every Species, except the first, the Orders are number'd from the Polygons, or Sums of the Arithmetical Series, whence they proceed: Whereas in the first Species they are number'd from the Series of Units; so that their Number is always more by 2 than that of the other Orders, at the same Distance from the Polygons. Wherefore in finding the preceding Polygonal of the first Species, and of the given Order, from Polygons, add 2 to the Number of

of the given Order, and find the Polygonal for that Order, according to the preceding Rules, which will give a Polygonal at the same Distance from the first and simple Polygons as the given Order is.

Exam. To find the 4th Term of the 3d Order of the 5th Species; I find the 3d and 4th Terms of the 5th Order of the 1st Species, which are 15, 35; then the Number of the given Species is 5, and $15\times4=60$; to which add 35, the Sum is 95, the 4th Term of the 3d Order of the 5th Species. (*See the Table.*)

Demon. The Reason of this Rule is manifest from *Coroll.* to *Theor.* II. and needs no farther Explication.

Scholium, *relating to* Problem I *and* II.

From these general Rules of *Probl.* I, and II. we may easily deduce particular Rules for particular Series: I shall apply them to two Cases, by which all others will be easily understood.

1°. To find the Sum of the Series of Triangles to any Number (n) of Terms, *i. e.* to find the n Term of the Series of Triangular Pyramids or Polygonals of the 1st Species, and 4th Order from the Series of Units (which is the 2d Order from the simple Polygons.) The Rule is,

To twice the Side or Place of the Term sought, add its Cube, and thrice its Square; the 6th Part of the Sum is the Term sought, *viz.* $\frac{n^3+3n^2+2n}{6}$; for by the general Rule of *Problem* I. this Term sought is $1\times\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}=\frac{n^3+3n^2+2n}{6}$.

2°. To find the Sum of Polygons of the 2d Species, or Squares, to any Number of Terms, *i. e.* to find any Term of the Series of square Pyramids, or Polygonals of the 2d Species and 4th Order from the Series of Units (which is the 2d Order from the simple Polygons.) The *Rule* is this:

To the Number of Terms added, or Place of the Term sought, add thrice its Square, and also double its Cube; the 6th Part of the Sum is the Number sought. Thus, if the given Number of Terms is n, the Number sought is $\frac{2n^3+3n^2+n}{6}$: The *Investigation* of which is this; the n Term of the 4th Order of Triangulars (from the Series of Units) is $=\frac{n^3+3n^2+2n}{6}$ (by the last) and the preceding, or $n-1$ Term is $1\times\frac{n-1}{1}\times\frac{n}{2}\times\frac{n+1}{3}=\frac{n^3-n}{6}$; but, by *Coroll. Theor.* II. the n Term of the 2d Species is the Sum of the n and $n-1$ Terms of the 1st or Triangular Species; *i. e.* $\frac{n^3+3n^2+2n}{6}+\frac{n^3-n}{6}=\frac{2n^3+3n^2+n}{6}$.

Theorem IV.

Take the Series of simple Polygons of any Species, after the 1st, to any Number of Terms; and take the Series of Triangulars of any Order after the 1st (numbering the Orders of these from the Series of Units) to the same Number of Terms; place these reversely under the other, and multiply the corresponding Terms (as they are plac'd) of the one into those of the other; the Sum of the Products is equal to a Term standing in the Place express'd by the given Number of Terms, of the Order express'd by 1 more than the given Order of Triangulars, and of the given Species of Polygonals.

Exam.

1	.	6	.	15	.	28	
35	.	15	.	5	.	1	
35	+	90	+	75	+	28	= 228

Exam. The Sexangulars to the 4th Term of the 1st Order, are 1 . 6 . 15 . 28; the Triangulars of the 5th Order (from Units) are 1 . 5 . 15 . 35; which placed under the other, and multiplied as in the Margin, produces 228; which is the 4th Sexangular of the 6th Order from the simple Hexagon (which is here the 1st Order) for this is 1 . 6 . 15 . 28; the 2d Order is 1 . 7 . 22 . 50; the 3d Order is 1 . 8 . 30 . 80; the 4th Order is 1 . 9 . 39 . 119; the 5th Order is 1 . 10 . 49 . 168; the 6th Order is 1 . 11 . 60 . 228.

Demon. The Reason of this is plain from the following Table; wherein, if 1 . *a* . *b* . *c* . *d* . &c. represent the Polygons of any Species, the several Orders of Sums proceeding from these, are evidently as in this Table.

1	1	1	1
a	$1+a$	$2+a$	$3+a$
b	$1+a+b$	$3+2a+b$	$6+3a+b$
c	$1+a+b+c$	$4+3a+2b+c$	$10+6a+3b+c$
d	$1+a+b+c+d$.	$5+4a+3b+2c+d$	$15+10a+6b+3c+d$
&c.	&c.	&c.	&c.

And it is plain also, that the Terms or Sums in each Order are according to the *Theorem*; because in the 2d Order they are the Sums of the Series of the first Order, multiplied by a Series of Units (which is the 1st Order in Triangulars) then the Multipliers in all that follow, are manifestly the Sums of the preceding continually, from the Series of Units.

Coroll. Hence we learn a new Practice for finding the Polygonal, in any Place, of any Order and Species after the 1st, *viz.* by having the Series of the 1st Order of any Species (after the 1st) and the Series of the Triangulars of the Order 1 less than the other: But this not being so easy a Practice as that in the preceding *Problem*, I have chosen to express the Rule in the Manner of a *Theorem*, regarding it only in general, as a Connection discovered betwixt the Triangulars and the other Species of Polygonals.

THEOREM V.

If we take the Progression, 1 . 2 . 3 . 4 &c. and the Series of Triangles, which are the Sums of the former, 1 . 3 . 6 . 10 . 15 &c. then take the Series of Ratios of the several Terms of the 1st Series, comparing each Term to the following, in a continued Order, as, 1 : 2, 2 : 3, 3 : 4, &c. Also take the Series of Ratios of the 2d Series, beginning at the 2d Term, and proceeding discontinuedly; *thus*, 3 : 6, 10 : 15, &c. These two Series of Ratios are the same; thus, 1 : 2 : : 3 : 6, 2 : 3 : : 10 : 15, and so on.

1 : 2, 2 : 3, 3 : 4, &c.
3 : 6, 10 : 15, 21 : 28, &c.

Demon. I have in the Margin placed the 2 Series according to the proposed Correspondence of their Ratios; and so far as it is carried, the Truth of the *Theorem* is plain. But to shew the Reason of it, and that it must be so for ever; in the first Place *observe*, that the Antecedents in the several Ratios of the 1st Series (1 . 2 . 3 . &c.) express the Places of these Terms from the Beginning; and the several Antecedents of the Ratios taken in the 2d Series, stand in the several even Places of the Series, *i. e.* in the 2d, 4th, 6th, *&c.* Places; but the Series of even Numbers, 2 . 4 . 6 &c. are the Doubles of the respective Terms of the natural Progression, 1 : 2 : 3 : &c. which being all Antecedents of the Ratios taken in the 1st Series, it follows, that the Antecedents of the several

several Ratios in the 2d Series are in such Places of that Series as are expressed by double the Antecedent of the correspondent Ratio (number'd from the Beginning of the Series of Ratios) in the 1st Series; *thus*, 3 : 4 is the 3d Ratio of the 1st Series; and the 3d Ratio of the 2d Series, is 21 : 28, whose Antecedent, 21, stands in the 6th ($=2\times3$) Place of the Series. Now to shew that these correspondent Ratios are equal, take any two Terms adjacent in the 1st Series; they may be expressed $n : n+1$, which make the nth Ratio in the Order of Ratios, as they are taken out of the 1st Series: And by what is last shewn, the Antecedent of the nth Ratio of the 2d Series, according to the Manner of taking them there, stands in the $2n$ Place of that Series; and consequently it is the Sum of the 1st Series to the $2n$th Term; which, by the Rule of Progressions, is $\overline{2n+1}\times\frac{2n}{2}=\overline{2n+1}\times n=2n^2+n$; and the Consequent or next greater Term in the Series of Sums must be this Sum, with the following Term of the 1st Series, which is $2n+1$; so the Consequent is $2n^2+n+\overline{2n+1}=2n^2+3n+1$. Then *lastly*, $n : n+1 :: 2n^2+n : 2n^2+3n+1$; because the Product of Extremes and Means are equal, *viz.* $2n^3+3n^2+n$.

THEOREM VI.

Take the natural Series, 1, 2, 3, &c. also the Series of its Sum, or Series of Triangles, 1 . 3 . 6 . &c. the Series of Ratios proceeding from the Comparison of every Term of the 1st Series to the 2d from it, or next but one, as 1 : 3, 2 : 4, 3 : 5, &c. are the same as these, which proceed from the Comparison of every Term of the 2d Series to the next, as 1 : 3, 3 : 6, 6 : 13, &c.

Series.

1 . 2 . 3 . 4 . 5 . 6.
1 . 3 . 6 . 10 . 15 . 21.

Ratios.

1 : 3, 2 : 4, 3 : 5, 4 : 6.
1 : 3, 3 : 6, 6 : 10, 10 : 15.

DEMON. Let n be any Term of the 1st Series, and $n+2$ the 2d above it; then is the n Term of the 2d Series $\frac{\overline{n+1}\times n}{2}$, and the next Term above it, or the $n+1$ Term, is $\frac{\overline{n+2}\times\overline{n+1}}{2}$; but it's plain, that $n : n+2 :: n\times\overline{n+1} : \overline{n+2}\times\overline{n+1}$; or as $\frac{n\times\overline{n+1}}{2} : \frac{\overline{n+2}\times\overline{n+1}}{2}$; the Thing to be proved.

THEOREM VII.

Take any three adjacent Triangles, and betwixt the lesser and the middle one, place the Number next lesser (*viz.* by 1) than that middle one; and these four are Geometrically Proportional: Thus for *Example*, 6 . 10 . 15, are three adjacent Triangles, and 6 : 9 :: 10 : 15.

DEMON. Take three Triangles standing in the $n-1$, n, and $n+1$, Places; they are $\frac{n\times\overline{n-1}}{2}$; $\frac{\overline{n+1}\times n}{2}$; $\frac{\overline{n+2}\times\overline{n+1}}{2}$; from the middle one $\frac{\overline{n+1}\times n}{2}$ take 1, the Remainder is $\frac{\overline{n+1}\times\overline{n-2}}{2}$; and $\frac{n\times\overline{n-1}}{2} : \frac{\overline{n+1}\times\overline{n-2}}{2} :: \frac{\overline{n+1}\times n}{2} : \frac{\overline{n+2}\times\overline{n+1}}{2}$, as will appear from the equal Product of Extremes and Means; and to do this more easily, because the Denominators are all equal, we may cast them all out; and then also observe, that $\overline{n+1}\times n : \overline{n+2}\times\overline{n+1} :: n : n+2$; wherefore we need only try the Proportionality of these, $n\times\overline{n-1} : \overline{n+1}\times\overline{n-2} :: n : n+2$, *i. e.* $nn-n : nn+n-2 :: n : n+2$, in which $\overline{nn-n}\times\overline{n+2}=\overline{nn+n-2}\times n$; therefore the 4 are $::l$, *i. e.* the Numbers proposed are $::l$.

COROLL. The Product of any two Triangles, betwixt which there lies but one other Triangle, is an Oblong, whose greater Side is that interjacent Triangle.

LEMMA.

Take the natural Progression, 1, 2 3, &c. and after the 1st Term 1, take the Sum of every two successive Terms, *thus*, 1, 2+3, 4+5, 6+7 &c. you have hereby a Series of Numbers, ÷*l* with the common Difference, 4. Thus the preceding Series is 1 . 5 . 9 . 13 . &c.

DEMON. The Reason is plain from this, That every Term in the natural Series, 1 . 2 . 3 . &c. exceeding the preceding, by 1, betwixt any Term and the next but 1 (or the 2d after it) the Difference is 2; consequently the Difference of the Sum of any two adjacent Terms, and the Sum of the next two adjacent Terms, must be 4; therefore the Series of these Sums are in a constant Difference of 4; which is also the Difference of 1, and the 1st Sum 2+3.

THEOREM VIII.

Every *Hexagon* is also a *Triangle*; and particularly, all the Triangles in odd Places, as the 1st, 3d, 5th, &c. make the complete Series of Hexagons: As here,

	1 .	2 .	3 .	4 .	5 .	6 .	7 .	8 .	9	
Triangles	1 .	3 .	6 .	10 .	15 .	21 .	28 .	36 .	45	*&c.*
Hexagons	1	:	6	:	15	:	28	:	45	
	1	.	5	.	9	.	13	.	17	

DEMON. Hexagons are the Sums of a Series ÷*l*, whose first Term is 1, and the common Difference 4 (as 1 . 5 . 9 &c.) and the Triangles are the Sums of the natural Series (1 . 2 . 3 . &c.) but taking this last Series in the Manner mentioned in the preceding *Lemma*, viz. 1 : 2+3 : 4+5 : &c. we have a Series beginning with 1, and proceeding with the Difference 4; consequently the Sums of this Series are Hexagons; but it's plain, that they are also Sums of the natural Progression taken to every odd Number of Terms; for they are 1, 1+2+3, 1+2+3+4+5, &c. and consequently they are all the Triangles in odd Places.

Otherwise *thus*; If you number the odd Places of any Series by themselves, and compare the Number in any odd Place, with the Place of that Term, as it's number'd with all the Terms of the Series; then if to the Number of the Place, in which any odd Term stands in the whole Series, be added 1, the Half of the Sum expresses what Place it stands in among the odd Terms number'd by themselves; *thus*, The 9th Term, in the Whole, is the 5th $\left(=\frac{9+1}{2}\right)$ Term of the odd Places numbered by themselves; the Reason of which is obvious. Again, Any odd Number may be expressed $2n+1$, and if the Sum of the natural Series is taken to the $2n+1$ Term, it is $\overline{2n+2} \times \frac{\overline{2n+1}}{2} = \frac{4nn+6n+2}{2} = 2nn+3n+1$, which is a Triangle. Then to the last Term (or Number of Terms) added in this Sum, *viz.* to $2n+1$, add 1; the Sum is $2n+2$, whose Half is $n+1$; which, by what's shewn, is the Place of the Number $2n+1$, among the odd Places of the natural Series, number'd by themselves; wherefore, find the $n+1$ Hexagon, and it is $2nn+3n+1$, which is the Triangle already found in the $2n+1$ Place: For Hexagons proceed from a Series ÷*l*, whose Difference is 4; where-

wherefore the $n+1$ Term of that Series is $1+4n$; and hence the Sum of the Series to the $n+1$ Term is $\overline{2+4n}\times\frac{n+1}{2}=\frac{4nn+6n+2}{2}=2n^2+3n+1$.

THEOREM IX.

If the Sums of the odd Series, 1 . 3 . 5 . 7 &c. are taken continually from the Beginning, they are the Squares of the natural Progression, 1 . 2 . 3 . 4 . &c. Or thus; Every Square Number is the Sum of the Terms of the odd Series taken from 1, to a Number of Terms equal to the Root of that Square.

Odd Series ————	1	3	5	7	9	11	13	15	&c.
Their Sums ————	1	4	9	16	25	36	49	64	&c.
Square Roots of the Sums	1	2	3	4	5	6	7	8	&c.

DEMON. The Truth of this Proposition you see so far as the Series are carried; and that it will be so for ever we have already demonstrated, in *Cor.* 4. *Probl.* V. *Ch.* II, *&c.* where it is shewn, that the Sum of the odd Series, 1 . 3 . 5 . &c. is the Square of the Number of Terms.

But there is another more natural Demonstration of this Truth, deduced from the Consideration of square Numbers, and their Composition. Thus;

1°. Take the natural Progression, 1 . 2 . 3 . &c. and the odd Series, 1 . 3 . 5 . &c. the several Terms of the natural Series, 1 . 2 . 3 . &c. express the Number of Terms from 1 to any Term of the odd Series, or to any Term of the Series of their Sums; but from the Nature of Progressions, and particularly of this odd Series, any Term of it is equal to the Sum of 1 (the lesser Extreme) and 2 (the common Difference) multiplied by the preceding Term of the natural Progression (which is the Number of Terms less 1.)

2°. The Difference of any two square Numbers, whose Roots differ by 1 (and such are every two adjacent Terms in the Series, 1 . 2 . 3 . &c.) is equal to the Sum of 1, and double the lesser Root; thus, $\overline{a+1}^2=a^2+2a+1$; so that a^2, and $\overline{a+1}^2$, differ by $2a+1$. Hence,

3°. If the Sum of the odd Series, carried to any Number of Terms, is the Square of the Number of Terms (*i. e.* of the correspondent Term of the natural Progression, 1 . 2 . 3 . &c.) so will it be if carried to one Term more; because that next odd Term is equal to the Sum of 1, and double the preceding Root (or Term of the Series, 1 . 2 . 3 . &c.) which is also the Difference of two Squares, whose Roots differ by 1; as it is in the present Case. But we see the Truth proposed as far as we have carried the Series; therefore it must go on so for ever.

COROLLARIES.

1*st*. The Difference of any two integral Squares is equal to some one, or the Sum of some two or more Terms of the odd Series: More particularly, it is equal to the Sum of all the Terms of the odd Series comprehended betwixt that Term (inclusive) whose Place in the Series is the Root of the greater Square, and that Term (exclusive) whose Place is the Root of the lesser Square, *i. e.* all the Terms from that one (inclusive) which stands over the greater Square, and that one (exclusive) which stands over the lesser. So if n represents the Place of any Term in the odd Series, and m the Place of any lesser Term; then are n, $m-1$, the Roots of two Squares; which differ by the Sum of all the Terms comprehended betwixt these Extremes, including both. *Exam.* $49-9=7+9+11+13$. Hence,

2*d*. 2 cannot be the Difference of any two Squares; becauſe it is not any Term, nor the Sum of any Terms of the odd Series, the two leaſt being $1+3=4$.

3*d*. If the Difference of two Squares is equal to the Sum of all the Terms of the odd Series, from 1 to any aſſigned Term; the greater of theſe Squares cannot be that correſponding to the aſſigned Term, *i. e.* it cannot have for its Root the Place of (or Number of Terms from the Beginning to) that Term; becauſe no leſſer Square can differ from that greater one, by the Sum of all the odd Series, from 1 to that greater. Hence again,

4*th*. 1 cannot be the Difference of two Squares, nor 4; becauſe $1+3$ cannot be the Difference of two Squares, whereof the greater correſponds to 3; nor is 4 any Term of the odd Series, or the Sum of any two or more Terms of the odd Series, other than $1+3$.

5*th*. Every odd Number above 1, and the Sum of any Number of Terms adjacent in the odd Series, whereof the leſſer is greater than 1, is the Difference of ſome two Squares, whoſe Roots are found as in the firſt *Corollary*.

$1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6$
$11 \,.\, 9 \,.\, 7 \,.\, 5 \,.\, 3 \,.\, 1$

$11+18+21+20+15+6=$
$1+4+9+16+25+36=$
91.

6*th*. If we take the natural Series, 1 . 2 . 3 . &c. to any Number of Terms, and under it ſet the Series of odd Numbers, 1 . 3 . 5 . &c. in a reverſe Order; then multiply each Term of the one into the correſponding of the other; the Sum of theſe Products is equal to the Sum of the Squares of all theſe Terms of the Series, 1 . 2 . 3 . &c.

Schol. That neither 1 . 2, or 4, can be the Difference of any two Squares, may be eaſily ſhewn otherwiſe; thus, n^2 and $\overline{n+1}^2$ $(=n^2+2n+1)$ differ by $2n+1$, which, it's plain, can neither be 1 . 2, nor 4; and the leaſt it can be, is 3, *viz.* when $n=1$. If we take two Roots differing more than 1, as n, $n+d$, their Squares differ by $2nd+dd$; which, it's manifeſt, exceeds 4; for dd is here, at leaſt, 4; d being greater than 1.

A particular Uſe and Application of ſome of theſe Corollaries you'll find afterwards.

Theorem X.

Take the Series of Triangles, 1 . 3 . 6 . 10 . &c. Then take the Sum of every two adjacent Terms continuedly, thus, $1+3$, $3+6$, &c. The Sums are the Series of Squares of the natural Progreſſion after 1; as in the Margin.

1	2	3	4	5	6	*Nat. Series*
1	3	6	10	15	21	*Triangle*
1	4	9	16	25	36	*Squares*

Demon. This follows from the laſt, compared with *Cor.* 1. *Theor.* II. For the Sums of the Series, 1 . 3 . 5 . &c. are the Squares of the natural Series, 1 . 2 . 3 . &c. by the laſt, and by *Cor.* 1. *Theor.* II. the Sum of every two adjacent Triangles is the Term of the Quadrangles (or Squares) collateral with the greater of theſe Triangles.

But I ſhall alſo demonſtrate this *Theorem* otherwiſe; thus, Triangles are the Sums of the natural Series, 1 . 2 . 3 . &c. the Sums of which, to n, and $n+1$ Terms, are $\frac{nn+n}{2}$, and $\frac{nn+3n+2}{2}$, which are therefore the two adjacent Triangles in the n and $n+1$ Places; but their Sum is $\frac{2nn+4n+2}{2}=nn+2n+1=\overline{n+1}^2$.

Theo-

THEOREM XI.

Some Triangles are alſo Squares: And if beginning with the Numbers, 2 : 3, you make another Couplet out of them, thus, Take the Sum of theſe two for one Term (*viz.* $2+3=5$) then to that Sum add the leſſer Term of the ſame Couplet (*viz.* 2.) and make this Sum ($5+2=7$) the other Term: Then out of this laſt Couplet, 5 : 7, make another Couplet in the ſame Manner as before; which will be 12 : 17 (*viz.* $5+7=12$, and $12+5=17$.) And go on in this Manner for ever. Again, Take the Squares of both Terms of each of theſe Couplets; the Products of the Squares of the two Terms of every Couplet, are all Numbers which are both Squares and Triangles: As in the following Scheme; where 36 is a Square, whoſe Root is 6; and it is a Triangle, the Sum of the 1ſt 8 Terms of the natural Series.

	2 : 3,	5 : 7,	12 : 17,	&c.
Squares	4 : 9,	25 : 49,	144 : 289,	&c.
Products	36 .	1225 .	41616 .	&c.

DEMON. 1°. The Numbers here produced are Squares; becauſe they are the Products of two Squares (*Theor. B.* III. *Ch.* I.)

2°. They are alſo Triangles, or the Sums of the natural Series to a certain Number of Terms; which I prove by theſe Steps.

(1.) If any two Numbers, a, b, are ſuch, that $b=2a+1$, or $2a-1$; then is ab the Sum of the natural Series, from 1 to $2a$, in the Caſe of $b=2a+1$; or from 1 to b, if $b=2a-1$: For if $b=2a+1$, then are $2a$, and b, two adjacent Terms in the natural Series; and the Sum of the Series to $2a$, is ab; becauſe $b=2a+1$ the Sum of Extremes, and a is the Half of $2a$, the Number of Terms: Again, If $b=2a-1$, then alſo are b, $2a$, two adjacent Terms in the natural Series, and the Sum to b is ab; for $2a$ is the Sum of the Extremes, whoſe Half, a, multiplied into the Number of Terms, b, gives the Sum: What remains to be ſhewn is, that the Squares of each Couplet (as 4 : 9, or 25 : 49) are ſuch Numbers as a, b are here ſuppoſed to be; which is made out thus:

(2.) If any two Numbers, a, b, (which now repreſent any of the Couplets of the firſt Line) are ſuch, that $2aa+1$ is $=bb$; then let a, b, again repreſent the next Couplet; and here it will be $2aa-1=bb$: Or, if it was $2aa-1=bb$ in the former, it will be $2aa+1=bb$ in this. Which I thus demonſtrate: The Terms of one Couplet being called a, b; the Terms of the next, made out of the former according to the Propoſition, are $a+b$, and $2a+b$; and I ſay, that $2\times\overline{a+b}^2-1$, or $2\times\overline{a+b}^2+1$ is $=\overline{2a+b}^2$; according as $2aa+1$, or $2aa-1$ is $=bb$ in the former Couplet: For $2\times\overline{a+b}^2-1=2aa+4ab+2bb-1=6aa+4ab+1$ (by ſubſtituting $2aa+1$ for bb) alſo $\overline{2a+b}^2=4aa+4ab+bb=6aa+4ab+1$ (by the ſame Subſtitution.) Hence it is plain, that $2\times\overline{a+b}^2-1=\overline{2a+b}^2$. Again, $2\times\overline{a+b}^2+1=2aa+4ab+2bb+1$ $=6aa+4ab-1$ (by ſubſtituting $2aa-1$ for bb) and $\overline{2a+b}^2=4aa+4ab+bb$ $=6aa+4ab-1$ (by the ſame Subſtitution) whence $2\times\overline{a+b}^2+1=\overline{2a+b}^2$.

(3.) The 1ſt Couplet, 2, 3, is ſuch, that putting $2=a$ and $3=b$, then is $2aa+1$ $=bb$: And therefore in the next it is $2aa-1=bb$; and ſo on alternately, by what's ſhewn in the laſt Step.

Laſtly, From all theſe Premiſes it follows, That the Product of the Squares of each Couplet is a Triangle; for by the firſt Step it is ſhewn, That if $2a+1$, or $2a-1$ is

is $=b$, then is ab a Triangle. Or, which is the same Thing, substitute aa, bb, for a, b; and since it is every where $2aa+1$, or $2aa-1=bb$ (by the 2d and 3d Steps) therefore $aa\times bb$ is a Triangle (by the 1st Step.)

Schol. As the Truth contained in this Theorem is a plain and direct Solution of this Problem, *viz. To find a Square Number, which is also the Sum of a certain Number of Terms of the natural Series* 1 . 2 . 3 . &c. So we have here also learned the Solutions of the following Problems.

Coroll. 1. (*Problem*) To find a Number of Terms, to which if the natural Series is carried, the Sum is a Square Number.

Rule. Take any of the Couplets of Squares of the preceding Scheme, as 4 : 9, or 25 : 49. Double the lesser Square, and if this Double is less than the other Square, it is the Number sought; but if it's greater, then the other Square is the Number sought. Thus, 8 $(=2\times 4)$ is such a Number as is required, because 8 is less than 9: Again, 49 is such a Number, because 50 $(=2\times 25)$ is greater than 49; and so on through all these Couplets, double of the lesser Square, and the greater Square, are alternately Solutions of this Problem.

The Reason of this *Rule* is plainly contained in the Demonstration of the preceding Theorem; for it's shewn that the Product of any of these Couplets of Squares, as $a^2\times b^2$, is a Triangle, or the Sum of a certain Number of Terms of the natural Series; which was deduced from the Supposition that $b^2=2\times a^2+1$, or $2\times a^2-1$; whereby it's plain, that b^2 and $2a^2$ differ by 1, and consequently stand next together in the natural Series, $2a^2$ being less than b^2 in the first Case, but greater in the other; whence the Sum of the Series to $2a^2$ Terms in the one Case, and to b^2 Terms in the other, is, by the Rules of Progression, $a^2\times b^2$.

Coroll. 2. (*Probl.*) To find two Squares (or two Numbers whose Squares are) such, that the greater Square, and the double of the lesser, differ by 1.

Rule. The Solution and Reason of this Problem is plainly contained in the Theorem; for the several Couplets of Squares (or their Roots) whose Products are both Squares and Triangles, solve this Problem, because it's shewn, that $a^2\times b^2$ is a Triangle, for this very Reason, that $2aa+1$, or $2aa-1$ is $=b^2$.

Coroll. 3. (*Probl.*) To find a Number, which added to its Square, the half Sum is also a Square.

Rule. Any Number which solves the Problem in *Cor.* 1. solves this also; for there it is shewn, that if $a^2\times b^2$ is a Triangle, it's the Sum of $2a^2$ Terms of the natural Series, supposing $b^2=2\times a^2+1$; or the Sum of b^2 Terms, if $b^2=2a^2-1$; but the Sum of the natural Series carried to any Number, as n Terms, is $\frac{nn+n}{2}$, which in the present Case is also a Square.

§ 3. *Of Prismatick Numbers* (see the Tables after the 9th *Defin.* § 1.)

Theorem XII.

THE Collaterals in any Place, of any the same Degree and Order of Prismaticks, are in Arithmetical Progression.

Demon. 1°. It's so in the Collaterals of every Degree of the 1st Order, because they are, by the Construction, Products of the Collaterals of the same Place of the 1st Order of Polygonals, multiplied by such a Power of the Side or Place whose Index is the Degree of the Prismaticks. But the Collateral Polygons are in Arithmetical Progression; and any such Progression being equally multiplied, the Products are also $\div l$.

2°. Since

2°. Since the Collaterals of the 1ft Order are $\div l$, it follows from *Lemma* 1. that it is fo in all the Orders of Sums proceeding from each of thefe Degrees.

Theorem XIII.

The Prifmatick in any Place of the 1ft, or Triangular Species, and 1ft Order of any Degree, is equal to the half Sum of thefe two Powers of the given Side or Place, whofe Indexes are the given Degree $+1$, and $+2$; thus in the 8th Place of the 3d Degree, it is the $\frac{1}{2}$ Sum of the 4th and 5th Powers of 8.

Degrees	1	3	6	10	*Triangles*
1ft	1	6	18	40	*Triangular*
2d	1	12	54	160	*Prifmaticks*
3d	1	24	162	640	*of the 1ft Or-*
4th	1	48	486	2560	*der.*

Demon. Let n be the Side or Place of any Triangle, and the Triangle it felf is $\frac{nn+n}{2}$; alfo by the Conftruction of the Prifmaticks, that in the n Place of the 1ft Degree, and 1ft Order is $\frac{nn+n}{2}\times n=\frac{n^3+n^2}{2}$. In the n Place, 2d Degree and 1ft Order, it is $\frac{n^3+n^2}{2}\times n=\frac{n^4+n^3}{2}$; and fo on. *Univerfally*, in the n Place of the 1ft Order and m Degree it is $\frac{n^{m+2}+n^{m+1}}{2}$. Of which take Examples in the annex'd Scheme.

Theorem XIV.

Take the Series of Prifmaticks of the 1ft Order of any Species, and any Degree, to any Number of Terms; alfo, take the Series of Triangulars of any Order [numbering from the Series of Units] to the fame Number of Terms; place this Series under the other, in a reverfe Order, and multiply the correfponding Terms together, the Sum of the Products is the Prifmatick of the given Species and Degree, which ftands in the Place expreffed by the given Number of Terms, and of the Order expreffed by 1 more than the given Order of Triangulars; or it is the Sum of the Series of Prifmaticks of the given Species, Degree and Number of Terms, and of the fame Order as that of the Triangulars.

$$\begin{array}{l} 1 : 12 : 54 \\ 6 : 3 : 1 \\ \hline 6+36+54=96 \end{array}$$

Example. The Prifmaticks of the 1ft Order, 2d Degree, 1ft Species, to 3 Terms, are 1, 12, 54; the Triangulars of the 3d Order (from Units, which is the Order of fimple Triangles) are 1 . 3 . 6; and thefe multiplied reverfely into the other, produce 96, the 3d Term of the Prifmaticks of the 1ft Species, 2d Degree and 4th Order, as you'll find by carrying on the Sums; for thefe of the 1ft Order being 1 . 12 . 54, of the 2d Order they are 1 : 13 : 67; of the 3d, 1 . 14 . 81; and of the 4th, 1 . 15 . 96.

Demon. The Reafon of this is the fame, as what has been explained in *Scholium* to *Probl.* 1. for finding the Polygonal in any Place of any Species, and any Order after the 1ft; as you'll eafily perceive by fuppofing the Expreffions there affumed, *viz.* 1 . a . b . c . d, &c. to reprefent the Series of Prifmaticks of the 1ft Order of any Species and Degree; for fince the different Orders are the continual Sums of the preceding in Prifmaticks, the fame way as in Polygonals, the Conclufion muft be the fame too.

THEOREM XV.

The Difference in any Column of Collateral Prifmaticks of the 1ft Order, and of any Degree, is equal to the Product of the precedent Triangle by fuch a Power of the Side of the given Collaterals, whofe Index is the given Degree of the Prifmaticks. Or alfo, it is equal to the half Difference of thefe two Powers of the Number expreffing the given Place, whofe Indexes are the given Degree more 1 and more 2.

Exam. The common Difference in the Collateral Prifmaticks in the 8th Place, 1ft Order and 4th Degree, is equal either, 1°. To the Product of the 7th Triangle multiplied by the 4th Power of 8. Or, 2°. To the half Difference of the 5th and 6th Powers of the Number 8.

DEMON. 1°. The Difference in any Column of Collateral Polygons, is the precedent Triangle (*Theor.* II.) and the correfponding Collateral Prifmaticks of the 1ft Order, and of any Degree, are the Products of thefe Collateral Polygons, by fuch a Power of the Side or Place, whofe Index is the Degree of the Prifmaticks (by the Conftruction of Prifmaticks.) Hence the Difference in thefe laft Collaterals muft be the Product of the Difference in the former (*viz.* of the precedent Triangle) by the fame Multiplier.

2°. For the 2d Part, fince by the 13th *Theorem*, the Prifmatick in the n Place of the 1ft Species and 1ft Order of the m Degree is $\frac{n^{m+2}+n^{m+1}}{2}$, and that in the n Place of the 2d Species is n^{m+2}, fubftracting the former from this, the Difference is $n^{m+2}-\frac{n^{m+2}+n^{m+1}}{2}=\frac{2n^{m+2}-n^{m+2}-n^{m+1}}{2}=\frac{n^{m+2}-n^{m+1}}{2}$.

COROLL. The Prifmatick in any Place of the 1ft Species, and 1ft Order, of any Degree, is an Arithmetical Mean betwixt thefe in the fame Place of the 2d Species and 1ft Order, of the fame and the preceding Degrees; for as that in the given Degree is n^{m+2}, fo that in the preceding, or $m-1$ Degree, is $n^{m-1+2}=n^{m+1}$, and the half Difference of thefe is the Difference betwixt any one of them and their Arithmetical Mean.

THEOREM XVI.

Take any Term of the Triangulars of any Order [numbered from the fimple Triangles 1 . 3 . 6 . *&c.*] and the Collateral Terms of the fame Order, of the 1ft, or Triangular Species, of as many Degrees as you pleafe, from the 1ft fucceffively of Prifmaticks; place thefe orderly in a Series; and under them fet the Series of the Coefficients belonging to that Power of a Binomial Root, whofe Index is the laft Degree taken of the Prifmaticks; multiply the correfponding Terms of thefe two Series together; the Sum of the Products is the common Difference in the Column of Collateral Prifmaticks of the next higher Place of the fame Order, and laft Degree taken in the Prifmaticks.

Exam. The Triangular in the 3d Place and 2d Order is 10 (*viz.* 1+3+6) Alfo the Prifmaticks in the 3d Place, 2d Order, 1ft Species, and of the 1ft, 2d, and 3d Degrees, are 25, 67, 187. Again, the Coefficients of the 3d Power, are 1 . 3 . 3 . 1; which multiplied into the former produce 473, the common Difference in the Collateral Prifmaticks of the 4th Place, 2d Order and 3d Degree; as you'll prove by carrying on the Tables of Prifmaticks to the 3d Degree and 2d Order.

$$\begin{array}{cccc} 10\ . & 25\ . & 67\ . & 187 \\ 1\ . & 3\ . & 3\ . & 1 \\ \hline 10+ & 75+ & 201+ & 187=473 \end{array}$$

DEMON.

Demon. 1°. The common Difference in any Column, as that in the n Place of Collateral Priſmaticks, 1ſt Degree and 1ſt Order, is equal to the Product of the preceding Triangle multiplied by (n) the Side of the Collateral Priſmaticks (by *Theor.* XV.) But that precedent Triangle multiplied by $n-1$ produces the precedent, or $n-1$ Priſmatick, 1ſt Species, 1ſt Order, and 1ſt Degree; alſo if that precedent Triangle, a, is added to it's Product by $n-1$ (*i. e.* to the precedent Priſmatick 1ſt Species) the Sum is the Product of that precedent Triangle by n; or, it is the Difference in theſe Collateral Priſmaticks.

2°. Let the Triangle in the n Place be called a, and the ſeveral Priſmaticks in the n Place of the 1ſt Species, 1ſt Order, of the ſeveral Degrees from the 1ſt, be b, c, d, &c. then by the Conſtruction it will be $b=na$. $c=nb$. $d=nc$, &c. Now the Difference in the $n+1$ Column of Collateral Priſmaticks, 1ſt Order, 1ſt Degree, being $a+b$ (by the 1ſt Article) the Difference in the $n+1$ Column of the 2d Degree, is the Product of the laſt Difference $a+b$, by the Side $n+1$ [becauſe the Terms of this Column are Products of $n+1$ by the Terms of that Column in the 1ſt Degree, whoſe common Difference is $a+b$] *i. e.* it is $\overline{a+b}\times n+\overline{a+b}=na+nb+a+b$; but $na=b$, and $nb=c$; therefore it is $=a+2b+c$. Again, the Difference in the $n+1$ Place of the 3d Degree is the Product of the laſt Difference by $n+1$, which is $\overline{a+2b+c}\times n+\overline{a+2b+c}=an+2nb+nc+a+2b+c=a+3b+3c+d$, and ſo on, as you ſee ordered in the annex'd Scheme; the Manner of continuing which, ſhews plainly the Truth of the Theorem; for the Numbers multiplying a, b, c, d, &c. in the ſeveral Differences, are evidently the Coefficients of the Powers, whoſe Indexes expreſs the Degrees of the Priſmaticks; ſo in the 1ſt, a, b, are multiplied by 1, 1, the Coefficients of the Root or 1ſt Power of a Binomial. In the 2d Degree, a, b, c, are multiplied by 1, 2, 1, the Coefficients of the 2d Power. In the 3d Degree, a, b, c, d are multiplied by 1, 3, 3, 1, the Coefficients of the 3d Power; and ſo it's plain they muſt go on for ever, by the Order of Conſtruction.

a	a	a	a
$b+na$	$2b+na$	$3b+na$	$4b$
. nb	$c+2nb$	$3c+3nb$	$6c$ &c.
.	. nc	$d+3nc$	$4d$
.	.	. nd	e
1ſt	2d	3d	4th

3°. That the ſame Thing muſt be true in the 2d and all the following Orders of any Degree of Priſmaticks, is evident from the Conſtruction of theſe Orders, *viz.* from their being the Sums of the preceding; with this Conſideration, that the Difference in the Collaterals of any Order is the Sum of the Differences in all the Collateral Columns of the preceding Order from the correſponding one backwards.

Schol. Beſides the Method of finding the Difference in any Column of Collaterals, contained in this Theorem, there is another Method deducible from *Theorem* XIV, which is this: Inſtead of the Series of Priſmaticks of any Degree, 1ſt Species and 1ſt Order, take the Series of Differences in the ſeveral Columns of the 1ſt Order of any Degree, and multiply them by the Series of Triangulars mentioned in that Theorem, and in the Manner there explained; the Sum of the Products is the Difference ſought: The Reaſon of which is the ſame as that for finding the Priſmatick of that Degree, Order and Place; becauſe the Difference in any Column is the Sum of all the Differences of the Columns of the preceding Order from the correſponding Place backwards, and therefore have the ſame Connection and Dependence as the Priſmatick Numbers themſelves; ſo that the ſame Demonſtration may be apply'd to this Caſe, only the firſt Difference being o, we are to keep out the 1, which is the 1ſt Term in that Demonſtration,

and make a, b, c, &c. represent the several Differences in the preceding Order; and then the Demonstration will be in all respects the same.

COROLLARIES.

1. The Difference in any Column of Collateral Prisms (or Prismaticks, 1st Degree, 1st Order) is the Product of its Side into the preceding Triangle; for that Side being n, and the preceding Triangle a, the preceding Prism is $\overline{n-1}\times a$, and the Difference in the Collateral Prisms of the n Place, is, by this Theorem, $a+\overline{n-1}\times a=a+na-a=na$. Whence again,

2. If to the Triangular Prism in any Place be added the Product of these 3 Numbers, *viz.* the preceding Triangle, the Side of the given Prism, and the Distance of any of its Collaterals, the Sum is the Collateral Prism at that Distance.

THEOREM XVII.

If to any Triangular Prism be added double of its Collateral Triangle, the Sum is equal to 3 times the Collateral Triangular Pyramid.

Exam. The 4th Triangular Prism is 40; the 4th Triangle is 10, whose Double is 20; then $40+20=60=3\times20$, and 20 is the 4th Triangular Pyramid. Take other Examples out of the annex'd Table.

Triangles	1	3	6	10	15
Pyramids	1	4	10	20	35
Prisms	1	6	18	40	75

DEMON. The Triangle in the n Place is $\frac{nn+n}{2}$ (for it is the Sum of n Terms of the natural Progression,) and the correspondent Prism is $\frac{nnn+nn}{2}$, to which add double the Triangle, *viz.* $nn+n$, the Sum is $\frac{nnn+nn}{2}+n^2+n.=\frac{nnn+nn+2nn+2n}{2}=\frac{nnn+3nn+2n}{2}$. Again, the Triangular Pyramid in the n Place is, by Problem 1st, $1\times\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}=\frac{nnn+3nn+2n}{6}$, Triple of which is equal to $\frac{nnn+3nn+2n}{2}$.

COROLL. Hence we have a particular Rule for finding the Sum of the Series of Triangles, *i. e.* any Term of the Series of Triangular Pyramids, which is this; Take the third Part of the Sum of the corresponding Triangular Prism, and double the Triangle, it is the Sum sought.

THEOREM XVIII.

If to any Triangular Prism be added its corresponding Triangle, the Sum is equal to the Sum of the Collateral Triangular Pyramid, and Square Pyramid.

Exam. 40 is the 4th Triangular Prism, to which add 10, the 4th Triangle; the Sum is $50=30+40$, the 4th Square and Triangular Pyramid.

Triangles	1	3	6	10	
Squares	1	4	9	16	
Tr. Pyram.	1	4	10	20	&c.
Sq. Pyram.	1	5	14	30	
Tr. Prisms	1	6	18	40	

DEMON. The n Triangular Prism is $\frac{n^3+n^2}{2}$, to which add $\frac{n^2+n}{2}$, the n Triangle; the Sum is $\frac{n^3+2n^2+n}{2}$. Again, the n Triangular Pyramid is $1\times\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}=\frac{n^3+3n^2+2n}{6}$, and the n Square Pyramid is

2n³

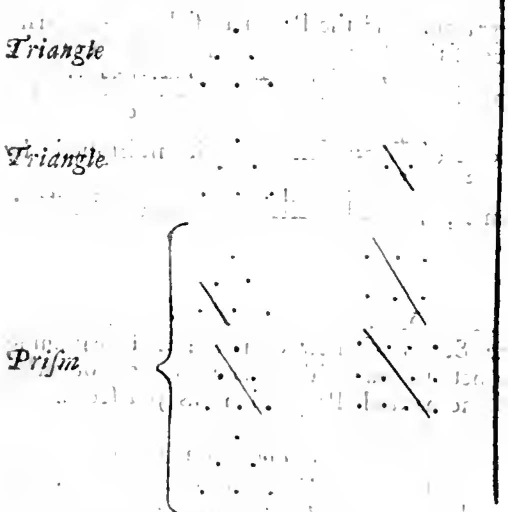
$\frac{2n^3+3n^2+n}{6}$ (as you see in *Schol. Probl.* II.) the Sum of these two is $\frac{3n^3+6n^2+3n}{6}$ $=\frac{n^3+2n^2+n}{2}$, by dividing Numerator and Denominator both by 3.

Coroll. Hence may be deduced a particular Rule for finding the Sum of the Series of Squares 1, 4, 9, &c. *i. e.* for finding any Square Pyramid (or Polygonal of the 2d Species and 2d Order from the Squares.) *Thus*, find the corresponding Triangular Pyramid (by the particular Rule explained in the *Coroll.* of last *Theorem*) and substract this from the Sum of the corresponding Triangle and its Prism; the Difference is the Number sought.

Scholium.

As the particular Rules contained in the Corollaries to this and the preceding *Theorem*, for finding the Sum of Triangles and Squares, depend immediately upon these Theorems, so they suppose the Truth of some other Rules for finding the same Numbers; which Rules are used in the Demonstration of these Theorems; and therefore to have these Rules demonstrated independently of other Rules for the same Problems, these Theorems must be demonstrated another way; but what I design here is only to observe, That among the antient Writers there is no such Thing as a general Rule for all the Orders and Species of Polygonal Numbers; they have only these two particular Rules for Triangles and Squares, and these they deduce from the same two Theorems, which they demonstrate from the Contemplation of the Schemes or Figures into which the Numbers are disposed; as they have been already explained in the Definitions, and which I shall here represent in the Manner they are formed, to make out these Demonstrations.

For Theorem XVII.



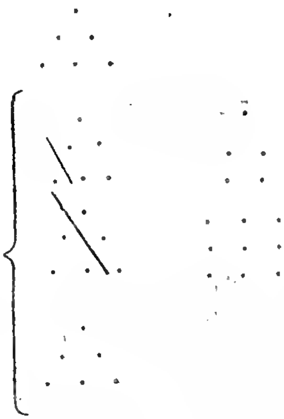
In the 1st Part of this Scheme you have the Triangle 6 taken twice, with its Prism 18; the lower Triangle in its Prism, with the Parts cut off from the upper, lying towards the left Hand, do plainly make the Collateral Triangular Pyramids; and the remaining Units on the right, with the two upper Triangles added, make two Pyramids, each equal to the former; as the 2d Part of the Scheme shews, in which the same Lines of Points are only disposed in another Form.

For

For Theorem XVIII.

Triangle

Prism



In the 1st Part of this Scheme you have the Triangle 6, and its Prism 18; the lower Triangle of which, with the Points cut from the left Hand of the two upper, make the Collateral Pyramid; and the remaining Points, with the Triangle added, make the Collateral Square Pyramid, as the 2d Part of the Scheme shews.

THEOREM XIX.

The Series of Triangular Prisms is the same as the Series of Pentangular Pyramids, as in the annex'd Scheme.

Series $\div l$	1	2	3	4
Triangles	1	3	6	10
Prisms	1	6	18	40

Series $\div l$	1	4	7	10
Pentagons	1	5	12	22
Pyramids	1	6	18	40

DEMON. That those two Series will continue to be the same for ever, I thus shew: The Triangular Prism in the n Place is $\frac{n^3+n^2}{2}$. Again, the Pentagonal Pyramid in the n Place, is the Sum of the n Triangular Pyramid, and the Product of the $n-1$ Triangular Pyramid multiplied by 2, the Distance of the Pentagon from the Triangle (*Cor.* 1. *Theor.* II.) But the n Triangular Pyramid is $1\times\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}=\frac{n^3+3n^2+2n}{6}$ and the $n-1$ Triangular Pyramid is $1\times\frac{n-1}{1}\times\frac{n}{2}\times\frac{n+1}{3}=\frac{n^3-n}{6}$, which multiplied by 2 produces $\frac{2n^3-2n}{6}$, to which add the former, *viz.* $\frac{n^3+3n^2+2n}{6}$, the Sum is $\frac{3n^3+3n^2}{6}$ $=\frac{n^3+n^2}{2}$.

THEOREM XX.

Take the Series of Triangles; then the Series of their Squares; and beginning with 1, take the Series of the Differences betwixt each Term and the next of these Squares; they are the Series of Cubes of the natural Progression, as you see in the annex'd Scheme.

Natural Progession	1	2	3	4
Triangles	1	3	6	10 &c.
Their Squares	1	9	36	100
Their Differences	1	8	27	64

which are the Cubes of the 1st Series.

DEMON. The Sums of the 1st Series taken to $n-1$, and n Terms, are $\frac{n^2-n}{2}$ and $\frac{n^2+n}{2}$; and the Squares of these are $\frac{n^4-2n^3+n^2}{4}$, and $\frac{n^4+2n^3+n^2}{4}$; whose

whoſe Difference is plainly $\frac{4n^3}{4}=n^3$. Now, ſince $n-1$, n, may be any two adjacent Terms of the natural Progreſſion, the univerſal Truth of what is propoſed is demonſtrated.

COROLLARIES.

1. As every Cube Number is the Difference of ſome two Squares, (*viz.* whoſe Roots are the Sum of a Number of Terms of the natural Series, equal to the Root of the given Cube, and the next leſſer Sum) ſo were it required to find two Squares, whoſe Difference ſhall be ſome Cube Number, which is neither known nor aſſumed, we have here a plain Rule for it; and obſerve, that I ſuppoſe the Difference neither known nor aſſumed, to make this Problem different from another, which you'll find afterwards, wherein two Squares are found, whoſe Difference is any given Number.

2. The Sum of the Cubes of the natural Series 1 . 2 . 3 . *&c.* to any Number of Terms, is equal to the Square of the Sum of the ſame Series taken to the ſame Number of Terms. Or thus, take the Cubes of the natural Series continually from the beginning, the Sums are all ſquare Numbers, whoſe Roots are the Sums of the ſame natural Series taken ſo far.

Schol. As the laſt Corollary follows plainly from the Theorem, ſo if that Corollary is demonſtrated another way, the Theorem will as clearly follow from it; and as to the Demonſtration of the Corollary independently of this Theorem, I have found one, which, though not ſo ſimple as the preceding Demonſtration of the Theorem, yet is curious enough, and therefore worth the explaining here.

The Thing then to be demonſtrated is this, *viz.* that $1^3+2^3+3^3+4^3$, *&c.* $=\overline{1+2+3+4}^2$, *&c.*

1*ſt.* From the Nature of Multiplication it's plain, that the Square of any Multinomial Root (*i. e.* a Root conſiſting of many Parts) is equal to the Sum of the Squares of each Part of the Root, and twice the Product of every Pair of Members of the Root; as in the annex'd Scheme, wherein $a+b+c+d$, &c. being ſquared, the Square is the Sum of all the Squares and Products ſet under it; and by the Order in which they are diſpoſed, you ſee the Square is compoſed of as many Members as the Root, each of which is equal to the Square of one Member of the Root, and the Sum of the double Products of that into all the preceding Members (on the left Hand) which evidently comprehends the double Products of every Pair of Members of the Root.

$$\begin{array}{rrrrl}
a & +b & +c & +d, & \text{\&c.} \\
a & +b & +c & +d, & \text{\&c.} \\
\hline
a^2 & +b^2 & +c^2 & +d^2 & \\
 & 2ab & +2bc & +2cd & \\
 & & 2ac & +2bd & \\
 & & & 2ad &
\end{array}$$

2*d.* If the Parts of the Multinomial Root are the ſeveral Terms of the natural Series 1 . 2 . 3 . *&c.* then ſuppoſe the Square of it is taken in the Manner of the preceding Scheme, I ſay that each Member of it (*i e.* the Sum of the Square of each Term of the Root, and the double Products of that Term into all the preceding Terms) is equal to the Cube of that Term. For *Exam.* The Cube of 4 is 64, equal to the Square of 4 (*viz.* 16) more $2\times3\times4+2\times2\times4+2\times1\times4$ $(=24+16+8=48.)$ And that it is ſo univerſally I thus prove, (1°.) It's plain, that to multiply any Term of the Root into each preceding Term ſeverally, the Sum of the Products is equal to the Product of that Term into the Sum of all the preceding Terms; and becauſe we muſt double all theſe Products, therefore we muſt take double the Sum of the preceding Terms, and multiply it into the given Term. Then, (2°.) This Product is equal to the Square of the given Term multiplied

$$\begin{array}{rrrrl}
1 & +2 & +3 & +4, & \text{\&c.} \\
1 & +2 & +3 & +4, & \text{\&c.} \\
\hline
1 & +4 & +9 & +16 & \\
 & 4 & +12 & +24 & \\
 & & 6 & +16 & \\
 & & & 8 & \\
\hline
1 & +8 & +27 & +64 &
\end{array}$$

into

into the next preceding Term, by the Rules of Progressions; for *Example*, $6\times 2\times \overline{1+2+3+4+5} = 6\times 6\times 5$; because $\overline{1+2+3+4+5} = \frac{6\times 5}{2}$; and this multiplied by 2, produces 6×5. *Universally*, Call the Term next preceding any given one, l, that given one must be $1+l$; which is also the Sum of the Extremes of that Series, whereof l is the greatest Extreme, *i. e.* of the Series to be summed; but this Sum is $\overline{1+l}\times\frac{l}{2}$ (l being here the Number of Terms) and the Double of this is $\overline{1+l}\times l$, which multiplied into the given Term, $1+l$, produces $\overline{1+l}^2\times l$. But, (3°.) It's manifest that $\overline{1+l}^2\times l$ (the Sum of twice the Product of $1+l$, the given Term, into all the preceding Terms) added to $\overline{1+l}^2$ (the Square of the given Term) makes $\overline{1+l}^2\times l+\overline{1+l}^2=\overline{1+l}^2\times\overline{1+l}=\overline{1+l}^3$, the Thing to be shewn.

3*d*. Since the Square of the Multinomial, $1+2+3+4$, &c. is resolveable into as many Members as the Root, each of which is proved to be equal to the Cube of a different Member of the Root; therefore that Square is equal to the Sum of the Cubes of the several Members of the Root, *i. e.* $1^3+2^3+3^3$ &c. $=\overline{1+2+3}^2$ &c.

Theorem XXI.

Take the Series of square Pyramids (*i. e.* the Sums of the natural Series of Squares) multiply each of them by 3; and from the Series of Products, take the Series of the Sums of the 2 Series of Triangles and Squares, as in the Margin; the Series of the Differences is the Series of Cubes of the natural Progression.

1	2	3	4	5	*Natural Series*
1	3	6	10	15	*Triangles*
1	4	9	16	25	*Squares*
1	5	14	30	55	*Square Pyramids*
3	15	42	90	165	*Their Product by* 3
2	7	15	26	40	*Sum of Trian. and Squ.*
1	8	27	64	125	*Cubes*

Demon. The n Square Pyramid is $\frac{2n^3+3n^2+n}{6}$ (by *Probl.* II. *Schol.*) which multiplied by 3 produces $\frac{2n^3+3n^2+n}{2}$. The n Triangle is $\frac{nn+n}{2}$, and n Square is nn: Then $n^2+\frac{nn+n}{2}=\frac{3n^2+n}{2}$; which taken from the former, leaves plainly $\frac{2n^3}{2}=n^3$. And because n may be any Term, therefore the *Theorem* is true.

Theorem XXII.

Take the Series of Heptangular Pyramids, and to them add the Series of Triangular Pyramids in this Manner, *viz.* The 1st of the Triangulars to the 3d of the other; the 2d Triangular to the 4th of the other; and so on; *i. e.* universally, the n Heptangular Pyramid added to the $n-2$ Triangular. All these Sums are Cubes; and prefixing the first two Heptangular Pyramids, 1 . 8 (which are Cubes) you have the whole Series of Cubes.

Demon.

Series $\div l$	1 . 2 . 3 . 4 . 5
Triangles	1 . 3 . 6 . 10 . 15
Series $\div l$	1 . 6 . 11 . 16 . 21
Heptag.	1 . 7 . 18 . 34 . 55
Hept. Pyramids	1 . 8 . 26 . 60 . 115
Triang. Pyramids	1 . 4 . 10
Cubes	1 . 8 . 27 . 64 . 125

Demon. The n Triangular Pyramid is $\frac{n^3+3n^2+2n}{6}$, and the $\overline{n-1}$ Term is $\frac{n^3-n}{6}$ (*Probl.* II. *Schol.*) But the Distance of the Heptagon from the Triangle, is 4; therefore (by *Probl.* II.) the n Heptangular Pyramid is $\frac{n^3+3n^2+2n}{6}+\frac{n^3-n}{6}\times 4=\frac{n^3+3n^2+2n+4n^3-4n}{6}=\frac{5n^3+3n^2-2n}{6}$. Again the $n-2$ Triangular Pyramaid is $1\times\frac{n-2}{1}\times\frac{n-1}{2}\times\frac{n}{3}=\frac{n^3-3n^2+2n}{6}$; to which add the n Heptangular Pyramid, *viz.* $\frac{5n^3+3n^2-2n}{6}$; the Sum is $\frac{6n^3}{6}=n^3$.

Theorem XXIII.

Take the Series of Octangular Pyramids, and from them substract the Series of Triangles; thus, The 1st Triangle from the 2d Octangular Pyramid, and so on, *i. e.* universally, the $n-1$ Triangle from the n Pyramid; the Series of Differences is the Series of Cubes after 1.

Series $\div l$	1 . 7 . 13 . 19
Octogons	1 . 8 . 21 . 40
Octog. Pyramids	1 . 9 . 30 . 70
Triangles	1 . 3 . 6
Cubes	1 . 8 . 27 . 64

Demon. The n Triangular Pyramid is $\frac{n^3+3n^2+2n}{6}$, and the $\overline{n-1}$ Triangular Pyramid is $\frac{n^3-n}{6}$: Therefore the n Octangular Pyramid is (by *Probl.* II.) $\frac{n^3+3n^2+2n}{6}+\frac{n^3-n}{6}\times 5=\frac{n^3+3n^2+2n+5n^3-5n}{6}=\frac{6n^3+3n^2-3n}{6}$; from which, substract the $n-1$ Triangle, which is $\frac{n^2-n}{2}$; the Difference is $\frac{6n^3+3n^2-3n}{6}-\frac{3n^2-3n}{6}=\frac{6n^3}{6}=n^3$.

0 . 6 . 12 . 18
0 . 6 . 18 . 36
0 . 6 . 24 . 60
1 . 2 . 3 . 4
1 . 8 . 27 . 64

Coroll. From this it is plain, That if we take a Series, $\div l$, beginning with 0, and proceeding by the Difference 6; then take the Sums of this Series; and then the Sums of these Sums; and to this last Series of Sums add the natural Series, 1 . 2 . 3 . &c. the last Sums make the Series of Cubes. The Deduction of the universal Truth of which from the present *Theorem*, is easily made; thus, In the Scheme of the *Theorem*, each Term of the 1st Series is 1 more than its Collateral in the 1st Series of the present Scheme; therefore each Term of the 2d Series of the former Schemes, exceeds the Collateral of the 2d Series of this Scheme, by as many Units as the preceding Number of Terms. Hence again; Each Term of the 3d Series, former Scheme, exceeds the Collateral in the 3d Series of this Scheme, by the Sum of as many Terms of the natural Series, as the Number of preceding Terms: Whence all the rest is plain.

From

0	.	6	.	12	.	18
1	.	7	.	19	.	37
1	.	8	.	27	.	64

From this laſt, *again*, it will eaſily appear, That if we take the Series, 0 . 6 . 12 . &c. and taking the Sum of it, add 1 to each Sum; and then of this Series of Sums take the Series of Sums; theſe are the Cubes of the natural Progreſſion; for by taking the Sums of the Series, 0 : 6 : 18 : 36 : &c. after 1 is added to each (*i. e.* the Sums of 1 . 7 . 19 . &c.) it's plain the Sums at every Step, will be more than the Sums taken without theſe Units (*i. e.* than the Sums of the Series, 0 . 6 . 18 . &c.) by as many Units as the Number of Terms added: So that it will be the ſame as if taking the Sums of this Series, 0 . 6 . 18 . 36, we add to the Series of the Sums the natural Series, 1 . 2 . 3; as in the preceding Scheme.

THEOREM XXIV.

Take the Series of odd Numbers, 1 . 3 . 5 . &c. and out of this make another Series; thus, Take the 1ſt Term; then the Sum of the next 2 Terms; again, the Sum of the next 3 Terms, after the laſt; and ſo on, taking in at every Step, one Term more; ſo that the Number of Terms taken at every Step, are the natural Series, 1 . 2 . 3 . &c. The Series thus compoſed, is the Series of Cubes of the natural Series; each Sum being the Cube of the Number of Terms added together.

$$1 \cdot \underbrace{3 \cdot 5} \cdot \underbrace{7 \cdot 9 \cdot 11} \cdot \underbrace{13 \cdot 15 \cdot 17 \cdot 19} \cdot \underbrace{21 \cdot 23 \cdot 25 \cdot 27 \cdot 29}$$

Cubes	1 .	3 .	27 .	64 .	125	&c.
Roots	1 .	2 .	3 .	4 .	5	

DEMON By *Theor.* XX. *Cor.* 2. the Sum of the Series of Cubes, is the Square of the Sum of the natural Series taken to the ſame Number of Terms: But the Sums of the odd Series are the Squares of the ſeveral Number of Terms ſummed (*Theor.* IX.) And in this Scheme there are always as many odd Terms from the Beginning, as the Sum of all the Roots; therefore the Sum of all theſe odd Numbers being the Square of the Sum of all the Cube Roots, is the Sum of all the Cubes. Again, That the Sum of the ſeveral Terms, diſtinguiſhed in the Series of odd Numbers, are the particular Cubes of the Number of theſe Terms, is evident; becauſe 1 is the Cube of 1, and $1+3+5$, the Sum of the Cubes of 1 and 2, by what's firſt ſhewn; therefore $3+5$ is the Cube of 2. Then ſince $1+3+5+7+9+11$ is the Sum of the Cubes of 1, 2, 3, therefore $7+9+11$ is the Cube of 3. And ſo on.

Another DEMON. This *Theorem* may alſo be demonſtrated independently of *Theor.* XX. by the Conſideration of the odd Series. Thus,

1°. If the Root of any Cube is an odd Number, its Square is alſo an odd Number; and is therefore a Term of the odd Series: Then taking as many adjacent Terms of the odd Series, as the Root of any Cube expreſſes, and whereof the Square of the ſame Root is the middle Term; it's plain the Sum of all theſe Terms will be the Cube; becauſe, being an Arithmetical Progreſſion, the Sum of every two Terms, equally diſtant from the middle one, is double that middle one, and conſequently the whole Sum is equal to as many Times the middle Term as the Number of Terms expreſſes; but that middle Term is the Square; and the Number of Terms the Root; therefore the Whole is the Product of the Square by the Root, *i. e.* the Cube.

2°. If the Root is an even Number, then we can find in the odd Series two adjacent Terms, the one exceeding the Square by 1, and the other wanting 1 of it; conſequently, taking as many Terms as the Root expreſſes, whereof theſe two now mentioned are the two middle Terms; the Sum of the Whole is the Cube; for the Sum

Sum of the two middle, and of every other two equally diſtant from them, are each double the Square; and ſo the Whole is equal to the Product of the Square by the Root, *i. e.* is the Cube.

3°. Repreſenting the two preceding Caſes in the annex'd Scheme [wherein a being any odd Number or Root, the 1ſt Series repreſents a Number of adjacent odd Numbers, whoſe Sum is a^3; and a being an even Number, the 2d Series is a Number whoſe Sum is a^3; underſtanding theſe Series to contain as many Terms as the Root has Units, whereof a^2 is the middle one in the 1ſt, and a^2-1, a^2+1, the two middle ones in the 2d Caſe.] Hence it will be evident, that the leaſt and greateſt Terms in the particular Series of adjacent odd Numbers, whoſe Sum is the Cube of

&c. $a^2-6 : a^2-4 : a^2-2 : a^2 : a^2+2 : a^2+4 : a^2+6$: &c.
&c. $a^2-5 : a^2-3 : a^2-1 : a^2+1 : a^2+3 : a^2+5$: &c.

the Number of Terms, are thus expreſſed, *viz.* The leaſt a^2+1-a, and the greateſt a^2+a-1: For the Root being odd, the Number of Terms is odd; and there are as many Terms on each Hand of the Middle, a^2, as Half the Root -1 or $\frac{a-1}{2}$; and the common Difference being 2, the Number ſubſtracted from, or added to a^2 in the Extremes, is equal to 2, taken as oft as $\frac{a-1}{2}$ expreſſes: But $\frac{a-1}{2}\times 2 = a-1$; therefore the Extremes are $a^2-\overline{a-1}$, and $a^2+\overline{a-1}$, which are a^2-a+1, and a^2+a-1. Again, If a is an even Number, the two middle Terms being always a^2-1, a^2+1, and the Number of Terms below a^2-1, and above a^2+1, being half of $a-2$, or $\frac{a-2}{2}$; it follows, that the Extremes are a^2-1, wanting the Product of 2 by $\frac{a-2}{2}$, which Product is $a-2$; and a^2+1, with the ſame Number added to it; which Extremes are therefore $a^2-1-\overline{a-2}=a^2-1-a+2=a^2-a+1$, and $a^2+1+a-2=a^2+a-1$.

4°. Take any two Roots differing by 1, as a, and $a+1$; the greateſt Extreme of the Series of odd Numbers, whoſe Sum is the Cube of a, is, by the laſt, a^2+a-1. And the leaſt Extreme of the Series, whoſe Sum is the Cube of $\overline{a+1}$, is (by ſubſtituting $a+1$ inſtead of a, in this Expreſſion, a^2-a+1) $=\overline{a+1}^2-\overline{a+1}+1=a^2+2a+1-a$ $=a^2+a+1$. Compare this with the former, *viz.* a^2+a-1, it's plain their Difference is 2, *i. e.* the greateſt Term of theſe adjacent odd Numbers whoſe Sum is a^3, and the leaſt of theſe whoſe Sum is $\overline{a+1}^3$, differ by 2; and ſo they are two adjacent odd Numbers; conſequently the Series for the one Cube begins at the Term next after that one with which the other ends; and therefore the Series of Cubes are found in the Manner preſcribed in the *Theorem*.

SCHOL. As this *Theorem* is demonſtrated by Means of *Theor.* XX. ſo that is demonſtrable by Means of this, and *Theor.* IX.

THEOREM XXV.

Take the Series of Pentagonal Pyramids, and multiply each of them by 3; then take the Series of Squares, and multiply each of them by 2: Subſtract the laſt Series of Products from the former; the Differences are the Series of Pentagonal Priſms.

Series $\div l$	1 .	4 .	7 .	10
Pentagons	1 .	5 .	12 .	22
Pent. Pyramids	1 .	6 .	18 .	40
Products by 3	3 .	18 .	54 .	120
Squares	1 .	4 .	9 .	16
Products by 2	2 .	8 .	18 .	32
Pent Prisms	1 .	10 .	36 .	88

DEMON. Pentagons proceed from the Series, 1 . 4 . 7 &c. whose common Difference is 3; of which therefore the nth Term is $1+\overline{n-1}\times 3=3n-2$; and the Sum of the Extremes is $\overline{3n-1}$. *Lastly*, The Sum of n Terms is $\overline{3n-1}\times\frac{n}{2}=\frac{3n^2-n}{2}$, which is the n Pentagon; and this multiplied by n, produces $\frac{3n^3-n^2}{2}$, the n Pentagonal Prism. Again, The nth Triangular Pyramid is $\frac{n^3+3n^2+2n}{6}$, and the $n-1$ Triangular Pyramid is $\frac{n^3-n}{6}$; the Distance of the Triangle and Pentagon is 2, and $\frac{n^3-n}{6}\times 2=\frac{2n^3-2n}{6}$; wherefore (by *Cor. Theor.* II.) the n Pentagonal Pyramid is $\frac{n^3+3n^2+2n}{6}+\frac{2n^3-2n}{6}=\frac{3n^3+3n^2}{6}=\frac{n^3+n^2}{2}$; which multiplied by 3, makes $\frac{3n^3+3n^2}{2}$; from which take $2n^2$, the Remainder is $\frac{3n^3-n^2}{2}$, the n Pentagonal Prism.

General SCHOLIUM.

Though there be no general *Canon*, that I know, for finding the Sum of any Series of Prismaticks of any Degree, Species and Order, excepting what is contained in the XIVth *Theorem*, which supposes the Series of Prismaticks of the 1st Order of the same Species and Degree, and to the same Number of Terms, as that whose Sum is sought; yet for the Sums of the Prismaticks of the 2d Species, 1st Order, of any Degree, *i. e.* for the Sums of Powers of the natural Progression, from the Cubes and upwards, we can, by Means of the preceding *Theory* of Polygonals, investigate particular *Canons* for every different Power; whereby the Sum may be found, having only the Number of Terms.

In order to which *observe*, That as the Invention of these *Canons*, for any Degree, depends upon the *Canons* for the preceding Degrees; so, though Squares are Polygons, and not Prismaticks, yet it will be necessary to take the Squares within the following *Problem*, that we may more easily, by the two first and most simple Powers, understand the Method of Investigation for all the superior Powers.

Again, *observe*, That we have already explained the particular Canons for the Sums of Squares and Cubes; that for Squares, being $\frac{2n^3+3n^2+n}{6}$, as you see in the 2d Article of the *Scholium* after *Problem* II. and that for Cubes, being $\frac{n^4+2n^3+nn}{4}$; for by *Theorem* XX. *Coroll.* 2. the Sum of the Cubes of the natural Progression, to the nth Term, is the Square of the Sum of the Roots; but this Sum is $\frac{n^2+n}{2}$, whose Square is $\frac{n^4+2n^3+n^2}{4}$. But the Method of investigating Rules for the higher Powers being different from the Method, by which these Rules for Squares and Cubes have already been invented, and depending also upon a new Method, by which the same two Rules may be investigated: Therefore we must explain this other

other Method for Squares and Cubes; and this being done, the univerſal Method for all other Powers will be eaſily underſtood.

Obſerve in the laſt Place, That there are ſeveral Methods of finding theſe Canons, ſo as to make the Invention of the Rule for any Degree, depend upon the Rules for the preceding Degrees; but I confine my ſelf to that which is moſt natural in this Place, *viz.* which depends upon the univerſal Rule for the ſeveral Orders of Polygonals of the Triangular Kind. You'll find another curious Method in *Ronayne*'s Algebra.

PROBLEM III.

How to *inveſtigate Rules* for finding the Sums of the Series of Powers of any Degree of the natural Progreſſion, 1 . 2 . 3 . &c. having the Number of Terms only given.

Solution.

1*ſt*. For the Sum of the Squares, $1+4+9+$&c. to the nth Term. The nth Triangular of the 3d Order (from Units) *i. e.* the Sum of the *Triangles*, to the nth Term, is $1\times\frac{n}{1}\times\frac{n+1}{2}=\frac{n^2+n}{2}$ (by the 2d Rule for *Probl.* I. with the *Coroll.* to *Theorem* III. Or, ſee the *Scholium* after that Rule.) Again, Take n, ſucceſſively equal to 1 . 2 . 3 . &c. and applying this Canon, we have $\frac{1^2+1}{2}$, $\frac{2^2+2}{2}$, $\frac{3^2+3}{2}$, &c. which expreſs the Series of Triangulars of the 3d Order, to any Number of Places given; which Number being called n, the Sum of this Series is the nth Triangular of the 4th Order; but the n Triangular of the 4th Order is, by the ſame general Rule, $1\times\frac{n}{1}\times\frac{n+1}{1}\times\frac{n+2}{3}=\frac{n^3+3n^2+2n}{6}$, which is therefore equal to $\frac{1^2+1}{2}+\frac{2^2+2}{2}+\frac{3^2+3}{2}$ &c. carried to n Terms; and this is $=\frac{1}{2}$ of $\underline{1^2+1+2^2+2+3^2+3}$ &c. $=\frac{1}{2}$ of $\underline{1+2^2+3^2}$ &c. $+\frac{1}{2}$ of $\underline{1+2+3}$ &c. Now $1+2+3$ &c. to n Terms, is $=\frac{nn+n}{2}$; therefore $\frac{1}{2}$ of it is $=\frac{nn+n}{4}$; conſequently $\frac{n^3+3n^2+2n}{6}=\frac{1}{2}$ of $\underline{1+2^2+3^2}$ &c. $+\frac{n^2+n}{4}$; and ſubſtracting $\frac{nn+n}{4}$ from both Sides, the Remainders are equal, *viz.* $\frac{1}{2}$ of $\underline{1+2^2+3^2}$ &c. $=\frac{n^3+3n^2+2n}{6}-\frac{n^2+n}{4}=\frac{4n^3+12n^2+8n-6n^2-6n}{24}=\frac{4n^3+6n^2+2n}{24}$; and multiplying the firſt and laſt Expreſſions, both by 2, the Products are $1+2^2+3^2$ &c. $=\frac{8n^3+12n^2+4n}{24}=\frac{2n^3+3n^2+n}{6}$; which is the Rule for the Sum of the Squares, *viz.* $1+2^2+3^2$ &c.

2*d*. For the Sum of the Cubes, or $1^3+2^3+3^3+$&c. to the nth Term. The nth Triangular of the 4th Order (from Units) *i. e.* the Sum of the Triangulars of the 3d Order to the nth Term, is $1\times\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}=\frac{n^3+3n^2+2n}{6}$; and taking n, ſucceſſively equal

 to

to 1, 2, 3, &c. the Series of Triangulars of the 4th Order are, $\frac{1^3+3\times1^2+2\times1}{6}$, $\frac{2^3+3\times2^2+2\times2}{6}$, $\frac{3^3+3\times3^2+2\times3}{6}$, &c. to any Number of Terms; which Number being called n, the Sum of this Series is the nth Triangular of the 5th Order; which, by the general Rule, is equal to $1\times\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{n+3}{4}=\frac{n^4+6n^3+11n^2+6n}{24}$; but the former Series (equal to this) is the Sum of these 3 Parts, *viz.* $\frac{1}{6}$ of $1+2^3+3^3$ &c. $+\frac{3}{6}$ of $1+2^2+3^2$ &c. $\left(=\frac{2n^3+3n^2+n}{6}\right.$, by the last) $+\frac{2}{6}$ of $1+2+3$ &c. $\left(=\frac{nn+n}{2}\right)$ and taking the last two Parts from both, we have $\frac{n^4+6n^3+11n^2+6n}{24}-\frac{3}{6}$ or $\frac{1}{2}$ of $\frac{2n^3+3n^2+n}{6}-\frac{2}{6}$ or $\frac{1}{3}$ of $\frac{n^2+n}{2}=\frac{1}{6}$ of $1+2^3+3^3$ &c. And reducing the first Part to the most simple Expression, by the common Rules of Fractions; and then multiplying both Parts by 6, there will come out $1+2^3+3^3$ &c. $=\frac{n^4+2n^3+n^2}{4}$; the Rule for Cubes.

Now for all superior Powers, the Investigation of Rules, by the same Method, will be plain and obvious to such as understand the first two Cases, how they depend upon one another, and upon the General Canon for Triangular Numbers: For we gradually express, by that Canon, the nth Triangular of the Order (from Units) expressed by 1 more than the Index of the Power (as above, we took the 3d Order for Squares, and the 4th Order for Cubes;) and taking n, gradually equal to 1, 2, 3, &c. we express the several Terms of the Triangulars of that Order, according to that Canon, to any Number of Terms, as n; and the Sum of all these Terms will resolve into as many other Series (connected by Addition, and in some Cases, some of them substracted) as there are Members in the Numerator of the General Canon, and whereof one of them will be some Multiple or *aliquot* Part of the Sum of the Series of Powers sought, to n Terms; and the other Parts are some Multiple or *aliquot* Part of the Sums of as many Terms of some of the inferior Powers (of the same natural Progression, 1 . 2 . 3 . &c.) which we must express by the Rules already invented for these inferior Powers; then we consider that the Sum of this Series, thus expressed, is the nth Term of the next Order of Triangulars; and this we express by the same General Rule as before we did the nth Term of the preceding Order: And then comparing these two different Expressions of the same Number, we find, after a due Reduction, the most simple Expression for the Sum of the Powers sought.

I shall here give you the Canon for Biquadrates, and leave the Investigation and Proof of it to your Exercise.

The *Canon* is, $1+2^4+3^4+4^4$ &c. $=\frac{6n^5+15n^4+10n^3-n}{30}$.

§. III. *Of* Oblongs, *and some remaining curious Propositions concerning* Squares.

THEOREM XXVI.

If we take the Series of even Numbers, 2 . 4 . 6 . &c. and then the Series of their Sums, 2 . 6 . 12 . &c. these are the Series of *Oblongs*.

DEMON.

	2	.	4	.	6	.	8	
		2	.	6	.	12	.	20
1	.	2	.	3	.	4	.	5

Demon. The 1st even Number is also the 1st Oblong, 2. And that all the other Sums are the other Oblongs, is thus plain. By the Nature of $\div l$ Progressions, the Sum of the even Numbers to n Terms is $nn+n=\overline{n+1}\times n$, which is therefore an Oblong; and taking n, gradually greater by 1, it's plain that $\overline{n+1}\times n$ becomes gradually the next Oblong.

Theorem XXVII.

Double the several Terms of the Series of Triangles (*i. e.* of the Sums of the natural Series 1 . 2 . 3 . *&c.*) and the Products are the Series of Oblongs.

1 . 2 . 3 . 4	
1 . 3 . 6 . 10	*Triangles*
2 . 6 . 12 . 20	*Oblongs*

Demon. Take any 2 adjacent Numbers n, $n+1$; in the natural Series; their Product is an Oblong, and it is also double the Sum of the natural Series, to n Terms; for that Sum is $\frac{n\times\overline{n+1}}{2}$; consequently taking n, gradually 1 more, we have the whole Series of Oblongs.

Theorem XXVIII.

The Arithmetical Means betwixt (or half Sums of) every two adjacent Oblongs, make the Series of natural Squares after 1, the Root of each Square being the lesser Side of the greater Oblong, and the greater Side of the lesser; and is also the common Difference betwixt the Square and each of these Oblongs.

	1 . 2 . 3 . 4 . 5 . 6
Obl.	2 . 6 . 12 . 20 . 30 *&c.*
	1 . 4 . 9 . 16 . 25

Demon. Take 3 Terms in the natural Series $n-1$. n. $n+1$. the two adjacent Oblongs produced from these, are $\overline{n-1}\times n=n^2-n$, and $n\times\overline{n+1}=n^2+n$, and their Sum is $2n^2$, whose half is n^2, the Arithmetical Mean. Now n being the middle Term of 3, assumed in the natural Series, which in the first 3, (*viz.* 1 . 2 . 3) is 2, whose Square is 4; and every successive Term of the natural Series being the middle Term of the next three, from which the two next adjacent Oblongs proceed; hence the universal Truth of the *Theorem* is clear.

COROLLARIES.

1. The Sum and Difference of any Square and its Root, are the 2 Oblongs next greater and lesser than that Square.

2. Take the Series of Oblongs, and of Squares from 4; the Differences of the corresponding Terms of these Series are the natural Progression of Numbers, from 2; thus, $4-2=2$. $9-6=3$. $16-12=4$. and so on.

Theorem XXIX.

The Series of Oblongs is the Series of Geometrical Means betwixt every two adjacent Squares, (see the preceding Scheme.) The common Ratio of that Mean, and these Extremes, being that of the Roots of these Squares, which are the Sides of the Oblong.

Demon. Betwixt any two Squares AA, and BB, the Geometrical Mean is AB (for AA : AB : : AB : BB, the Ratio being A : B) wherefore if the Roots A, B, differ by 1, the Geometrical Mean AB is an Oblong; and thus consequently we have the whole Series of Oblongs.

Coroll. Take the Series of Oblongs and of the Squares; compare the 1st Square, 1, to the 1st Oblong 2; and the 2d to the 2d, and so on. The Ratios are the continued Ratios of the natural Progression, *viz.* 1 : 2. 2 : 3. 3 : 4. *&c.*

Theorem XXX.

Take the Series of Squares and of Oblongs; out of these make another Series, thus; Take the 1st Square 1, then to this Square add the 1st Oblong, then to the 1st Oblong add the 2d Square, and so on, adding each Square to the next Oblong, and each Oblong to the next Square; I say the Sums are the Series of Triangles, each Sum being a Triangle, which is the Sum of such a Number of Terms of the natural Series, as is equal to the Sum of the Places of the Square and Oblong added, in their respective Series: Thus, 16 is the 4th Square, and 12 the 3d Oblong, and their Sum is $16+12=28$, the 7th Triangle.

Nat. Series	1	.	2	.	3	.	4	.	5
Their Squares	1	.	4	.	9	.	16	.	25
Oblongs		2	.	6	.	12	.	20	
Triangles	1	3	6	10	15	21	28	36	45

Demon. It is plain that the Places of the two Terms added in their several Series, are alternately equal, and then differing by 1, the Place of the Square being the greatest Number; for the 1st Square is added to the 1st Oblong, then the 1st Oblong to the 2d Square, and so on; therefore the Thing to be proved is, universally that the n Square $+$ n Oblong, is the Sum of $2n$ Terms of the natural Series; and that the n Oblong $+\overline{n+1}$ Square, is the Sum of $2n+1$ Terms of the natural Series: Which is shewn thus; the n Oblong is $n\times\overline{n+1}=nn+n$, to which add nn, the Sum is $2nn+n$; and the Sum of the $2n$ Terms of the natural Series is $\overline{2n+1}\times\frac{2n}{2}=\overline{2n+1}\times n=2nn+n$; which is the first Thing. Again, the n Oblong being $nn+n$, and the $\overline{n+1}$ Square being $nn+2n+1$, their Sum is $2nn+3n+1$; and the Sum of $2n+1$ Terms of the natural Series is $\overline{2n+2}\times\frac{2n+1}{2}=\overline{2n+1}\times\overline{n+1}=2n^2+3n+1$; which is the second Thing.

Or the Demonstration may be made thus; each Square is the Sum of as many Terms of the odd Series, as the Root expresses (*Theor.* IX.) and the Oblong in the next lower Place is the Sum of all the intermediate even Numbers; also the Oblong of the same Place is the Sum of all the intermediate even Numbers, and the next greater even Number (*Theor.* XXVI.) whence the *Theorem* is evident. See the following Scheme, wherein $9=1+3+5$, and $6=2+4$, therefore $9+6=1+2+3+4+5$; also $16=1+3+5+7$, and $20=2+4+6+8$, therefore $16+20=1+2+3+4+5+6+7$; and so of others.

Odd Series	1	.	3	.	5	.	7	.	9
Even		2	.	4	.	6	.	8	
Squares	1	.	4	.	9	.	16	.	
Oblongs		2	.	6	.	12	.	20	
Triangles	1	3	6	10	15	21	28	36	45

Theorem XXXI.

Take the Sum of every two adjacent Oblongs, and add it to the Double of the Arithmetical Mean (or Interjacent Square) these Sums make the Series of Squares of the even Numbers after 2. Thus, $2+6+8=16$, the Square of 4; then $6+12+18=36$, the Square of 6.

Demon.

Squares	1 .	4 .	9 .	16
Oblongs		2 .	6 .	12 . 20
Squares		16 .	36 .	64
Roots		4 .	6 .	8

Demon. The Sum of every two adjacent Oblongs being double the Arithmetical Mean, or interjacent Square, which call nn; therefore double that Square added to that Sum, is 4 Times that Square or $4nn$; but 4 and nn being both Squares, their Product $4nn$ is alſo a Square, whoſe Root is $2n$, an even Number; alſo the firſt Value of n being 2, in that Caſe $2n$ is $=4$; and the following Values of n being gradually 1 more (becauſe the Roots of the Squares added are 1 . 2 . 3, &c.) therefore $2n$ is at every Step 2 more than in the preceding; and ſo they make the even Series, 4 . 6 . 8, &c.

Theorem XXXII.

Take the Sum of every two adjacent Squares, and twice the interjacent Oblong (or Geometrical Mean) the Sums make the Series of the Squares of all odd Numbers after 1; thus $1+4+4=9$, the Square of 3; alſo, $4+9+12=25$, the Square of 5.

Squares	1 .	4 .	9 .	16 .	25
Oblongs		2 .	6 .	12 .	20
Squares		9 .	25 .	49 .	81
Roots		3 .	5 .	7 .	9

Demon. Let n, $n+1$, be two adjacent Numbers in the natural Series, their Squares are nn, $nn+2n+1$, and the Sum of theſe two is $2nn+2n+1$; again, the interjacent Oblong being the Geometrical Mean, is the Product of the Roots $n\times\overline{n+1}=nn+n$, the Double of which is $2nn+2n$, which added to $2nn+2n+1$, the Sum is $4nn+4n+1$; and this is the Square of $2n+1$, an odd Number, becauſe $2n$ is even; but $2n+1$ being in the 1ſt Step 3, and n encreaſing gradually by 1, therefore $2n$ encreaſes gradually by 2, and ſo alſo muſt $2n+1$; conſequently the ſeveral Values of $2n+1$ are the Terms of the odd Series, 3, 5, 7, &c.

Theorem XXXIII.

To the Product of every two adjacent Oblongs add the interjacent Square, the Sums make the Series of Biquadrates, or 4th Powers, of the natural Series after 1; thus, $2\times6+4=16$, the 4th Power of 4; alſo $6\times12+9=81$, the 4th Power of 3.

Squares	1 .	4 .	9 .	16 .	25	
Oblongs		2 .	6 .	12 .	20 .	30
Biqua.			16 .	81 .	256 .	15000
Roots			2 .	3 .	4 .	.

Demon. $n-1$, n, $n+1$, expreſs any 3 adjacent Numbers in the natural Series, and $n\times\overline{n-1}=nn-n$ alſo $n\times\overline{n+1}=nn+n$, two adjacent Oblongs, and their Product is n^4-n^2, to which add nn, the Sum is n^4; but in the 1ſt Caſe n is $=2$, and it encreaſes gradually in all the following Steps by 1; whence the Truth propoſed is clear.

Theorem XXXIV.

The Product of two adjacent Oblongs is an Oblong, whoſe greater Side is the Square of the leſſer Side of the greater Oblong multiplied, (or of the greater Side of the leſſer Oblong.)

Demon. $n-1$, n, $n+1$ being 3 adjacent Numbers, then are $\overline{n-1}\times n=nn-n$, and $n\times\overline{n+1}=nn+n$, two adjacent Oblongs; and their Product is $n^4-n^2=\overline{n^2-1}\times n^2$, which is an Oblong whoſe greater Side is nn, the Square of n, the leſſer Side of the greater, and greater Side of the leſſer Oblong.

Or thus, Let a, b, c, be 3 Numbers differing by 1, then are ab, bc, two adjacent Oblongs, and their Product is $ab\times bc=ac\times bb$; but from the Nature of Arithmetical Progreſſions, when the common Difference is 1, then is $ac=bb-1$, for a, b, c may be repre-

represented thus, $a, a+1, a+2$. then is $a\times\overline{a+2}=a^2+2a$. and $\overline{a+1}^2=a^2+2a+1$, whence $a^2+2a=a^2+2a+1-1$, that is, $a^2=bb-1$, therefore $ac\times bb$ is an Oblong.

PROBLEM IV.

One Oblong given, to find another such, that the two admit of one Geometrical Mean, *i. e.* such that their Product is a square Number.

Rule. Multiply the given Number by 4, and its Square by 16 (the Square of 4) the Sum of these Products is the Number sought. Thus, the given Oblong being n, that sought is $16n^2+4n$.

DEMON. $16n^2+4n$ is $=4n\times\overline{4n+1}$, which is plainly Oblong; and to shew that n, $16n^2+4n$ admit one Geometrical Mean; or that $n\times\overline{16n^2+4n}$ $(=16n^3+4n^2)$ is a square Number; let us suppose $n=ab$, and $a=\overline{b+1}$; then is $4n=4ab=2a\times 2b$; but since $a=b+1$, then is $a-b=1$, therefore $2a-2b=2$; and an Arithmetical Mean betwixt $2a$ and $2b$ is $a+b$; so that $2a-\overline{a+b}=1$ $(=a-b.)$ Then also is $\overline{a+b}^2=2a\times 2b+1=4ab+1$ (by taking $2a+1=a+b$, and $a+2=2b$) wherefore $4ab\times\overline{a+b}^2$ is an Oblong. Also $4ab\times 4ab\times\overline{a+b}^2$ is a Square, whose Root is $4ab\times\overline{a+b}$, so that $4ab\times\overline{a+b}^2$ is the Number sought, when $4ab$ is the Number given: But $4ab=4n$, and $\overline{a+b}^2=4ab+1=4n+1$, therefore $4ab\times\overline{a+b}^2=4n\times\overline{4n+1}=16n^2+4n$.

SCHOL. As to the Invention of this Rule, it may be traced in this Manner: The Sides of a given Oblong being a, b, then, 1°. To find a Number which multiplied into ab will produce a Square; it is obvious that if we multiply ab into any square Number, as nn, the Product $abnn$, is a Number which multiplied into ab produces a Square, *viz.* $\overline{ab}^2\times n^2=\overline{abn}^2$. But then, 2°. The Question is, whether $ab\times nn$ be also an Oblong; which depends upon the Choice of nn; as to which, it is plain in the first Place, that ab, nn, cannot be Sides of an Oblong; for, by the Nature of Oblongs, and their Connection with Squares, the next Square to ab is $ab-b$, or $ab+a$ (Vid. *Theorem* XXVIII. *Cor.* I.) either of which has a greater Difference from ab than 1; wherefore if nn is the greater Side of an Oblong, the other must be greater than ab: Also, in order that the Product of that Oblong by ab may be a Square, the other Side of the Oblong sought must be the Product of ab by some square Number, which if we suppose to be xx, then the two Sides of the Oblong are $abxx=ax\times bx$, and nn. But 3°. The Question still remains, What Numbers we shall chuse for x and n, so as $ax\times bx$, and nn, be Sides of an Oblong? In order to which it may readily occur, that if 3 Numbers are $\div l$, differing by 1; then the Product of the Extremes, and Square of the Mean differ by 1, [as in the Demonstration of *Theor.* XXXIV. we see $ac=bb-1$] and so are Sides of an Oblong; wherefore it follows, that in the present Case, ax, n, bx, must be $\div l$ differing by 1; and consequently x must be 2, because $a-b=1$, and hence $2a-2b=2$; so that betwixt $2a$, $2b$ there is one Arithmetical Mean in Integers, *viz.* $a+b$, the common Difference being 1; wherefore it's plain, that n must be an Arithmetical Mean betwixt $2a$ and $2b$, *i. e.* $n=a+b$.

From this Investigation the Rule may be expressed in this Manner, *viz.* Take the given Oblong, *viz.* ab, and multiply it by $a+b$ $(=n)$ the Sum of the Sides (which is the Arithmetical Mean betwixt the Doubles of the Sides, *viz.* $2a$, $2b$,) the Square of the Product is the Oblong sought, $\overline{abn}^2$. But observe, that this Rule requires that the Sides of the given Oblong be known, whereas the former Rule requires only the Oblong it self.

PROBLEM V.

To find 3 square Numbers in Arithmetical Progression, *i. e.* that the middle one exceed the least as much as the greatest exceeds the middle; thus, we are to find a^2, b^2, c^2, such that $a^2-b^2=b^2-c^2$.

Rule.

Rule. Take any two Square Numbers, which call c^2, d^2, and let them be such that $2c^2$ be greater than d^2; then is the Difference, *viz.* $2c^2-d^2$, the lesser of the Roots sought; to the Sum of the same Numbers, *viz.* $2c^2+d^2$, add $2dc$, and the Total $2c^2+dd+2dc$, is the next greater Root sought. Again, to this last Root add also $2dc$, and the Sum $2c^2+dd+4dc$ is the greatest of the Roots sought.

Exam. Take $c=2$, and $d=1$, then is $c^2=4$ and $d^2=1$: Again, $2c^2-d^2=8-1=7$, the lesser Root. To the Sum of $8+1=9$ add $2dc=4$, the Total is 13, the next Root. *Lastly*, To this Root 13 add $2dc=4$, the Sum 17 is the other Root sought: For $7\times7=49$, $13\times13=169$, and $17\times17=289$; and $289-169=169-49=120$.

Demonstration and Investigation of this Rule.

If you take the 3 Numbers $2c^2-d^2$; $2c^2+d^2+2dc$; $2c^2+d^2+4dc$, and find their Squares by the common Operations; then it will be found to be in Arithmetical Progression; but the Truth of this we shall see by the following *Investigation.*

Suppose the 3 Roots sought represented thus, *viz.* a, $a+b$, $a+b+c$; their Squares are a^2; a^2+b^2+2ab; $a^2+b^2+c^2+2ab+2ac+2bc$; the Difference of the 1st and 2d is b^2+2ab, and of the 2d and 3d is $c^2+2ac+2bc$; and these two Differences are, by Supposition, equal, *i. e.* $b^2+2ab=c^2+2ac+2bc$; whence this Proportion is evident, *viz.* $b+2a : c+2a+2b :: c : b$, but the 2d Term is greater than the 1st, and therefore so is b greater than c; and consequently $2ab$ is greater than $2ac$. In the next Place then, from each of these Equals, *viz.* $b^2+2ab=c^2+2ac+2bc$, take b^2+2ac, the Remainders are equal, *viz.* $2ab-2ac=c^2+2bc-b^2$; divide both these by $2b-2c$, or $2\times\overline{b-c}$, the Quotes are equal, *viz.* $a=\frac{c^2+2bc-b^2}{2b-2c}$, which is the lesser Root sought. But now to make this a possible Solution, it's manifest, that c^2+2bc must be greater than b^2, and if we chuse b and c, so as they have this Condition, and b also be greater than c, it's certain from what is shewn, that we have all the 3 Roots sought, *viz.* a, $a+b$, $a+b+c$. What remains then is to shew, how to chuse b, c, with these Conditions, in order to which suppose $b=c+d$, then is $b^2=c^2+d^2+2cd$, $bc=c^2+cd$, and $2bc=2c^2+2cd$; hence $c^2+2bc-b^2=c^2+2c^2+2cd-c^2-d^2-2cd=2c^2-d^2$; and because $b-c=d$, therefore $2b-2c=2d$, so that $a=\frac{c^2+2bc-b^2}{2b-2c}=\frac{2c^2-d^2}{2d}$, wherein $2c^2$ must be greater than d^2; and to chuse c, d so, has in it no manner of difficulty; for it's easily done, whether we take c greater or lesser than d; wherefore having assumed c, d according to the Conditions mentioned, we have all the 3 Roots, for $a=\frac{2c^2-d^2}{2d}$, then because $b=c+d$, therefore $a+b=\frac{2c^2-d^2}{2d}+c+d=\frac{2c^2-d^2+2dc+2d^2}{2d}=\frac{2c^2+d^2+2dc}{2d}$; also, $a+b+c=\frac{2c^2+d^2+2dc}{2d}+c=\frac{2c^2+d^2+2dc+2dc}{2d}=\frac{2c^2+d^2+4dc}{2d}$. In the last Place, since the Squares of these fractionally expressed Roots, are in Arithmetical Progression, so must the Squares of the Numerators be, because the Denominators are the same; but these Numerators are the same Expressions as in the Rule, which is therefore demonstrated.

SCHOL. If in all Cases we make $d=1$, then the 3 Roots are, $2c^2-1$; $2c^2+2c+1$; $2c^2+4c+1$; that is, assume any Number c, and from double its Square take 1, the Remainder is the lesser Root sought; to double its Square add double the Root, and 1, the Sum is the next Root sought; then to double its Square add 4 Times its Root, and 1 (*i. e.* to the preceding Root found add double the assumed Number) the Sum is the greatest of the Roots sought.

LEMMA.

The Difference of two Integral Squares is either ſome odd Number greater than 1; or, it is an even Number greater than 4, and alſo a Multiple of 4.

DEMON. (1°.) If it's an odd Number it muſt be greater than 1; this is ſhewn in *Cor.* 4. *Theor.* IX. (2°.) If it's an even Number it muſt be greater than 4, as is ſhewn in *Cor.* 2d and 4th. *Theor.* IX. But again, it muſt be a Multiple of 4, which is demonſtrated thus, by *Coroll.* 1. *Theor.* IX. The Difference of any two Integral Squares is either ſome one, or the Sum of ſome two or more Terms adjacent in the odd Series, whereof the leſſer is greater than 1. If it's one of theſe Terms, or the Sum of an odd Number of them, then it is an odd Number; and if it is the Sum of an even Number of Terms, then it is an even Number; and I ſay all theſe are Multiples of 4. For any two adjacent Terms in that Series, whereof the leſſer is above 1, may be repreſented thus, $2a+1$, and $2a+3$, whoſe Sum is $4a+4=4\times\overline{a+1}$; which ſhews the Truth propoſed, if the Squares differ by the Sum of any two adjacent Terms above 1. If the Difference is the Sum of any other even Number of Terms, then the Sum of every Pair of them at equal Diſtance from the middle Pair, is equal to the Sum of the middle Pair, becauſe they are in Arithmetical Progreſſion; but the Sum of the middle Pair being adjacent Terms, is a Multiple of 4, conſequently ſo is the Sum of each of the other Pairs. And hence, *Laſtly*, The total Sum of all the Pairs muſt be a Multiple of 4, ſince each of the Parts, *i. e.* each of the Pairs is ſo.

PROBLEM VI.

Having any Integral Number, to find two Integral Squares, whoſe Difference is equal to that given Number.

Rule. The given Number being an odd Number greater than 1, or an even Number greater than 4, and meaſurable by 4 (as required in the preceding *Lemma*) take any two different Integers, whoſe Product is equal to the given Number, *i. e* any two of its reciprocal Meaſures [and mind, that if the given Number is odd, we admit of 1 for a Meaſure, whoſe reciprocal Meaſure is the given Number it ſelf; but if the given Number is even, the reciprocal Meaſures muſt be both even] then the half Sum and half Difference of theſe two Numbers, are the Roots of two Squares, whoſe Difference is the given Number; and thus by taking every Pair of reciprocal Meaſures of the given Number, you'll find all the poſſible Solutions of the Problem.

Exam. 1. Given $15=1\times15=3\times5$; there are two Solutions, *viz.* $\frac{15-1}{2}=7$, and $\frac{15+1}{2}=8$; or $\frac{5-3}{2}=1$; and $\frac{5+3}{2}=4$. For $7\times7=49$, $8\times8=64$, and $64-49=15$; alſo, $4\times4=16$ and $16-1=15$.

Exam. 2. Given $105=1\times105=3\times35=5\times21=7\times15$, which make 4 Solutions, *viz.* $\frac{105-1}{2}=52$, and $\frac{105+1}{2}=53$, or $\frac{35-3}{2}=16$, and $\frac{35+3}{2}=19$, or $\frac{21-5}{2}=8$, and $\frac{21+5}{2}=13$, or $\frac{15-7}{2}=4$, and $\frac{15+7}{2}=11$.

Exam. 3. Given $20=2\times10$; whence we have this one Solution $\frac{10-2}{2}=4$, and $\frac{10+2}{2}=6$; no other Pair of the reciprocal Meaſures of 20 being both even Numbers.

Exam. 4. Given, $112=2\times56$, $=4\times28$, $=8\times14$; whence we have three Solutions, *viz.* 27 and 29, or 12 and 16, or 3 and 11.

DEMONSTRATION.

1. Let the given Number be called D, and any of its reciprocal Meaſures a, n, that is, $D=an$; then are $\frac{a-n}{2}$, $\frac{a+n}{2}$, two Roots, whoſe Squares differ by $D=an$; for theſe Squares are $\frac{aa+2an+nn}{4}$ and $\frac{aa-2an+nn}{4}$, whoſe Difference is plainly an.

2. Again, it is evident that as a, n, are ſuppoſed to be qualified, $\frac{a-n}{2}$, $\frac{a+n}{2}$, will be both Integers: For D being an odd Number, all its Meaſures, as, a, n, are odd Numbers, and ſo $a-n$, $a+n$, are both even (as explained in *Ch.*I. §.4.) conſequently $\frac{a-n}{2}$, $\frac{a+n}{2}$, are both Integers; if D is even, a, n, muſt both be even, for if the one is even, and the other odd, the Sum and Difference are both odd, and ſo have not an Integral half; but a, n, being both even, the Sum and Difference are both even, and conſequently their Half is an Integer.

3. Thus it is demonſtrated that all the Numbers found by this Rule are true Solutions of the Problem; but obſerve, that it does not from hence follow, that there can be no other Solutions than theſe, *i.e.* if $D=x^2-y^2$ (two Integral Squares) it remains to be demonſtrated, that x, y, muſt be one of them the half Sum, and the other the half Difference of ſome two Integers, a, n, ſuch that $an=D$ (the given Number) which I thus demonſtrate.

Let x be the greater Root ſought, the other, y, may be expreſſed $x-n$, whoſe Square is $x^2-2xn+n^2$; hence $x^2-\overline{x-n}^2=2xn-n^2=D$, and dividing both by n, it is $2x-n=\frac{D}{n}$, which muſt be an Integer, becauſe x, n, are ſuppoſed to be ſo, *i.e.* n meaſures D. Suppoſe then that $\frac{D}{n}=a$ (ſo that $D=an$) then is $2x-n=a$, and $2x=a+n$, and $x=\frac{a+n}{2}$, which is the one Root; alſo $y=x-n=\frac{a+n}{2}-n=\frac{a-n}{2}$, the other Root, according to the Rule.

SCHOLIUM I.

In the preceding Demonſtration you have alſo the *Inveſtigation* of this Rule after that Method which is called Analytical, by ſuppoſing x, and $x-n$, the two Roots ſought. And we have alſo a Demonſtration of the Rule, as a Solution of the Problem taken more univerſally, without reſpect to Integers; for whether D be Integral or Fractional, any two Numbers, a, n, ſuch that $an=D$, make $\frac{a-n}{2}$, $\frac{a+n}{2}$, a true Solution; and all the poſſible Solutions are got in this Method, becauſe x and $x-n$ may repreſent all the poſſible Roots.

But there are other Methods whereby this Rule might be diſcovered, of the Kind we call Synthetical, wherein we proceed from ſome known Truth; and theſe being alſo worth knowing, I ſhall explain them.

Second Method of Investigating the preceding Rule.

This Rule might be owing to the Consideration of the Squares of a Binomial and Residual Root. Thus, having observed that the Square of $a+n$ is $aa+2an+nn$; and the Square of $a-n=aa-2an+nn$; also, that the Difference of these two Squares is $4an$; it was obvious to conclude, that if the Difference of two Squares was a Multiple of 4, it may be expressed by $4b$, and b be expressed $a\times n$, (a, n being any two Numbers, whose Product is $=b$.) Again, it might easily occur, that though the Difference of every two Squares is not a Multiple of 4, yet the Difference of any two Squares may be represented by the Product of two Numbers, as an; and then, considering that $\frac{4an}{4}=an$; which being the 4th Part of the preceding Difference $4an$ ($=4b$) therefore it must be the Difference of the 4th Parts of the preceding Squares, *viz.* of $\frac{aa+2an+nn}{4}$ and $\frac{aa-2an+nn}{4}$, whose Roots are $\frac{a+n}{2}$, $\frac{a-n}{2}$; whence an universal Rule is discovered for finding two Squares, whose Difference is any given Number, provided always that a, n, be different Numbers, for otherwise $a-n$ is nothing. What other Conditions are necessary, upon Supposition that D is Integer, are also obvious, *viz.* that a, n be both Integers, and both odd, or both even.

Third Method of Investigating the preceding Rule.

The last Method seems to be rather by Chance, than that one is naturally directed to the Consideration of Binomial and Residual Squares, as a first Principle that should lead to the Solution of the Problem.

But after the Invention of *Theor.* IX. we have a more easy and natural Principle afforded us for the Solution of the Problem, as limited to Integers; which, though tedious, yet is very curious and not difficult; thus,

By *Coroll.* 1. *Theor.* IX. The Difference of any two Integral Squares is either some one, or the Sum of some 2 or more Terms of the odd Series, 1 . 3 . 5 . &c. and the Roots of these Squares are the Indexes of the Places of the greater Term, and of that next below the lesser of these Terms, whose Sum is equal to the given Number. It is hence evident, that we have no more to do, but from the given Number, and the known Properties of an Arithmetical Progression, to seek an Expression in known Numbers for these Roots; and all the Variety of them that have the same Difference; and this we must distribute into 2 Cases.

Case I. The given Number being an odd Number greater than 1. In this Case the Problem has at least one Solution; for D being a Term of the odd Series, whose Place call n, then is D the Difference of two Squares, whose Roots are n and $n-1$; and these being expressed according to the Rules of Arithmetical Progressions, are $n=\frac{D+1}{2}$, and $n-1=\frac{D-1}{2}$; for if the greater Extreme of an Arithmetical Series is l, the lesser a, and the common Difference d, then is the Number of Terms $n=\frac{l-a+d}{d}$. But in the present Case $l=D$, $a=1$, and $d=2$; hence $n=\frac{D-1+2}{2}=\frac{D+1}{2}$, and $n-1=\frac{D+1}{2}-1=\frac{D-1}{2}$; and these Roots $\frac{D-1}{2}$, $\frac{D+1}{2}$, are manifestly two Examples of the preceding Rule, because $1\times D=D$.

Again,

Again, To find if there can be any other Solutions, and what they are: Consider, That if there is another Solution, then, by *Coroll.* 1. *Theor.* IX. the given odd Number, D, must be equal to the Sum of some odd Number of Terms greater than 1, standing next together in the odd Series, and whereof also the least Term cannot be 1: And in this Case the Place of (or the Number of Terms from 1, to) the greatest of these Terms, and the Place next below the lesser of these Terms, are two Roots, whose Squares have the given Difference (which is the Sum of all these Terms.) Wherefore it's plain, That as many different odd Numbers of Terms as there are, whose Sums are each equal to D (of which the least is greater than 1) so many, and no more, Solutions has the Problem: And to discover these, consider *again*, That from the Nature of Arithmetical Progressions, when the Number of Terms is odd, it's an *aliquot* Part of the Sum; for the Sum being D, and Number of Terms n, the middle Term is $\frac{D}{n}$; which being, by Supposition, an Integer, n must measure, or be an *aliquot* Part of D; whereby it's evident, that as there can be no Solution of the Problem but one for every odd Number of Terms of the odd Series, whose Sum is $=D$, and whereof the least Term is greater than 1; so, because that Number of Terms is also an *aliquot* Part of D, it follows, that there cannot be more Solutions than there are different Measures of D. But yet again, I say that all the Solutions amount only to one for every Pair of Reciprocal Measures (and not one for every Measure:) And to demonstrate this, and also find the general Expression for them all, agreeable to the Rule, we proceed *thus*; D being the Sum of an odd Number, n, of Terms of the odd Series, $\frac{D}{n}$ is the middle Term; and then having $\frac{D}{n}$, the middle Term, and 2, the common Difference, we may find the Extremes thus; Suppose the middle Term, $\frac{D}{n}$, to be a lesser Extreme, with respect to the greater Extreme sought (which in general call l) and a greater, with respect to the lesser Extreme sought (which call A;) then, since the Number of Terms, from A to l, is n, the Number of Terms less 1, from A to $\frac{D}{n}$, and from $\frac{D}{n}$ to l, is $\frac{n-1}{2}$; and hence, by the common Rules, we find $l=\frac{D}{n}+2\times\frac{n-1}{2}=\frac{D}{n}+n-1$; and $A=\frac{D}{n}-2\times\frac{n-1}{2}$ $=\frac{D}{n}-n+1$; then the Places of these Terms in the odd Series, are $\frac{l+1}{2}$, the Place of l; which is also the Root of the greater Square sought; and $\frac{A+1}{2}$, the Place of A, from which take 1, the Remainder is $\frac{A-1}{2}$, the lesser Root sought; now putting instead of l, and A, their Equals, $\frac{D}{n}+n-1$ $(=l,)$ and $\frac{D}{n}+1-n$ $(=A)$ these Roots are $\frac{D+nn}{2n}$, and $\frac{D-nn}{2n}$; for $l+1=\frac{D}{n}+n=\frac{D+nn}{n}$; which divided by 2, Quotes $\frac{D+nn}{2n}$; and $A-1=\frac{D}{n}-n=\frac{D-nn}{n}$; which divided by 2, Quotes $\frac{D-nn}{2n}$. Again, Since n measures D, suppose $\frac{D}{n}=a$, then is $D=an$; and putting an for

D, the Roots are $\frac{an+nn}{2n}$, $\frac{an-nn}{2n}$; which by equal Division of Numerator and Denominator, by n, are $\frac{a+n}{2}$, $\frac{a-n}{2}$; which Expressions comprehend also the 1st Solution; wherein n being $=1$, D is $=a$: And hence $\frac{a-n}{2}$, $\frac{a+n}{2}$, are $=\frac{D-1}{2}$, $\frac{D+1}{2}$, the Rules given for that Case; so that D being $=an$, any two reciprocal Measures of D, all the possible Roots which solve the Problem, are expressed in general, by $\frac{a-n}{2}$, $\frac{a+n}{2}$, according to the preceding Rule: Then, because n must be less than a, to make $a-n$ possible; therefore of two reciprocal Measures of D, there can be a Solution only for the lesser of them, taken as a Number of Terms, whose Sum is $=$D. And thus is the whole Investigation and Demonstration finished, when D is an odd Number.

Case II. The given Number, D, being an even Number (greater than 4, and measurable by 4) then the Problem has, at least, one Solution; for by Supposition, $\frac{D}{2}$ is an even Number; therefore $\frac{D}{2}+1$, $\frac{D}{2}-1$, are two odd Numbers; which differing by 2, shews that they stand next together in the odd Series; and their Sum being $=$ to D, therefore the Places of $\frac{D}{2}+1$, and of the Term next below $\frac{D}{2}-1$, are two Roots, whose Squares differ by D. Now these Places are found by the common Rules to be $\frac{D+4}{4}$, and $\frac{D-4}{4}$. Again, Since D is a Multiple of 4, let it be $=4d=2\times 2d$; then $\frac{D+4}{4}=\frac{4d+4}{4}=\frac{2d+2}{2}$ $(=d+1)$ and $\frac{D-4}{4}=\frac{4d-4}{4}=\frac{2d-2}{2}$ $(=d-1)$ which shews us plainly that this is an Example of the preceding Rule.

Again, To find if there are any other Solutions, and what they are. Consider, That if there is another Solution, then D must be the Sum of an even Number of Terms greater than 2, standing next together in the odd Series, and whereof the least is not 1, (by *Cor.* 1. *Th.* IX.) and in this Case the Places of the greatest of these Terms, and of that next before the least, are 2 Roots, whose Squares differ by D, the Sum of all these Terms; wherefore it's plain, that as many different Numbers of Terms as there are, whose Sums are each $=$D, and of which the least Term is greater than 1, so many, and no more, Solutions has the Problem: And to discover these, consider, *again*, That from the Nature of a Progression of odd Numbers, if the Number of Terms is even, it's an *aliquot* Part or Measure of the Sum; for D being the Sum, n the Number of Terms, and the Extremes, A, L, then is D $=A+l\times\frac{n}{2}$. Hence $A+l=D\div\frac{n}{2}=\frac{2D}{n}$, and $\frac{A+l}{2}=\frac{D}{n}$; but A, l, are both odd, and so their Sum is even, *i. e.* $\frac{A+l}{2}$, is an Integer; consequently so is $\frac{D}{n}$, or n measures D: Hence, if there is any Number of Terms of the odd Series, whose Sum is $=$D, that Number measures D; whereby it's evident there can be no Solution of the Problem, but one, for every even Number of Terms, whose Sum is D, and whereof the least Term is greater than 1; and because that Number of Terms must measure D, it follows, that there cannot be more Solutions than there are even Measures of D: But yet again, I say, that there cannot be more Solutions than the Number of reciprocal Measures of D, or one for each Pair, which are both even Numbers: And

And to demonſtrate this, and find the general Expreſſion for them all, agreeable to the Rule, we proceed thus; D being the Sum of an even Number, n, of Terms of the odd Series, whoſe common Difference is 2, then $\frac{D}{n}-1$, $\frac{D}{n}+1$, expreſs the two middle Terms; for the Sum of the two middle Terms is equal to the Sum of the Extremes, $A+l$, which is equal to $\frac{2D}{n}$; becauſe $\overline{A+l}\times\frac{n}{2}=D$, and hence $A+l=\frac{2D}{n}$; therefore the Sum of the two middle Terms is alſo $=\frac{2D}{n}$. But it is a known Truth, that the half Sum, more or leſs the half Difference of two Numbers, is equal to the greater or leſſer of them; wherefore $\frac{2D}{n}$ being the Sum of the middle Terms, $\frac{D}{n}$ is the half Sum; alſo 2 being their Difference, 1 is the half Difference; therefore $\frac{D}{n}+1$ is the greater, and $\frac{D}{n}-1$ the leſſer middle Term: From which, *again*, we may find the two Extremes, A, l, *thus*; Suppoſe the greater middle Term, $\frac{D}{n}+1$, to be a leſſer Extreme, with reſpect to l; and $\frac{D}{n}-1$ to be a greater Extreme, with reſpect to A; then the Number of Terms, from A to l, being n, that from A to $\frac{D}{n}-1$, and from $\frac{D}{n}+1$ to l, will be $\frac{n}{2}$; from which take 1, the Remainder is $\frac{n}{2}-1=\frac{n-2}{2}$; and from the common Rules, l is $=\frac{D}{n}+1+2\times\frac{n-2}{2}=\frac{D}{n}+n-1$; alſo, $A=\frac{D}{n}-1-2\times\frac{n-2}{2}=\frac{D}{n}+1-n$. Now theſe being the ſame Expreſſions, as for l and A, in the 1ſt Caſe; all the reſt of the Inveſtigation is the ſame as there; whereby the two Roots ſought are $\frac{a-n}{2}$, $\frac{a+n}{2}$; which comprehend alſo the 1ſt Solution, wherein $n=2$, $D=4d=2d\times2$; whence $2d=a$, and $\frac{a-n}{2}=\frac{2d-2}{2}=d-1$; alſo $\frac{a+n}{2}=\frac{2d+2}{2}=d+1$, which are the Rules of that firſt Solution: So that D being $=an$ (any two reciprocal Meaſures of D) all the Solutions of the Problem are univerſally expreſſed by $\frac{a-n}{2}$, $\frac{a+n}{2}$, upon Condition alſo that a, n, be both even Numbers; for elſe the Roots cannot be Integers: They muſt alſo be different Numbers; elſe $a-n=0$: Whence, in the laſt Place it is clear that all the Solutions amount only to one for every Pair of the reciprocal Meaſures of D, which are both even Numbers.

SCHOLIUM II.

In the preceeding Problem, the given Difference being called D, will, in ſome Solutions, be leſs than $\frac{a-n}{2}$ ſquar'd, and in ſome greater; and indeed, in ſome Values of D, there will be no Solution in which D will be leſs than $\frac{a-n}{2}$ ſquared: And in ſome

Caſes

Cafes there will be no Solution, wherein D is greater than $\frac{a-n}{2}$ fquar'd. In other Cafes you will find Solutions of both Kinds. But I fhall alfo explain thefe Things more particularly.

1ft. If D is an even Number, then (1.) If it's lefs than 24, it muft be 8, 12, 16, or 20; for no other even Number lefs than 24, is a Multiple of 4: And in all thefe there is but one Solution, *viz.* Where 2 is one of the reciprocal Meafures; becaufe no other Pair of reciprocal Meafures of any of thefe Numbers, are both even Numbers; and the leffer Roots in thefe feveral Examples are $\frac{4-2}{2}$, $\frac{6-2}{2}$, $\frac{8-2}{2}$, $\frac{10-2}{2}$ (*viz.* when $8=4\times2$, $12=6\times2$, $16=8\times2$, $20=10\times2$, are the Values of D) and the Squares of thefe are all lefs than D, or 24; for they are, 1, 4, 9, 16. (2.) If D is 24, or greater, then in fome Values of it there will be but one Solution; and in that, D $(=a\times n)$ will be lefs than $\frac{a-n}{2}$ fquared: And fuch are all thefe Values of D, whofe 4th Part is an odd Number; for fince D muft be a Multiple of 4, let it be $=4d$; if its 4th Part, d, is an odd Number, it follows, that it can have no Pair of even Numbers for its reciprocal Meafures, but 2 and $2d$; fo that there is but this one Solution: And here the Roots are $\frac{2d-2}{2}$, $\frac{2d+2}{2}$, which are equal to $d-1$, and $d+1$; whofe Squares are both greater than D. That $\overline{d-1}^2$ is greater than $D=4d$, is thus proved; $\overline{d-1}^2=dd-2d+1$; add $2d$ both to this Square and to $4d$; the Sums are $dd+1$, and $6d$; divide both by d, and the Quotes are $d+\frac{1}{d}$, and 6: But $4d$ being equal to 24, or greater, it follows, that D muft be $=6$, or greater; and confequently $d+\frac{1}{d}$ is greater than 6; and hence $dd+1$ is greater than $6\times d$; alfo $dd-2d+1$, greater than $6\times d-2d=4d$; that is, $\overline{d-1}^2$ greater than $D=4d$.

In all other Values of D (greater than 24, and alfo an even Number) there will be, at leaft, two Solutions; for we fuppofe now, that the 4th Part of D is an even Number; and therefore D may be reprefented, $4\times2d=2\times4d$, which make two Solutions; and according as d is varioufly compounded, fo will there be a Variety of other Solutions; in fome of which D will be lefs, and in fome greater, than the leffer of the two Squares.

2d. If D is an odd Number, then (1.) If it's a prime Number, there is but one Solution, and the Roots fought are $\frac{D-1}{2}$, and $\frac{D+1}{2}$. And here, if D is 3 or 5, it is greater than the leffer Square; for this, in thefe Cafes, is 1, or 4. But if D is greater than 5, then it's always lefs than the leffer Square fought, *viz.* $\frac{D-1}{2}$ Square. For $\overline{D-1}^2=D^2-2D+1$, and $\frac{D-1}{2}$ Square, $=\frac{D^2-2D+1}{4}$; compare this with D, thus, multiply both by 4, and the Products are D^2-2D+1, $4D$; add $2D$ to both, the Sums are D^2+1, $6D$; divide both thefe by D, the Quotes are $D+\frac{1}{D}$, 6;

6; but D is fuppofed to be an odd Number greater than 5, that is, 7 at leaft, and confequently $D+\frac{1}{D}$ is greater than 6: And hence going backwards, D^2+1 is greater than $6D$; alfo D^2+1-2D, greater than $6D-2D=4D$; and $\frac{D^2+1-2D}{4}$ greater than D, or D lefs than $\frac{D-1}{2}$ fquar'd. (2.) If D is an odd compofite Number, there will be feveral Solutions, according to the Variety of the Compofition; and in all Cafes, there will be at leaft, two Solutions, whereof the Roots of one will be $\frac{D-1}{2}$, $\frac{D+1}{2}$, both their Squares being greater than D; becaufe D is greater than 5; as already fhewn.

3*d*. Another Thing to be obferved here, is, That having found all the reciprocal Meafures of D; if you begin with that Pair, which confifts of the leaft and greateft Meafures of D, *i. e.* that Pair which have the greateft Difference; or alfo with that Pair which have the leaft Difference; and fo proceed in Order: Then, if the firft Solution makes $\frac{a-n}{2}$ fquared, lefs or greater than D, go on till you have a Solution making $\frac{a-n}{2}$ fquared, contrarily greater or lefs than D; and after this, all the reft will be of the fame Kind. The Reafon is plain; for let the reciprocal meafures of D be reprefented as in the Margin; wherefore if a is the leaft, and A the greateft Meafure of D, then it is plain that $A-a$ is greater than $B-b$, which is greater than $C-c$ &c. Therefore if $\frac{A-a}{2}$ fquared is greater than D, then having proceeded till we find a Solution, wherein $\frac{x-y}{2}$ fquared (x, y, being any Pair of reciprocal Meafures of D) is lefs than D, all that follow will be fo too; becaufe the Difference of the two reciprocal Meafures grows ftill lefs and lefs. Again, if $A-a$ is lefs than $B-b$, &c. (*i. e.* if a is the greateft Meafure of D, which is lefs than its reciprocal Meafure) then if $\frac{A-a}{2}$ fquared is lefs than D, having proceeded till we find a Solution, in which $\frac{x-y}{2}$ fquared is greater than D, all after this will be fo too; becaufe the Differences increafe.

$$\begin{array}{l} a\ .\ b\ .\ c\ .\ \&c. \\ A\ .\ B\ .\ C\ .\ \&c. \\ \hline aA=bB=cC=\&c.=D \end{array}$$

PROBLEM VII.

To find 3 Integral Squares (or 3 Integral Roots, whofe Squares are) fuch, that the greater is equal to the Sum of the two leffer.

Rule. Affume any fquare Number, N^2, greater than 4; and take any two of its reciprocal Meafures, as a, n; then are $\frac{a-n}{2}$, $\frac{a+n}{2}$, the Roots of two Squares, which with N^2 folve the Problem.

Exam. 1. If we affume $9=1\times 9$; the 3 Squares fought are 9, 16, 25.
Exam. 2. If we affume $64=2\times 32=4\times 16$; the 3 Squares fought are 64, 225, 289, or 64, 36, 100.

DEMON. The universal Reason of this is contained in the Demonstration of the preceding Problem; for the Difference of two Squares may be any odd Number greater than 1, or any even Number greater than 4, and measurable by 4, (by the preceding *Lemma*) therefore it may be any Square Number greater than 4; because any such Square is either an odd Number greater than 1, or it is an even Number measurable by 4 (as all even Squares are, by *Theor.* XXIX. § 4. *Chap.* 1.)

SCHOLIUMS.

I. By this Rule, the Square assumed is always one of the two lesser, because it is the Difference of the other two; and if the Problem be limited to this Condition, *viz.* That of the three Squares, the assumed (or given) one be the least, or also the middle one of the three, then it may be done thus:

(1°.) To make the assumed Square the least of the Three; if you assume N^2, an odd Square, then as it must be greater than 1, so the other 2 Roots will be $\frac{N^2+1}{2}$, and $\frac{N^2-1}{2}$, which will be greater than N; as is shewn in the second Article of *Schol.* 2*d.* to the preceding *Problem*; where it's shewn, that if D (an odd Number) is greater than 5, then $\frac{D-1}{2}$ squared is greater than D, *i.e.* in the present Case, $\frac{N^2-1}{2}$ squar'd is greater than N^2; for here N^2 is at least 9. Again, if you assume N^2 an even Number, as it must be a Multiple of 4, in order to be the Difference of 2 Squares, so it must be the Multiple of it by some Square Number, for else the Product of it by a Square 4 could not produce a Square Number (by *Coroll.* 2. *Theor.* II. *Book* III. *Chap.* I.) Suppose now, that $N^2=4\times a^2=2\times 2a^2$, then are $\frac{2a^2-2}{2}$ $(=a^2-1)$ and $\frac{2a^2+2}{2}$ $(=a^2+1)$ two Roots, which solve the Problem in all Cases wherein N^2 is greater than 16; as appears by the first Article of *Schol.* 2. to the preceding Problem; for there it's shewn, that when D (or in this Case N^2) is greater than 24 (as it must be if it's a Square Number greater than 16) then it is less than $\overline{d-1}^2$, the lesser of the Squares sought (which is in this Case a^2-1 squar'd.) But if $N^2=16$, this will not solve the Problem, because then $4\times a^2$ $(=N^2)=16$, and consequently $a^2=4$, and $a^2-1=3$, less than $4=N$.

(2°.) To make the assumed Square Number the middle one of the Three; then find all its reciprocal Measures, and take that Pair which have the least Difference, suppose them to be x, y; then if $\frac{x-y}{2}$ is less than N, the Problem is possible, and $\frac{x-y}{2}$, $\frac{x+y}{2}$, are two Roots which solve it; but if $\frac{x-y}{2}$ is greater than N, the Problem is impossible; for the Difference of all the other Pairs of reciprocal Measures being, by Supposition, greater than $x-y$, therefore $\frac{x-y}{2}$ will be less than the half of the Difference of these others; so that $\frac{x-y}{2}$ being greater than N, and the half Difference of all the others being greater than $\frac{x-y}{2}$, they must still be greater than N, and consequently N^2 is the least, and not the middle of the three Squares.

II. If

II. If the given Square Number is proposed as the greatest of the three, [which is the same Problem as proposing to divide a given Integral Square into two other Integral Squares] the Limitation is still more difficult; and the preceding Rule is of no use here; nor do I indeed know any Rule better or easier than taking all the Numbers less than N, and adding together the Squares of every one of them, whereby you'll find overy Pair of Squares less than N^2, whose Sum is equal to N^2.

But in some Cases the Impossibility of this Problem may be discovered without the Application of this tedious Rule; thus, divide the given Square by 4; and if the Remainder is 3, the Problem is impossible; the Reason of which you have in *Cor.* 4. *Theor.* XXIX. § 4. *Chap.* I. Yet observe, that though the Remainder is not 3, or if there is no Remainder, as in all even Squares, it does not follow that therefore the given Square is divisible into two Squares.

THEOREM XXXV.

If 3 Numbers are such that the Square of the greater is equal to the Sum of the Squares of the other two, the same will be true also of any the like Multiples, or *aliquot* Parts of these Numbers.

Exam. 3, 4, 5, are such Numbers as proposed; and so are their Doubles, 6, 8, 10, and their Triples, 9, 12, 15; as you'll find by Calculation.

DEMON. Let A, B, C, be such that $C^2=A^2+B^2$, then are nA, nB, nC, such also, *viz.* $\overline{nC}^2=\overline{nA}^2+\overline{nB}^2$, for $\overline{nC}^2=n^2\times C^2$, $\overline{nB}^2=n^2\times B^2$, $\overline{nA}^2=n^2\times A^2$; but $C^2=A^2+B^2$, therefore their Equimultiples are also equal, *viz.* $n^2C^2=n^2A^2+n^2B^2$. The same is true of like *aliquot* Parts, which is but the Reverse of the former, since nA, nB, nC, may represent any 3 Numbers which have a like *aliquot* Part denominated by n.

COROLLARIES.

1. If it is proposed to find 3 Integral Numbers, such that the Square of the greater be equal to the Sum of the other two Squares, and such too, that the Roots be in certain Ratio's to one another, then reduce these Ratios to a common Antecedent, *i. e.* find 3 Numbers, which taken in a Series, are gradually to one another in the proposed Ratios (by *Probl.* I. *Book* IV. *Chap.* IV.) and if these 3 Numbers answer the Problem, you have done, otherwise the Problem is impossible; for if there are any other 3 Numbers in these Ratios that can solve the Problem, they are either the least Terms of these Ratios, or like Multiples of them; if the least Terms, then the like Multiples will also solve the Problem: If you say they are some like Multiples of the least Terms, then also the least Terms, which are like *aliquot* Parts of them, will solve it; but the 3 Numbers first found must also be either the least Terms of the same Ratios, or like Multiples of them; and therefore if they solve not the Problem, it cannot be solved, because if any Terms of these Ratios solve it, all Terms must solve it.

2. Having any 3 Numbers, such that the Square of the greater is equal to the Sum of the Squares of the other two, we can find an infinite Number of other Examples of the same kind, by taking any like Multiples of the Numbers given.

THEOREM XXXVI.

If the Root of any Square Number is equal to the Sum of two Square Numbers, that Square is alſo equal to the Sum of two Square Numbers, whoſe Roots are, the Difference of the two Squares whoſe Sum the firſt Root is, and double the Product of their Roots.

Exam. $4+9=13$, and $13\times13=169=144+25$, two Squares, whoſe Roots are 12 ($=2\times2\times3$) and 5 ($=9-4$.)

DEMON. Take any two Squares, a^2, b^2, their Sum is a^2+b^2, and the Square of this Sum is $a^4+2a^2b^2+b^4=a^4-2a^2b^2+b^4+4a^2b^2$; alſo $a^4-2a^2b^2+b^4$ is a Square, whoſe Root is a^2-b^2; and $4a^2b^2$ is a Square, whoſe Root is $2ab$.

COROLL. Hence we learn how to find a Square Number, which is equal to the Sum of two Squares; or, another Rule for finding 3 Square Numbers, ſuch that the greater is equal to the Sum of the two leſſer. But *obſerve*, that though we can hereby find an infinite Number of Anſwers to this Problem, yet all the poſſible Solutions of the Problem cannot be found in this Method; becauſe all the Examples of this Rule are of a particular Kind, *viz.* where the Root of the greateſt Square is it ſelf the Sum of two Squares; whereas there is an infinite Number of other Examples not of this Kind; thus, 9, 12, 15, are ſuch Roots, and yet 15 is not the Sum of two Squares: But all the poſſible Examples of this Problem are to be found by *Probl.* VII. and therefore all the Examples found by this Rule muſt coincide with ſome of thoſe found by that univerſal Rule; which Coincidence you may ſee thus: Let the aſſumed Root of a Square be $2ab$, its Square is $4a^2b^2=2a^2\times2b^2$, and by the Rule of *Probl.* VII. $\frac{2a^2-2b^2}{2}=a^2-b^2$, $\frac{2a^2+2b^2}{2}=a^2+b^2$, are two Roots which ſolve the Problem; and of which a^2+b^2, the greateſt of the 3 Roots, is it ſelf the Sum of two Squares, the Roots of the other two being the Difference of theſe two Squares, *viz.* a^2-b^2, and double the Product of their Roots, *viz.* $2ab$; agreeable to the preſent Theorem.

CHAP.

CHAP. III.

Of Infinite Series.

DEFINITION.

AN *Infinite Series* is a Series, consisting of an Infinite Number of Terms, *i. e.* to the end of which 'tis impossible ever to come; so that let the Series be carried on to any assignable length, or number of Terms, it can be carried yet farther, without End or Limitation.

SCHOLIUM. A Number actually Infinite (*i. e.* all whose Units can be actually assign'd, and yet is without Limits) is a plain Contradiction to all our Ideas about Numbers; for whatever Number we can actually conceive, or have any proper Idea of, is always determinate and finite; so that a greater after it may be assign'd, and a greater after this, and so on, without a Possibility of ever coming to an end of the Addition or Encrease of Numbers assignable: Which Inexhaustibility, or endless Progression in the nature of Numbers, is all that we can distinctly understand by the *Infinity* of Number: And therefore, to say that the Number of any Things is Infinite, is not saying that we comprehend their Number, but indeed the contrary; the only Thing positive in this Proposition being this, *viz.* That the Number of these Things is greater than any Number which we can actually conceive and assign. But then, whether in Things that do really exist it can truly be said, that their Number is greater than any assignable Number; or, which is the same thing, That in the Numeration of their Units one after another 'tis impossible ever to come to an End; this, I say, is a Question about which there are different Opinions, with which we have no business in this place; for all that we are concern'd to know here is this certain Truth, That after one determinate Number, we can conceive a greater, and after this a greater, and so on without end. And therefore, whether the Number of any Things that do or can really exist all at once, can be such that it exceeds any determinable Number, or not, this is true, That of Things which exist, or are produced successively one after another, the Number may be made greater than any assignable one; because tho' the Number of Things thus produced that does actually exist at any time is *Finite*, yet it may be encreas'd without end. And this is the distinct and true Notion of the *Infinity of a Series*; *i. e.* of the Number of its Terms, as it is express'd in the *Definition*.

From hence again 'tis plain, That we cannot apply to an *Infinite Series* the common Notion of a Sum, *viz.* a Collection of several particular Numbers that are join'd and added together one after another, for this supposes that these Particulars are all known and determin'd; whereas the Terms of an *Infinite Series* cannot be all separately assign'd, there being no end in the numeration of its Parts, and therefore it can have no Sum in this Sense. But again, consider that the Idea of an *Infinite Series* consists of two Parts, *viz.* the Idea of something positive and determin'd, in so far as we conceive the *Series* to be actually carried on; and the Idea of an inexhaustible Remainder still behind, or an endless Addition of Terms that can be made to it one after another; which is as different from the Idea of a *Finite Series* as two Things can be: Hence we may conceive it as a Whole of its own Kind, which therefore may be said to have a total Value whether that be determinable, or not. Now in some *Infinite Series* this Value is finite or limited; *i. e.* a Number is assignable beyond which the Sum of no assignable Number of Terms of

the Series can ever reach, nor indeed ever be equal to it, yet may approach to it in such a manner as to want less than any assignable Difference; and this we may call the Value or Sum of the Series; not as being a Number found by the Common Method of Addition, but as being such a Limitation of the Value of the Series, taken in all its Infinite Capacity, that if it were possible to add them all one after another, the Sum would be equal to this Number.

Again: In other Series the Value has no Limitation; and we may express this by saying, *The Sum of the Series is Infinitely Great*; which indeed signifies no more than that it has no determinate and assignable Value; and, that the Series may be carried such a length as its Sum, so far, shall be greater than any given Number. In short, in the first Case we affirm there is a Sum, yet not a Sum taken in the common sense; in the other Case we plainly deny a Determinate Sum in any sense. What kind of Series have *Finite* or *Infinite* Sums in these senses, you'll learn in what follows.

THEOREM I.

In an *Infinite Series* of Numbers, encreasing by an equal Difference or Ratio [*i. e.* an Arithmetical or Geometrical encreasing Progression] from a given Number, a Term may be found greater than any assignable Number.

DEMON. 1°. If it's an Arithmetical Progression, let the Distance of any Term from the first be call'd n; the first, A; and the common Difference d; then any Term after the first is $A+nd$: And, that this may be found greater than any assign'd Number B, is thus prov'd: Suppose $B \div d = q$, then is $B = dq$ But whatever q is, since it is Finite, we can take a greater Finite Number; and therefore we may suppose or take n greater than q, so that nd will be greater than qd or B; and $A+nd$ greater than $A+qd$; but $qd = B$, therefore $A+nd$ is greater than $A+B$, and consequently yet greater than B.

2°. If it's a Geometrical Progression, the differences of its Terms make also a Geometrical Progression encreasing (*Theo.* 18, Ch. 3, B. 4.) But the thing propos'd being true, in case the Differences in a Series are equal, (as in an Arithmetical Progression) it must necessarily be so, and after a smaller number of Terms too, where the Differences do continually encrease (as in a Geometrical Progression). Or we may prove this independently of the other, thus: Let the first Term of a Geometrical Progression be A, the Ratio r, and the Distance of any Term from the 1st be n, then is that Term Ar^n. But we may take n greater than any assign'd Number; and it's plain that Ar^n will be yet much greater than that Number.

COROLLARY.

If the *Series* encrease by Differences that continually encrease, or by Ratio's that continually encrease, comparing each Term to the preceeding, it's manifest that the same thing must be true, as if the Differences or Ratio's continued equal.

THEOREM II.

In a *Series* decreasing *in infinitum* in a given Ratio, we can find a Term less than any assignable Fraction.

DEMON. The first Term being l, and the Ratio r, a whole or mix'd Number, the Series is $l : \frac{l}{r} : \frac{l}{r^2}$ &c. Wherein the Denominators continually encrease in the Ratio $1 : r$. Suppose then the assign'd Fraction is $\frac{a}{b}$, take $a : b :: l : m$, then is $\frac{a}{b} = \frac{l}{m}$.

But

But we can find a Power of r, as r^n, greater than any assignable Number m (by the last). Hence $\frac{l}{r^n}$ is less than $\frac{l}{m}$ or $\frac{a}{b}$.

COROLL.

If the Terms decrease, so as the Ratio's of each Term to the preceding do also continually decrease, then the same thing is also true as when they continue equal.

SCHOLIUM. Some may possibly think these two Theorems might have pass'd for Axioms, because the Notion of a Series continually encreasing or decreasing, may seem to include them; but you'll find in what follows, that a Series may have its Terms continually encreasing, yet so that no one of them can ever be actually found equal to a certain assignable Number. And for a decreasing Series, tho' its Terms continually decrease, yet it may be so, that no Term can ever be actually found so little as a certain assignable Number: And here, to distinguish these different Kinds of encreasing and decreasing Series, we may call such as encrease or decrease above or below any assignable Number, A Series encreasing or decreasing *infinitely*; and such as encrease or decrease continually, yet so as never to reach a certain assignable Number, we may call them *Infinite Series*, encreasing or decreasing *limitedly*; and when we say in general *an Infinite Series*, that may be taken indifferently, for either kind.

THEOREM III.

The Sum of an *Infinite Series* of Numbers all equal, or encreasing continually, by whatever Differences or Ratio's, is infinitely great; *i. e.* such a Series has no determinate Sum, but grows so as to exceed any assignable Number.

DEMON. 1°. If the Terms are all equal, as $A : A : A$, &c. then the Sum of any Finite Number of them is the Product of A by that Number, as An; but the greater n is, the greater is An; and we can take n greater than any assignable Number, therefore An will be yet greater than that assignable Number.

2°. Suppose the Series encreases continually, (whether it do so *infinitely* or *limitedly*) then its Sum must be infinitely great, because it would be so if the Terms continued all equal, and therefore will rather be so if they encrease. But if we suppose the Series encreases infinitely, either by equal Ratio's or Differences, or by encreasing Differences or Ratio's of each Term to the preceeding; then the Reason of the Sum's being Infinite will appear from the first Theorem; for in such Series a Term can be found greater than any assignable Number, and much more therefore the Sum of that and all the preceeding.

THEOREM IV.

The Sum of an *Infinite Series* of Numbers decreasing in the same Ratio is a *Finite Number*; equal to the Quote arising from the Division of the Product of the Ratio and first Term, by the Ratio less Unity; that is, the Sum of no assignable Number of Terms of the Series can ever be equal to that Quote; and yet no Number, less than it, is equal to the value of the Series, or to what we can actually determine in it; so that we can carry the Series so far, that the Sum shall want of this Quote less than any assignable Difference.

DEMON. To whatever assign'd Number of Terms the Series is carried; it is so far Finite; and if the greatest Term is l, the least A, and Ratio r, then the Sum is $S=\frac{rl-A}{r-1}$ (*Prob* 4, *Ch.* 3, *E.* 4.) Now, in a decreasing Series from l, the more Terms we actually raise, the last of them, A, becomes the lesser, and the lesser A

$rl - A$ is the greater, and so also is $\frac{rl - A}{r - 1}$ (for the greater the Dividend with the same Divisor, the greater is the Quote): But $rl - A$ being still less than rl, therefore $\frac{rl - A}{r - 1}$ is still less than $\frac{rl}{r - 1}$ *i. e.* the Sum of any assignable Number of Terms of the Series is still less than the Quote mention'd, which is $\frac{rl}{r - 1}$, and this the First Part of the Theorem.

Again: The Series may be actually continued so far, that $\frac{rl - A}{r - 1}$ shall want of $\frac{rl}{r - 1}$ less than any assignable Difference; for, as the Series goes on, A becomes less and less in a certain Ratio, and so the Series may be actually continu'd till A becomes less than any assignable Number (*Theor.* II.) Now $\frac{rl}{r - 1} - \frac{rl - A}{r - 1} = \frac{A}{r - 1}$ (by the common Rules) and $\frac{A}{r - 1}$ is less than A; therefore let any Number assign'd be call'd N, we can carry the Series so far till the last Term A be less than N: And because $\frac{rl - A}{r - 1}$ wants of $\frac{rl}{r - 1}$ the Differ. $\frac{A}{r - 1}$, which is less than A, which is also less than N, therefore the second Part of the Theorem is also true, and $\frac{rl}{r - 1}$ is the true Value of the Series.

Scholium. 1. The Sense in which $\frac{rl}{r - 1}$ is call'd *The Sum of the Series*, has been sufficiently explain'd; to which however I have this to add, That whatever Consequences follow from the Supposition of $\frac{rl}{r - 1}$ being the true and adequate Value of the Series taken in all its *Infinite* Capacity, as if the whole were actually determin'd and added together, can never be the Occasion of any assignable Error in any Operation or Demonstration where it is used in that sense; because if you say it exceeds that adequate Value, yet it's demonstrated, that this Excess must be less than any assignable Difference, which is in effect no Difference, and so the consequent Error will be in effect no Error: For if any Error can happen from $\frac{rl}{r - 1}$ being greater than it ought to be to represent the complete Value of the *Infinite Series*, that Error depends upon the Excess of $\frac{rl}{r - 1}$ over that complete Value; but this Excess being unassignable, that consequent Error must be so too; because still the less the Excess is, the less will the Error be that depends upon it. And for this Reason we may justly enough look upon $\frac{rl}{r - 1}$ as expressing the adequate Value of the *Infinite Series*. But we are further satisfied of the Reasonableness of this, by finding in Fact, that a Finite Quantity does actually convert into an *Infinite Series*, which we have already seen in the Case of *Infinite Decimals*. For Example; $\frac{2}{3} = .6666$, &c. which is plainly a Geometrical Series from $\frac{6}{10}$ in the Continual Ratio of $10 : 1$; for it is $\frac{6}{10} + \frac{6}{100} + \frac{6}{1000}$ &c.

And Reversely; If we take this Series, and find its Sum by the preceeding Theorem, it comes to the same $\frac{2}{3}$; for $l = \frac{6}{10}$, $r = 10$, therefore $rl = \frac{60}{10} = 6$; and $r - 1 = 9$; whence $\frac{rl}{r - 1} = \frac{6}{9} = \frac{2}{3}$

2. The

2. The ſame Variety of *Problems* may be made upon Infinite decreaſing Progreſſions, as before upon Finite Progreſſions; but with this conſiderable Difference, that in the *Infinites* the Number of Terms, and leaſt Term, depend ſo upon one another, that when the one is known, ſo is the other; for the Number of Terms is always Infinite (or greater than any aſſignable Number) and the leſſer Extream is o; [for there is not a leſſer Extream, it being inexhauſtible on the decreaſing ſide.] And hence all theſe Problems of *Finite Series*, wherein A and n are both given, will be unlimited, or capable of an infinite number of Anſwers in the *Infinites*. And in theſe, where only A, or n, is given, and the other ſought; then, becauſe that other is thereby alſo known in *Infinites*, there remains but one unknown Number to find. But we ſhall a little more particularly conſider them.

In *Probl.* 4, *Ch.* 2, *B.* 4. there are given A, l, r, to find S, n: But now if $A = o$, then is n Infinite; or if we call the *Series* Infinite, then A and n are both known; and becauſe $A = o$, there remain only l, r, by which to find S; as below.

In *Probl.* 5. there are given A, l, n to find S, r; but A being o, and n infinite, there remains only l to find S, r, which makes the Problem undetermin'd: For whatever l is, we may aſſume any Number greater than 1 for r, and then by r, l, find S, as below.

In *Probl.* 7. are given r, n with A or l, to find S with l or A: Now if r, n, A are given, alſo if $A = o$, we have only r to find S and l; and we aſſume l at pleaſure, and then find S.

Again; If r, n, l are given to find S, A, then if n is infinite, we have only S to find by r, l.

In *Problem* 6. are given A, L, S to find r, n; and if $A = o$, there remains only L, S to find r; as below.

In *Problem* 8. S, r, n are given to find A, L; and, if n is infinite, $A = o$, and we have S, r to find L; as below.

In *Problem* 9. we have r, S, with A or L, to find n with L or A: Now if r, S, A are given, and $A = o$, we find L by r, S: Again given r, S, L to find A, n. If you alſo ſay that the Series is Infinite, then there is nothing unknown, unleſs it be to examine whether all theſe Data are conſiſtent; and this we can do by taking r, L, S, and by them finding A, which being found equal to o, the Series to which the Data r, S, L belongs is truly Infinite, otherwiſe 'tis Finite.

Wherefore upon the Subject of Infinite, Decreaſing, Geometrical Progreſſions, all the Variety of determin'd Problems depends upon theſe three things, *viz.* the greateſt Term l, the Ratio r, and the Sum S; by any two of which the remaining one may be found: To which I ſhall joyn ſome other Problems, wherein $S - L$ is conſider'd as a diſtinct thing by it ſelf, *i. e.* without conſidering S and L ſeparately.

PROBLEM I.

Having l, r to find S, Rule. $S = \frac{rl}{r-1}$

Demon. This we have ſeen demonſtrated in *Theorem* 4.

Exa. $l = 6$, $r = 3$; then is $S = \frac{3 \times 6}{3 - 1} = \frac{18}{2} = 9$, the Series being $6 : 2 : \frac{2}{3} : \frac{2}{9} \cdot \frac{2}{27}$ &c.

Scholium. If the Ratio is a whole Number, we cannot expreſs the Rule for the Sum more ſimply than is done; but if it is a mix'd Number, or improper Fraction, then we may expreſs the Rule thus: Multiply the firſt Term l by the Numerator of the Ratio, and divide the Product by the Difference of the Numerator and Denominator. *Exa.* If the Ratio,

Ratio is $\frac{a}{b}$ then $S = \frac{a\,l}{a-b}$, the Reason of which is contain'd in the other Rule; for if $r = \frac{a}{b}$, then $r\,l = \frac{a\,l}{b}$, and $r - 1 = \frac{a}{b} - 1 = \frac{a-b}{b}$. Hence again, $\frac{r\,l}{r-l} = \frac{a\,l}{b} \div \frac{a-b}{b} = \frac{a\,l}{a-b}$. And by taking any Integer r fractionally thus, $\frac{r}{1}$ this Rule does comprehend the other Case also.

Observe, From this last expression of the Rule we also learn, that the Sum may be found by the first and second Term, thus: The second Term being B, the Ratio is $\frac{L}{B}$ and $S = \frac{L^2}{L-B}$, *that is*, the Sum is a 3*d* : : l to the Difference of the first and second Term, and the first Term; for because $S = \frac{L^2}{L-B}$, therefore $L-B : L :: L : S$.

COROLLARIES.

1. If the Ratio is a whole Number, the Sum is such an improper Fraction of the first Term, whose Numerator is the Ratio, and its Denominator the Ratio less Unity; for $\frac{r\,l}{r-1} = \frac{r}{r-1}$ of l: Particularly, if the Ratio is 2, the Sum is $2\,l$. If $r = 3$, then $S = \frac{3}{2}$ of l. If $r = 4$, then $S = \frac{4}{3}$ of l; and so on. Whence you see, that with no other Integral Ratio but 2 will the Sum be Multiple of l; for no Number but 1 can be contain'd precisely a certain number of times without a Remainder, in a Number which exceeds it only by 1; for in all such Cases the Quote is 1, and there is a Remainder of 1, thus, $\overline{A+1} \div A = 1$, and 1 remains. Hence again,

2. If the first Term is a Fraction, and the Ratio equal to the Denominator of it, then is the Sum equal to such an *aliquot* part of the Numerator, whose Denominator is $r - 1$. Thus, if the first Term is $\frac{A}{r}$ and the Ratio r, then is $S = \frac{1}{r-1}$ of A. Wherefore, lastly, if the first Term is $\frac{A}{r}$ and $A = r - 1$, then is $S = 1$.

3. If the Ratio is a mix'd Number (or its Equivalent improper Fraction) the Sum is such an improper Fraction of l, whose Numerator is that of the Ratio, and Denominator the Difference of the Numerator and Denominator of the Ratio; thus, if $r = \frac{a}{b}$, then is $S = \frac{a}{a-b}$, of l; for it is $\frac{a\,l}{a-b}$, by what's shewn in the preceeding *Scholium*. But $\frac{a\,l}{a-b} = \frac{a}{a-b}$ of l. Hence again,

4. If the Numerator of a mix'd Ratio is Multiple of the Difference betwixt the Numerator and Denominator, (or if the first Term is Multiple of its Excess over the second Term) the Sum is Multiple of the first Term.

And *Observe*, that this can happen in no Case, but when the Numerator and Denominator of the lowest Terms of the Ratio differ only by 1, (as indeed every Ratio is in its lowest Terms when the Numerator and Denominator differ only by 1) in which Case it is manifest that the Sum is equal to the Product of the first Term, multiply'd by the Numerator of the Ratio, thus: Let the Ratio be $\frac{a}{a-1}$, then is $S = \frac{a}{1}$ of l, or $a\,l$, by what's

already

already shewn (*Corol.* 3.) In Numbers, if $r = \frac{5}{4}$, then $S = 5\,l$. Again, let the Numerator exceed the Denominator by more than 1; as, suppose it $\frac{a}{a - n}$ then is $S = \frac{a}{n}$ of l, (*Corol.* 3.) Now, if the Sum is Multiple of l, then is $\frac{a}{n}$ an Integer.

Let $\frac{a}{n} = b$, then $a = b\,n$, Consequently the Ratio is $\frac{n\,b}{n\,b - n} = \frac{b}{b - 1}$, which is in its lowest Terms, because the Numerator and Denominator differ by 1. And this shews us, that as any mix'd Ratio may be expres'd $\frac{a}{a - n}$, so, if the Sum is Multiple of the first Term, the lowest Terms of the Ratio is a Fraction whose Numerator and Denominator differ only by 1; and therefore no other kind of mix'd Ratio's can have this Effect, since every Ratio having this Effect is of that kind whose lowest Terms differ by 1.

PROBLEM II.

Having S and r to find l, Rule. $l = \frac{S \times \overline{r - 1}}{r}$

Demon. By the last, $S = \frac{r\,l}{r - l}$ therefore $r\,l = S \times \overline{r - 1}$, and $l = \frac{S \times \overline{r - 1}}{r}$

Exa. $S = 9$, $r = 3$, then is $l = \frac{9 \times 2}{3} = 6$

Scholium. If the Ratio is $\frac{a}{b}$, then is $l = \frac{S \times \overline{a - b}}{a}$, for $S = \frac{a\,l}{a - b}$ by the last; so that $S \times \overline{a - b} = a\,l$, and therefore $l = \frac{S \times \overline{a - b}}{a}$

COROLLARIES.

1. If r is an Integer, l is such a proper Fraction of the Sum whose Numerator is $r - 1$, and its Denominator is r; thus, $l = \frac{r - 1}{r}$ of S; for $\frac{S \times \overline{r - 1}}{r} = \frac{r - 1}{r}$ of l.

2. If the Ratio is a mix'd Number, as $\frac{a}{b}$, then is L such a Fraction of S, whose Numerator is $a - b$, and its Denominator a; thus, $l = \frac{a - b}{a}$ of S: For $l = \frac{\overline{a - b} \times S}{a}$ (by *Schol.*) $= \frac{a - b}{a}$ of S.

Observe; l is an aliquot Part of S only when the Ratio is 2; or in case of a mix'd Ratio, when the lowest Terms of it differ only by 1, this being the Reverse of what is above demonstrated concerning S being a Multiple of l.

PROBLEM III.

Having S and l to find r. Rule. $r = \frac{s}{s - l}$

Demon. By *Probl.* 1, $S = \frac{r\,l}{r - 1} = \frac{r}{r - 1} \times l$; Hence $S : l :: \frac{r}{r - 1} : 1 :: r : r - 1$

And

And again divisively, $s - l : s :: r - \overline{r - 1}\ (= 1) : r$. Hence $r = \frac{s}{s - l}$.

Exa. $S = 9$, $l = 6$, and $r = \frac{9}{9 - 6} = 3$.

PROBLEM IV.

Having l and r to find $s - l$, without finding first s, (as might be done by *Probl.* 1)

RULE. $S - l = \frac{l}{r - 1}$

DEMON. $S = \frac{r\,l}{r - 1}$ hence $S \times \overline{r - 1} = r\,l$, and $S : l :: r : r - 1$, and $s - l : l :: r - \overline{r - 1}\ (= 1) : r - 1$. Wherefore $S - l = \frac{l}{r - 1}$

Exa. $l = 6$, $r = 3$, and $S - l = \frac{6}{2} = 3$

SCHOL. If the Ratio is $\frac{a}{b}$, then is $S - l = l \div \frac{a - b}{b} = \frac{b\,l}{a - b}$ for $\frac{a}{b} - 1 = \frac{a - b}{b}$

COROLLARIES.

1. If the Ratio is 2, then $s - l = l$; but in all other Cases $s - l$ is either greater or lesser than l. Reversely, if r is lesser than 2, or if $\overline{r - 1}$ is lesser than 1, then is $l \div \overline{r - 1}\ (= s - l)$ greater than l. Again; if r is greater than 2, $r - 1$ is greater than 1; and hence $l \div r - 1\ (= s - l)$ is less than l.

2. If the Ratio is a whole Number greater than 2, $s - l$ is such an aliquot Part of l, as $r - 1$ denominates; i.e. $S - l = \frac{1}{r - 1}$ of l.

But *Observe* also, that $s - l$ can never be an aliquot Part of l, except when r is a whole Number greater than 2: It must be greater than 2, else $s - l$ is not less than l; as we saw in the last *Corollary*. Again; it must be an Integer: For suppose it a mix'd Number, as $2 + \frac{a}{n} = \frac{2n + a}{n}$ (which may represent any Number greater than 2, according as we take $\frac{a}{n}$) Then is $r - 1 = \frac{2n + a}{n} - 1 = \frac{n + a}{n}$ So that $s - l = l \div \overline{r - 1} = l \div \frac{n + a}{n} = \frac{n}{n + a}$ of l. But if $\frac{n}{n + a}$ is an aliquot Part, then is n an aliquot Part of $n + a$; or n measures $n + a$, and because n measures n and $n + a$, therefore also it measures a; so that $\frac{a}{n}$ is an Integer; and $2 + \frac{a}{n}$ [the Ratio which makes $s - l$ an aliquot Part of l] is an Integer contrary to Supposition: Wherefore no mix'd Ratio can make a Series in which $s - l$ is an aliquot part of l.

3. If the Ratio is a mix'd Number $\frac{a}{b}$, whether greater or lesser than 2, $s - l$ is such a Fraction of l as $\frac{b}{a - b}$ expresses, i.e. $S - l = \frac{b}{a - b}$ of l; for, by the preceeding *Schol.* $S - l = \frac{b\,l}{a - b} = \frac{b}{a - b}$ of l. Whence again $s - l$ is a Multiple of l only in such Cases

wherein

wherein the loweſt Terms of the Ratio differ by 1. For ſuppoſe that is $\frac{a}{a-1}$ then is $s - l = \overline{a-1} + l$.

4. Hence again we learn another *Rule* for finding s by l and r; *viz.* firſt finding $s - l = \frac{l}{r-1}$, and then adding l to it, thus, $S = \frac{l}{r-1} + l$ (which added by the common Rules, comes to the former Rule $S = \frac{r\,l}{r-1}$)

SCHOLIUM. This *Problem* may be expreſs'd alſo in this manner, *viz.* of any Quantity l, let a certain Fraction, as $\frac{b}{a}$, be taken; then the ſame Fraction of this Fraction, and ſo on, continually taking the ſame Fraction of the preceeding; Then will the Sum of a Series of Quantities equal reſpectively to theſe ſeveral Fractions of l, be equal to $\frac{b}{a-b}$ of l; for the Series of the Fractions is $\frac{b}{a}$ of l : $\frac{b}{a}$ of $\frac{b}{a}$ of l, &c. or thus, $\frac{b\,l}{a}$, : $\frac{b^2\,l}{a^2}$: $\frac{b^3\,l}{a^3}$ &c. which is a Geometrical Series decreaſing in the Ratio $\frac{a}{b}$, whoſe Sum is therefore by the preceeding Rule, $\frac{b}{a-b}$ of l, or $\frac{b\,l}{a-b}$.

In what Caſes $\frac{b\,l}{a-b}$ is equal to l, or greater or leſſer, alſo when it is Multiple, or an aliquot Part of l, has been already explain'd.

PROBLEM V.

Having s and r to find $s - l$ (without finding firſt l, as we may do by *Problem* 2.)

RULE. $s - l = \frac{s}{r}$

DEMON. By *Probl.* 3d, $r = \frac{s}{s-l}$. Hence $r \times \overline{s-l} = s$, and $s - l = \frac{s}{r}$

Exa. $s = 9$, $r = 3$, and $s - l = \frac{9}{3} = 3$

SCHOLIUM. If we expreſs the Ratio thus, $\frac{a}{b}$, then $s - l = s \div \frac{a}{b} = \frac{s\,b}{a}$

COROLLARIES.

1. If r is an Integer, $s - l$ is an aliquot Part of s, *viz.* $\frac{1}{r}$ of s; but in all other Caſes $s - l$ is ſuch a proper Fraction of s as the Reciprocal of the Ratio expreſſes. So the Ratio being $\frac{a}{b}$, then $s - l = s \div \frac{a}{b} = \frac{b}{a}$ of s; for 'tis $\frac{b\,s}{a}$ by the *Schol.*

2. Hence we have another *Rule* for finding l by s and r; *viz.* Find $s - l = \frac{s}{r}$ and then ſubſtract $\frac{s}{r}$ from s, for $s - \overline{s-l} = l$; Wherefore $l = s - \frac{s}{r}$ which ſubſtracted, by the *Common Rules*, comes to the ſame expreſſion as before, *viz.* $\frac{s \times \overline{r-1}}{r}$

PROBLEM VI.

Having $r, s-l$ to find s, l, RULE. $s = r \times \overline{s-l}$. Then take $s-l$ from s, the Remainder is l; or find l independently of s, thus, $l = \overline{s-l} \times \overline{r-1}$.

DEMON. By the laſt $s-l = \frac{s}{r}$. Hence $s = r \times \overline{s-l}$. Again, by *Prob.* 4, $s-l = \frac{l}{r-1}$. Hence $l = \overline{s-l} \times \overline{r-1}$.

Exa. $r = 3$, $s-l = 3$: And $s = 3 \times 3 = 9$: Alſo $l = 3 \times 2 = 6$.

SCHOLIUM. If the Ratio is $\frac{a}{b}$, then is $s = \frac{\overline{s-l} \times a}{b}$. And, becauſe $r-1 = \frac{a-b}{b}$, hence $l = \frac{\overline{s-l} \times \overline{a-b}}{b}$.

COROL. If the Ratio is an Integer, then is s a Multiple of $s-l$; but the Ratio being a mix'd Number $\frac{a}{b}$, s is a mix'd Multiple of $s-l$, expreſs'd by the Ratio; *that is*, $s = \frac{a}{b}$ of $s-l$; and $l = \frac{a-b}{b}$ of $s-l$.

As for the different Caſes in which l is equal to $s-l$, or an aliquot Part or Multiple thereof, ſee *Corol.* 1, 2, 3. *Prob.* 4. where it's ſhewn in what Caſes $s-l$ is equal to l, or is a Multiple, or aliquot Part thereof.

TABLE *of the preceeding* Problems.

Given	Sought	Solutions.		
r, l	s	$s = \frac{rl}{r-1}$	Or ſuppoſing the Ratio $\frac{a}{b}$, or the 2d Term to be M, whereby the Ratio is $\frac{l}{M}$, then is.	$s = \frac{a}{a-b}$ of $l = \frac{l^2}{l-M}$
r, s	l	$l = \frac{s \times \overline{r-1}}{r}$		$l = \frac{a-b}{a}$ of $s = \frac{\overline{l-M} \times s}{l}$
l, s	r	$r = \frac{s}{s-l}$		
l, r	$s-l$	$s-l = \frac{l}{r-1}$		$s-l = \frac{b}{a-b}$ of $l = \frac{Ml}{l-M}$
s, r	$s-l$	$s-l = \frac{s}{r}$		$s-l = \frac{b}{a}$ of $s = \frac{Ms}{l}$
$r.\ s-l$	s, l	$\begin{cases} s = \overline{s-l} \times r \\ l = \overline{s-l} \times \overline{r-1} \end{cases}$		$s = \frac{a}{b}$ of $s-l = \frac{l \times \overline{s-l}}{M}$ $l = \frac{a-b}{b}$ of $s-l = \frac{\overline{l-M} \times \overline{s-l}}{M}$

THEO-

THEOREM V.

A Series of Numbers may encreafe continually, yet fo that no Term of it can ever be fo great as a certain affignable Number.

Alfo a *Series* may decreafe continually, yet fo that no Term of it can ever be fo fmall as a certain affignable Number.

DEMON. 1°. Take an *Infinite Series* decreafing from any Number l, in any conftant Ratio r, as, $l : \frac{l}{r} \therefore \frac{l}{r^2} \therefore \frac{l}{r^3}$ Then take the Sums of this Series, repeated always from the beginning, thus, $l : l + \frac{l}{r} \therefore l + \frac{l}{r} + \frac{l}{r^2} \therefore l + \frac{l}{r} + \frac{l}{r^2} + \frac{l}{r^3} \therefore$ &c. This is fuch an encreafing Series as was propos'd; for the Sum of the decreafing Series $l + \frac{l}{r}$ &c is limited to $\frac{r\,l}{r-1}$ (*Theor.* 4.) therefore no Term of the Series of its Sums can ever actually reach to $\frac{r\,l}{r-1}$.

2°. Take a decreafing Series $l : \frac{l}{r}$, &c. and fuppofe it fuch that all the Series wanting the firft Term is lefs than the firft Term; as it will be if r is greater than 2: (*Corol.* 1, *Prob.* 4.) then fubftract the fecond Term from the firft, and the third from the remainder, and fo on, and make a Series of thefe remainders, beginning with l, thus, $l : l - \frac{l}{r} \therefore l - \frac{l}{r} - \frac{l}{r^2} \therefore$ &c. it is fuch a decreafing Series as was propos'd; for the Sum of the Series $l : \frac{l}{r}$: &c. wanting then the firft term l, *i. e.* the Sum of the Series $\frac{l}{r} \therefore \frac{l}{r^2} \therefore \frac{l}{r^3} \therefore$ &c. is $\frac{l}{r-1}$ which is fuppos'd lefs than l; but fince that Series can never be exhaufted, it's plain, That no Term in the other Series, $l : l - \frac{l}{r}$ &c. can ever be fo fmall as the Difference betwixt l and $\frac{l}{r-1}$, becaufe we can never actually fubftract fo much from l as $\frac{l}{r-1}$, the Sum of the *Infinite Series* of Numbers fubftracted.

SCHOLIUMS.

1. We may take any Number A to begin a *Series*, and add to it fucceffively the Terms of any decreafing Series, $l : \frac{l}{r} \therefore \frac{l}{r^2}$ &c. thus, $A : A + l : A + l + r$, &c. and this will be fuch an encreafing Series as was propos'd.

Again; we may take any Number A greater than the Sum of any decreafing Series $l : \frac{l}{r}$ &c. and fubftracting this out of A, make this Series $A : A - l : A - l - \frac{l}{r}$ &c. it will be fuch a decreafing *Series* as was propos'd. But to begin with the firft Term of the decreafing Series $l : \frac{l}{r}$ &c. makes a more regular *Series*.

2. Thefe *Infinite Series* encreafing or decreafing limitedly, are not, and cannot be, in a conftant Ratio; for then they could not be limited, by *Theor.* 1 & 2. And far lefs do the Ratio's of each Term to the preceeding encreafe; for then they would fo much fooner than with an equal Ratio outreach any propos'd Limitation: Wherefore 'tis plain, that of a *Series* encreafing for ever, but limitedly, the Series of its Ratio's, comparing each

Term to the preceeding, is a Series of Numbers which do continually decrease: and of a Series decreasing for ever, but limitedly; the Series of its Ratio's, comparing each Term to the preceeding, is a Series of Numbers which do continually encrease. But *observe* again, that tho' of a *Series* whose Terms encrease continually, but limitedly, the Ratio's of each Term to the preceeding do continually decrease, yet the Reverse will not hold; *that is*, tho' a Series of encreasing Numbers is such that the Ratio's of each Term to the preceeding continually decrease, yet it does not follow that the *Series* encreases under a Limitation; which one Example demonstrates: For if we take any Encreasing Arithmetical Progression, as 1, 2, 3, 4, *&c.* its Ratio's, comparing each Term to the preceeding, are $\frac{2}{1} : \frac{3}{2} : \frac{4}{3}$ *&c.* which do continually decrease, and will do so in all such Cases (as has been shewn *Cor. Theor.* 17, Ch. 5, B. 4.) yet we can find a Term in the *Series* greater than any assignable Number. The like is also true as to a decreasing Series, *viz.* That they may decrease by encreasing Ratio's of each Term to the preceeding, and yet be unlimited in their decrease; of which we have Examples, by taking the reciprocal Ratio's of any *Arithmetical Series*, and making a *Series* decreasing according to these Ratio's; as in this *Example*, $1 : \frac{1}{2} : \frac{1}{3} : \frac{1}{4} : \frac{1}{5}$, *&c.* where the Ratio's of each Term to the preceeding are $1:2,\ 2:3,\ 3:4$.

COROL. A *Series* may decrease continually, and yet have a Sum infinitely great: For if it decreases Limitedly, then its Sum will be always greater than as many Terms equal each to the Limiting Number, since none of the Terms can ever be so little as that Number.

THEOREM VI.

Take any Geometrical Progression encreasing from 1, which may be universally represented, thus, $1 : r : r^2 : r^3 :$ &c. then

1°. Take the Sums of this *Series* continually from the beginning, thus, $1 : 1+r : 1+r+r^2 :$ &c, and divide each of these Sums by the last Term added in each, thus, $1 : \frac{1+r}{r} : \frac{1+r+r^2}{r^2}$ &c. and this is a *Series* encreasing limitedly, so as no Term can ever reach to $\frac{r}{r-1}$.

2°. Instead of the Sums, substract from each Term of the $\div\, l$ (*that is*, from each Power of r) all the preceeding Terms, and make this Series $1 : \frac{r-1}{r} : \frac{r^2-r-1}{r^2} : \frac{r^3-r^2-r-1}{r^3} :$ &c. 'Tis a *Series* decreasing limitedly, in which no Term can ever reach to $\frac{r-2}{r-1}$.

3°. Take the *Series* of the Reciprocals of the first Series, *viz.* take $1 : \frac{r}{1+r} : \frac{r^2}{1+r+r^2} :$ &c. 'tis a *Series* decreasing limitedly, in which no Term can ever reach to $\frac{r-1}{r}$.

4° Take

4°. Take the Series of the Reciprocals of the second Series, *viz.* take $1 : \frac{r}{r-1} : \frac{r^2 - r - 1}{r^2}$: &c. 'tis a Series encreasing limitedly, in which no Term can ever reach to $\frac{r-1}{r-2}$.

DEMON. 1°. For the first Series, $1 : \frac{1+r}{r} : \frac{1+r+r^2}{r^2}$: &c. 'tis no other than the sums of this decreasing Series, $1 : \frac{1}{r} : \frac{1}{r^2}$ &c. added by the Common Rules, thus, $1 + \frac{1}{r} = \frac{1+r}{r}$ and $\frac{1+r}{r} + \frac{1}{r^2} - \frac{1+r+r^2}{r^2}$: But the sum of this Series $1 : \frac{1}{r} : \frac{1}{r^2}$: &c. is $\frac{r}{r-1}$. Therefore,

2°. For the second Series, $1 : \frac{r-1}{r} : \frac{r^2 - r - 1}{r^2}$: &c. it is no other than the Effect of taking every Term after the first of this Series, $1 ; \frac{1}{r} : \frac{1}{r^2}$ &c. out of the first, and out of the succeeding Remainders; for $1 - \frac{1}{r} = \frac{r-1}{r}$, and $\frac{r-1}{r} - \frac{1}{r^2} = \frac{r^2 - r - 1}{r^2}$ &c. and the Sum of the Series $\frac{1}{r} : \frac{1}{r^2} : \frac{1}{r^3}$: &c. is $\frac{1}{r-1}$, which therefore can never be all taken away from 1; whence it's plain, that no Term of this *Series* of *Remainders* can ever reach so low as $1 - \frac{1}{r-1} = \frac{r-2}{r-1}$. And *Observe*, that we have in this Conclusion a plain Demonstration that the Numerators in each Term of the Series: $1 : \frac{r-1}{r} : \frac{r^2 - r - 1}{r^2}$ &c. are positive Numbers, *i. e.* that any Power of r (a whole or mix'd Number) is greater than the Sum of all the preceeding Terms, $+ 1$; for the Sum of the whole Series, $\frac{1}{r} : \frac{1}{r^2}$ &c. is $\frac{1}{r-1}$, which is less than 1, ;and therefore each Term of the Series $1 : \frac{r-1}{r}$ &c. is a real positive Number; which it cannot be, unless the Numerator is a positive Number, *i. e.* unless r^2 is greater than all the preceeding Terms. But this Truth may be also demonstrated from the Rules of a Geometrical Progression $1 . r . r^2$, &c.

3°. For the third Series, $1 : \frac{r}{1+r} : \frac{r^2}{1+r+r^2}$ &c. it decreases limitedly, so as never to reach to $\frac{r-1}{r}$; because if it did reach that Number, then 'tis evident that the Reciprocal Series $1 : \frac{1+r}{r} : \frac{1+r+r^2}{r^2}$ must reach to the reciprocal Number $\frac{r}{r-1}$; which it can never do by Article the First.

4°. For the fourth Series, $1 : \frac{r}{r-1} ; \frac{r^2}{r^2 - r - 1}$ &c. because 'tis the Reciprocal of the second, which can never decrease to $\frac{r-2}{r-1}$, therefore this encreases so, as it can never reach to $\frac{r-1}{r-2}$; for if it could, the other would decrease to $\frac{r-2}{r-1}$, which is shewn to be impossible in Article second.

Scho-

Scholiums.

1. If we compare the two Encreaſing Series, $1 : \frac{1+r}{r}$ &c. and $1 : \frac{r}{r-1}$ &c. then each Term of the firſt (after 1) is leſſer than the Correſponding Term of the ſecond; thus $\frac{1+r}{r}$ is leſſer than $\frac{r}{r-1}$; and ſo on. To *Demonſtrate* which univerſally, let any Number A be Denominator of a Fraction, and $A+B$ the Numerator, thus, $\frac{A+B}{A}$; alſo make A Numerator, and $A-B$ the Denominator, thus, $\frac{A}{A-B}$; then is $\frac{A+B}{A}$ leſs than $\frac{A}{A-B}$: For if we reduce them to a common Denominator, they are $\frac{A^2-B^2}{A^2-AB}\left(=\frac{A+B}{A}\right)$ and $\frac{A^2}{A^2-AB}\left(=\frac{A}{A-B}\right)$; whereby it's plain the firſt is leſs than the ſecond, becauſe A^2-B^2 is leſs than A^2: But A may repreſent any Power of r, as r^n, and B the Sum of all the preceeding Terms of the *Series*, (which Sum is always leſs than r^n); ſo that $\frac{A+B}{A}$ may repreſent any Term of the Series $1 : \frac{1+r}{r}$: &c. and $\frac{A}{A-B}$ any Term of the Series $1 : \frac{r}{r-1}$: &c. which finiſhes the Demonſtration.

Hence the Value of the firſt Series is leſs than that of the other, (*i. e.* the Value of any aſſignable Number of Terms of the one compar'd to as many of the other.)

2d. Compare the two Decreaſing Series $1 : \frac{r}{1+r}$ &c. and $1 : \frac{r-1}{r}$ &c. and each Term of the firſt (after the 1) is greater than the Correſponding of the other, *viz.* $\frac{r}{1+r}$ greater than $\frac{r-1}{r}$ &c. becauſe the Reciprocal of that is leſſer than of this; *i. e.* $\frac{1+r}{r}$ leſs than $\frac{r}{r-1}$ (from the nature of Fractions). Or we may demonſtrate this the ſame Way as the former; for $\frac{A}{A+B}$ is greater than $\frac{A-B}{A}$, becauſe being reduced to one Denominator, they are $\frac{A^2}{A^2+AB}=\frac{A}{A+B}$ and $\frac{A^2-B^2}{A^2+AB}=\frac{A-B}{A}$: Hence the Value of the firſt is greater than that of the other.

3d. If we take the Ratios of theſe ſeveral *Series*, comparing each Term to the preceeding, they make theſe regular Series:

$$\frac{1+r}{r} : \frac{1+r+r^2}{r+r^2} : \frac{1+r+r^2+r^3}{r+r^2+r^3}\text{ \&c. for the Series, } 1 : \frac{1+r}{r} : \frac{1+r+r^2}{r^2}\text{ \&c}$$

$$\frac{r-1}{r} : \frac{r^2-r-1}{r^2-r} : \frac{r^3-r^2-r-1}{r^3-r^2-r}\text{ \&c. for the Series, } 1 : \frac{r-1}{r} : \frac{r^2-r-r}{r^2}\text{ \&c.}$$

$$\frac{r}{1+r} : \frac{r+r^2}{1+r+r^2} : \frac{r+r^2+r^3}{1+r+r^2+r^3}\text{ \&c. for the Series, } 1 : \frac{r}{1+r} : \frac{r^2}{1+r+r^2}\text{ \&c}$$

$$\frac{r}{r-1} : \frac{r^2-r}{r^2-r-1} : \frac{r^3-r^2-r}{r^3-r^2-r-1}\text{ \&c. for the Series, } 1 : \frac{r}{r-1} : \frac{r^2}{r^2-r-1}\text{ \&c}$$

Where

Where you see the third and fourth are Reciprocals of the first and second, because the Series of which they are the Ratios, are so.

THEOREM VII.

Let there be two *Infinite Series* of Numbers $\div l$; And let their corresponding Terms be multiplied together, *i. e.* the first Term of the one by the first of the other, and so on; The products make an *Infinite Series*, $\div l$ whose Sum is in some cases Infinite, in others Finite. Thus,

1°. If each Series consists of Terms equal among themselves, or the one having equal Terms, and the other encreasing; or, lastly, both encreasing, the Sum of the Product is Infinite.

2°. If both Series decrease, or if the one Series has equal terms, and the other Decreases, the sum of the products is Finite; notwithstanding the Sum of the Series of equal Terms is Infinite.

3°. If the one encreases and the other decreases, the Sum of the Products is in some cases Infinite, and in some Finite, notwithstanding the Sum of the encreasing one be always Infinite: particularly, if the Ratio of the encreasing Series is equal to or lesser than the Reciprocal Ratio of the decreasing one, the sum of the products is Infinite; but if it's greater, the Sum is Finite.

DEMON. By *Theorem* 3. *Ch.* 4, *B* 4. The Series of Products is $\div l$ in the Ratio compounded of those of the Series Multiplied. So the Ratio of the one Series is

$$\left.\begin{array}{l} A : B : C : D : \&c. \\ L : M : N : O : \&c. \\ \hline AL : BM : CN : DO : \&c. \\ \hline \end{array}\right\}$$

$\frac{A}{B}$, and of the other it is $\frac{L}{M}$, whose product is $\frac{AL}{BM}$ the Ratio of the Series of Products; And because that Series either consists of equal terms, or it encreases or decreases in a constant Ratio, therefore it will accordingly have either a Finite or Infinite Sum: So that what remains to be shewn is only the Correspondence of the several Cases, in which it encreases or decreases or is equal, to the Theorem; which will easily appear thus, 1°. If both the Series encrease, or both consist of equal Terms, or the one equal, and the other encreasing, it's evident the Sum of any one of them is Infinite (by *Theorem* 3.) and much more is the Series of their Products so. Or thus, the two Ratio's $\frac{A}{B}$, $\frac{L}{M}$ are either both proper Fractions, or the one so, and the other equal to 1, or both equal to 1: Wherefore $\frac{AL}{BM}$ is either a proper Fraction, so that the Series of Products encreases in that constant Ratio, and the Sum is Infinite *(Theorem* 3.*)*; or it is equal to 1, and so the Products are equal, and here also the Sum is Infinite, *(Theorem* 3.*)*

2°. If both the Series decrease, or one decreases, and the other equal; then $\frac{A}{B}$ and $\frac{L}{M}$ are improper Fractions, and hence $\frac{AL}{BM}$ is an improper Fraction: Therefore the Series of Products decreases in that constant Ratio, and so the Sum is Finite *(Theorem* 4.*)*

3°. If the one Series encreases, and the other decreases; We shall suppose that $A : B : C :$ encreases, so that $\frac{A}{B}$ is a proper Fraction; and that $L : M : N :$ &c. decreases, so that $\frac{L}{M}$ is an improper Fraction: Now the Ratio of the compound Series $AL : BM$, &c.

is

is $\frac{AL}{BM}$, which, I say, is a proper Fraction when $\frac{A}{B}$ is less than $\frac{M}{L}$, but is an improper Fraction when $\frac{A}{B}$ is equal to, or greater than $\frac{M}{L}$; for these Fractions reduced to one Denominator, are $\frac{AL}{BL}\left(=\frac{A}{B}\right)$ and $\frac{MB}{LB}\left(=\frac{M}{L}\right)$ Wherefore, as AL is equal to, greater or lesser than MB, consequently $\frac{AL}{BM}$ (the Ratio of the Compound Series) is either $=1$; if $AL=BM$ (*i.e.* if $\frac{A}{B}=\frac{M}{L}$) and in this Case the Compound Series consists of equal Terms, and so the Sum is Infinite: Or $\frac{AL}{BM}$ is a proper Fraction, if AL is less than BM (*i. e.* if $\frac{A}{B}$ is less than $\frac{M}{L}$) and then the Series encreases, so that the Sum is here also Infinite: Or, lastly, $\frac{AL}{BM}$ is an improper Fraction, if AL is greater than BM, (i. e. $\frac{A}{B}$ greater than $\frac{M}{L}$) and so the *Series* decreases, and the Sum is consequently *Finite.*

Scholiums.

1. If we suppose each Term of the one *Series* is multiply'd into each Term of the other, the Sum of the Products will be *Infinite* in all Cases, except when the two *Series* do both decrease: for the Sum of all these Products is the Products of the Sums of the two *Series*; and if but any one of them is Encreasing, or Equal, its Sum is *Infinite*; Which therefore multiply'd into the other Sum, whether *Finite* or *Infinite*, must make an *Infinite Product*: but both the *Series* decreasing, their Sums are *Finite*, and consequently their Product is *Finite.*

2. Tho' we have only suppos'd *Series* encreasing or decreasing in one constant Ratio, yet we may consider other Kinds of *Series*: As, *first*, we may suppose any Kind of *Series* whose Terms continually Encrease, whether in one constant Ratio, or not; and any such *Series* being put instead of one which is $\div l$, in the first Article of the *Theorem*, the Conclusion will be the same, as is most evident, tho' the *Compound Series* will not be $\div l$. Again, in the second Article of the *Theorem*, we may suppose a *Series* decreasing limitedly, instead of one in $\div l$; and if two such be multiply'd together, or one such with a Series of equal Terms, the *Compound Series* will certainly have an Infinite Sum; because each Term of this suppos'd Series being greater than the Limiting Number, must have a greater Effect than a Series of equal Terms equal to that Number; but such a limited Series multiply'd into a Series decreasing in one Ratio, will have a Finite Sum, because the Series decreasing limitedly will have a lesser Effect than a Series of equal Terms equal to the first Term of this Series, which would make a Finite Sum in the Series of Products. Again, suppose two Series decreasing in Ratio's that do also decrease, (comparing each Term to the preceeding) the Sum of the Products is Finite; which it is also if one of the Series is such, and the other equal or decreasing limitedly. In the third Article of the *Theorem* let us suppose, first, a Series encreasing so, that the Reciprocal Ratio's do continually encrease, (and so the Ratio's themselves decrease) and another decreasing by a Constant Ratio; then, if the first Ratio of the Encreasing Series is equal to, or less than the Reciprocal of the decreasing one, the Sum of the Products is certainly Infinite; for they will make a Series encreasing, and whose reciprocal Ratio's will also encrease, because each Ratio of that

that *Series* will be gradually less and less than the reciprocal Ratio of the other: But tho' the first Ratio of the encreasing Series is greater than the reciprocal Ratio of the decreasing one, it will not be true that the Sum of the Products is Finite, unless all the following Ratios of the encreasing Series be also greater. Again, *secondly*, suppose a Series encreasing in a Constant Ratio, and another decreasing, so that the reciprocal Ratios do decrease, (and so the Ratio's themselves encrease) then if the Ratio of the encreasing Series is greater than the Reciprocal of the first Ratio of the decreasing one, it will be greater than all the following ones; and consequently the Series of Products will decrease in Ratios whose Reciprocals decrease, and so the Sum will be Finite. And if the Ratio of the Encreasing Series be equal to the first Reciprocal Ratio of the other, then it will be greater than all the following ones (which decrease); and here again the Sum of the Products will be Finite. But tho' the Ratio of the Encreasing Series be less than the first reciprocal Ratio of the other, the sum of Products is not Infinite, unless it be also less than each of the rest of the reciprocal Ratios of the other Series; for it may become equal to some one of them, or greater, and then the Sum of the Product will be Finite. *Thirdly*, If the one *Series* encreases by Ratios that decrease (or whose Reciprocals encrease) and the other by Ratios whose Reciprocals decrease, then if each Ratio of the first Series is equal to, or less than, the reciprocal Ratio of the Correspondent Terms of the second Series, the sum of the Products is Infinite; but if greater, the Sum is Finite. *Fourthly*, If the one Series encreases in whatever manner, and the other decreases limitedly, the Sum of the Products will be Infinite; because this decreasing Series is greater than a Series of equal Terms, equal to the limiting Number.

THEOREM VIII.

If from any quantity A we take away any proper Fraction of it, as $\frac{a}{b}$; and then of what remains take away rhe same Fraction, and so on continually; the Sum of the *Infinite Series* of Parts taken will be equal to the whole A, and the Sum of the *Infinite Series* of the Parts left at every Substraction is equal to $\frac{b-a}{a}$ of A.

$$\begin{cases}\text{Taken.} & \dfrac{aA}{b} : \dfrac{a\times\overline{b-a}\times A}{b} : \dfrac{a\times\overline{b-a}^2\times A}{b\times b^2} : \dfrac{a\times\overline{b-a}^3\times A}{b\times b} : \&c \\[2ex] \text{Left.} & \dfrac{\overline{b-a}\times A}{b} : \dfrac{\overline{b-a}^2\times A}{b^2} : \dfrac{\overline{b-a}^3\times A}{b^3} : \dfrac{\overline{b-a}^4\times A}{b^4} : \&c\end{cases}$$

DEMON. The Series of the Parts *taken* and *left* are evidently these express'd above; for the first Part taken being $\frac{a}{b}$ of A $(=\frac{aA}{b})$ what remains is $\frac{b-a}{b}$ of A, $\left(=\frac{\overline{b-a}\times A}{b}\right)$ for $\frac{a}{b}+\frac{b-a}{b}=\frac{b}{b}=1$. Then of this Remainder we take again $\frac{a}{b}$ Parts, which makes $\frac{a}{b}$ of $\frac{b-a}{b}$ of $A\left(=\frac{a\times\overline{b-a}\times A}{b\times b}\right)$ and there remains plainly the $\frac{b-a}{b}$ Parts of $\frac{\overline{b-a}\times A}{b}$, which is $\frac{\overline{b-a}^2\times A}{b^2}$: Whereby it's manifest, that every Series left is the $\frac{b-a}{b}$ Parts of the preceeding; and every Term of the Series taken is

the $\frac{a}{b}$ Parts of the preceding Term of the Series left, or also the $\frac{b-a}{b}$ Parts of the preceding Term of the same Series. *Now* these two Series proceed, each decreasing in a constant Ratio, which is the same in both, *viz.* $\frac{b}{b-a}$; for every Term being $\frac{b-a}{b}$ of the preceeding, the Ratio of that preceeding to the following is the reciprocal $\frac{b}{b-a}$, and applying the Rule of *Probl.* 1, the Sum of the Series taken is A; thus, $\frac{aA}{b} \times \frac{b}{b-a} = \frac{aA}{b-a}$ And this divided by $\frac{b}{b-a} - 1$, or $\frac{a}{b-a}$ quotes A. Again; the Sum of the Series left is $\frac{b-a}{a}$ of A; for $\frac{b-a \times A}{b} \times \frac{b}{b-a} = A$, which divided by $\frac{b}{b-a} - 1 = \frac{a}{b-a}$ gives $\frac{b-a}{a}$ of A.

Observe also, that the Series of Parts left being demonstrated to be an *Infinite Decreasing Series*, the other may be deduced from it, thus: Since the Series of Parts left may be carried on till there be a Term less than any assignable Quantity, hence it plainly follows, that the Sum of the Parts taken away shall want less of the whole A than any assignable Difference; which is all that's meant by saying, That that Sum is equal to A.

SCHOLIUMS.

1°. If the Correspondent Terms of these two *Series* of the Parts taken and left are multiply'd together, the Sum of the Compound Series is *Finite*, and equal to $\frac{b-a}{2b-a}$ of A^2; which is easily prov'd from the common Rules, thus: The common Ratio of both the *Series* is $\frac{b}{b-a}$, therefore the Ratio of the Compound Series is $\frac{b^2}{\overline{b-a}^2} = \frac{b^2}{b^2 - 2ab + a^2}$; but the first Term of the one Series is $\frac{a}{b}$ of A, and of the other it is $\frac{b-a}{b}$ of A. Hence the first Term of the Compound Series is the Product of these, *viz.* $\frac{a \times \overline{b-a}}{b^2}$ of A^2; or $\frac{ab-a^2}{b^2}$ of A^2. And, according to *Probl.* 1, we must multiply this by the Ratio $\frac{b^2}{b^2 - 2ab + a^2}$ the Product is $\frac{ab - a^2}{b^2 - 2ab + a^2}$ of A^2; which again divided by the Ratio less Unity, *viz.* $\frac{b^2}{b^2 - 2ab + a^2} - 1 = \frac{2ab - a^2}{b^2 - 2ab + a^2}$, the Quote is $\frac{ab - a^2}{2ab - a^2}$ of $A^2 = \frac{b-a}{2b-a}$ of A^2.

2°. If we suppose each Term of the one Series multiply'd into each of the other, the Sum of the Products is Finite, and it is particularly equal to $\frac{b-a}{a}$ of A^2: For the Sum of Parts taken away is A, and the Sum of the Parts left is $\frac{b-a}{a}$ of A, and their Product is $\frac{b-a}{a}$ of A^2.

3°. Hence

3°. Hence we have these Proportions, or Ratios of the total Values of these several Series *viz.* (1°.) the Sum of the parts taken is to the Sum of the remainders *viz.* $A : \frac{b-a}{a}$ of A, as $a : b-a$. (2°.) the Sum of the Series taken, to the Sum of the Products of the correspondent Terms of the two *viz.* $A : \frac{b-a}{2b-a}$ of A^2, as $1 : \frac{b-a}{2b-a}$ of A: Or $2b-a : \overline{b-a} \times A$. (3°.) The Sum of the Series taken, is to the Sum of the Products of all the Terms of the one Series, multiplied into all the Terms of the other, *viz.* $A : \frac{b-a}{a}$ of A^2, as $1 : \frac{b-a}{a}$ of A, or $a : \overline{b-a} \times A$. (4°) The Sum of the Series of Parts left, is to the Sum of the first Products, *viz.* $\frac{b-a}{a}$ of $A : \frac{b-a}{2b-a}$ of A^2, as $\frac{b-a}{a} : \frac{b-a}{2b-a}$ of A. (5°.) The Sum of the Parts left, is to the Sum of the second Products, *viz.* $\frac{b-a}{a}$ of $A : \frac{b-a}{a}$ of A^2 as $1 : A$ (6°.) The two Sums of Products, *viz.* $\frac{b-a}{2b-a}$ of $A^2 : \frac{b-a}{a}$ of A^2, as $a : 2b-a$.

THEOREM IX.

In the Arithmetical Progression 1, 2, 3, 4, *&c.* The Sum is to the Product of the last Term by the Number of Terms, *i. e.* to the Square of the last Term; in a Ratio always greater than that of 1, 2. But approaching infinitely near to it.

DEMON. The Sum of the Arithmetical Progression is $\frac{n^2+n}{2}$ *Probl.* 5. *Chap.* 2. *B.* 4.) And the Square of the last Term n is n^2, therefore the Sum is to that Square as $\frac{n^2+n}{n} : n^2 :: n^2+n : 2n^2 :: n+1 : 2n$. But $\frac{n+1}{2n} = \frac{n}{n2} + \frac{1}{n2} = \frac{1}{2} + \frac{1}{2n}$ And as the Number of Terms, or last Term n encreases, so does $\frac{1}{2n}$ decrease infinitely, therefore the Ratio approaches infinitely near to $\frac{1}{2}$.

Observe if the Arithmetical Series begins with 0, thus, 0, 1, 2, 3, then the Sum is to the Product of the last Term by the Number of Terms, exactly in every Step, as 1 to 2, for the Sum is in this Case, $\frac{n^2-n}{2}$, But the last Term is $n-1$, and its Product by the Number of Terms n is n^2-n, therefore the Sum is to the Product as $\frac{n^2-n}{2} : n^2-n ::$ $n^2-n : 2n^2-2n :: 1 : 2$.

THEOREM X.

Take the natural Progression beginning with 0, thus, 0, 1, 2, 3, *&c.* And take the Series of any the like powers of the former Series; As the Squares, 0, 1, 4, 9, *&c.* Or Cubes, 0, 1, 8, 27, *&c.* Then again take the sum of the Series of Powers to any Number of Terms, and also multiply the last of the Terms summed by the Number of Terms, (reckoning always 0 for the first Term.) The *Ratio* of that *Sum* to that Product is more than $\frac{1}{n+1}$ (n being the Index of the Powers) *i. e.* in the Series of Squares

it is more than $\frac{1}{3}$; in the Cubes more than $\frac{1}{4}$; and so on: But the Series going on *in infinitum*, we may take in more and more Terms without end into the Sum; and the more we take, the Ratio of the *Sum* to the *Product* mentioned grows less and less; yet so as it can never actually be equal to $\frac{1}{n+1}$ but approaches infinitely near to it, or within less than any assignable difference.

DEMON. The truth of this Theorem has hitherto, that I know of, been demonstrated only by an induction; or shewing that it is true in Squares, Cubes, and a few more where an actual examination of it has been made; and then its concluded that since it holds true in every Case where it has been actually tried, and no reason appearing against this being an universal Rule or Law in the nature of Numbers, therefore it is true in all other Cases. It must be acknowledged that where we find the same general Law observed in a Variety of Cases of different Powers, taken at Pleasure, as in the second, third, fourth, the eighth, the thirteenth, the twentieth, and many more taken up and down among the infinite Variety of Powers; we have great Reason to believe that it's a general Law in all Cases, tho' we don't see a direct and positive Reason or Demonstration for it; yet it's as certain that this is but an imperfect Proof, or a probability of its being true. *Again*, as to the Demonstrations given of the particular Cases whence the general Theorem is deduced, they are also of the same nature, *i. e.* they are taken only from seeing the thing proposed, to be true in as many Cases as have been actually tried, *i. e.* the Series of any Power being summed to 2 Terms, or 3, or 4 it's found to be always true, that the Ratio of the Sum to the Product mentioned is more than $\frac{1}{n+1}$ but still diminishing as the number of Terms becomes greater; and diminishing also in such a certain, constant Tenor, as shews that if it proceed so, it will approach infinitely near $\frac{1}{n+1}$. Now after having premised this, concerning the Method that has hitherto satisfied for the Demonstration of this Theorem, I shall shew how it appears in a few particular Cases, and how it may by the same Method be investigated in any other Cases at pleasure.

Example in Squares.

Arithmetical *Series.*	0	1	2	3	4.	&c.
Squares.	0	1	4	9	16.	&c.

$$\text{Sum of two Terms. Products.}\quad \frac{0+1}{2\times 1}=\frac{1}{2}=\frac{1}{3}+\frac{1}{6}$$

$$\text{Sum of three Terms. Products.}\quad \frac{1+4}{3\times 4}=\frac{5}{12}=\frac{1}{3}+\frac{1}{12}$$

$$\text{Sum of four Terms. Products.}\quad \frac{5+9}{4\times 9}=\frac{14}{36}=\frac{1}{3}+\frac{1}{18}$$

$$\text{Sum of five Terms. Products.}\quad \frac{14+16}{5\times 16}=\frac{30}{80}=\frac{1}{3}+\frac{1}{24}$$

And 1° for a Series of *Squares.*

By taking the Sums, and the Product of the last Term summed, multiplied by the Number of Terms; the Ratio is in all Cases equal to $\frac{1}{3}$ + a certain aliquot Fraction, which goes on decreasing by a constant addition of 6 (the Denominator of the first of these Fractions) to the Denominator of the preceding; which Fractions, because this Law is to be constantly observed, do therefore decrease infinitely; so that the Ratio of the Sum to the Product approaches infinitely near to $\frac{1}{3}$ by the infinite decreasing of the Fraction which is still actually joined with $\frac{1}{3}$ in every step.

Exa.

Exa. for *Cubes.*

$$\begin{array}{rl} \text{Arithm. Progress.} & 0 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \;\&c. \\ \text{Cubes.} & 0 \cdot 1 \cdot 8 \cdot 27 \cdot 64 \;\&c. \end{array}$$

$$\text{Sum of 2 Terms. Prod.} \quad \frac{0+1}{2 \times 1} = \frac{1}{4} = \frac{1}{4} + \frac{1}{4}$$

$$\text{3 Terms. Prod.} \quad \frac{1+8}{3 \times 8} = \frac{9}{24} = \frac{1}{4} + \frac{1}{8}$$

$$\text{4 Terms. Prod.} \quad \frac{9+27}{4 \times 27} = \frac{36}{108} = \frac{1}{4} + \frac{1}{12}$$

$$\text{5 Terms. Prod.} \quad \frac{36+64}{5 \times 64} = \frac{100}{320} = \frac{1}{4} + \frac{1}{16}$$

&c &c

2°. For *Cubes.* The Sums and Products being taken and compar'd, as in the Margin, the Ratio is continually $\frac{1}{4}$ + an aliquot Fraction, which goes on decreasing by a constant addition of 4 (the Denominator of the first of these Fractions) to the Denominator of the preceding; which Law being constantly observ'd, that Fraction join'd to $\frac{1}{4}$ becomes infinitely little; *i. e.* the Ratio becomes infinitely near to $\frac{1}{4}$.

Scholiums.

I. If we examine the superior Powers, the same General Truth will be found in them; but the Fractions adhering to the $\frac{1}{n+1}$ will not diminish in the same manner as they have been observ'd to do in the Squares and Cubes; *i. e.* by a constant addition of the Denominator of the first of these Fractions to the preceding Denominator: Yet this I have found, as far as I have examin'd them, that these Fractions decrease so, that their Ratios to one another, comparing each to the preceding, do also constantly decrease; which makes the Fractions themselves decrease so much quicker than if they decreas'd in one constant Ratio; Or if they decreas'd by Encreasing Ratios, as these in the Squares and Cubes do; which you'll easily observe do so decrease, that their Ratios are the Series of the Reciprocal Ratios of the natural Progression $1 : 2 : 3 : 4$. For $\frac{1}{6} : \frac{1}{12} :: 2 : 1$, and $\frac{1}{12} : \frac{1}{18} :: 3 : 2$, also $\frac{1}{18} : \frac{1}{24} :: 4 : 3$. and so also in the Cubes, $\frac{1}{4} : \frac{1}{8} :: 2 : 1$, and $\frac{1}{8} : \frac{1}{12} :: 3 : 2$, and $\frac{1}{12} : \frac{1}{16} :: 4 : 3$. And it's observable here too, that the first of these Fractions, in the *Squares* being $\frac{1}{6}$, and in the *Cubes* $\frac{1}{4}$, they decrease faster in those than in these; *i. e.* at the same distance from the beginning the Ratio is nearer equal to $\frac{1}{3}$ in the *Squares*, than it is to $\frac{1}{4}$ in the *Cubes.* But, in the 4th and 5th Powers, and others which I have actually examin'd, they decrease faster, because of their decreasing Ratios.

Another Way of proposing the Theorem.

II. Some Authors (particularly *Sturmius*) have propos'd this *Theorem* in another View, thus: They make the Arithmetical Progression begin with 1, and assert the same Truth concerning the Ratios of the Sums and Products, *viz.* that it approaches Infinitely near to $\frac{1}{n+1}$: But any I have met with give us no other kind of Demonstration than what has been given above, *i. e.* by *Induction*, or arguing from a few Particulars: And because their Method of investigating these particular Cases is somewhat different from the preceding, I shall here explain it.

1°. For

1°. For Squares.

		Sums.	Products.		
	1 . 2 . 3 . 4 . &c.				
	1 . 4 . 9 . 16 . &c.				
For 3 Terms.		14	: 27	::	$1 + \frac{1}{2} + \frac{1}{18} : 3$
For 6 Terms.		91	: 216	::	$1 + \frac{1}{4} + \frac{1}{72} : 3$

In the *Squares* take the first 3 Terms, and at every succeeding Step twice as many, and the Ratio of the *Sum* to the *Product* is found always greater than 1 : 3. But so as to decrease, and become infinitely near to it; because the two Fractions that adhere to 1 (in the lesser Term of the Ratio) are found to decrease in a Constant Ratio, which is $\frac{1}{2}$ in the first Fraction, and $\frac{1}{4}$ in the second Fraction.

Whence *Sturmy* concludes the Argument in this manner, *viz.* Since of the two Fractions adhering to 1 at every Step, the first is always $\frac{1}{2}$ of that in the preceding Step, and the second is $\frac{1}{4}$ of that in the preceding Step; therefore these two Fractions are in every Step the Effect of substracting from $\frac{1}{2} + \frac{1}{18}$, which belongs to the first Step, this Series, $\frac{1}{4} + \frac{3}{72} : \frac{1}{8} + \frac{3}{288}$, making 3 the Numerator of the second Part, because $\frac{1}{18} = \frac{4}{72}$, and $\frac{4}{72} - \frac{3}{72} = \frac{1}{72}$, and so of the rest: But this Series consisting of two Series, take their Sums separately, they are $\frac{1}{4} + \frac{1}{8} + \frac{1}{16}$: &c. $= \frac{1}{2}$: and $\frac{3}{72} + \frac{3}{288}$ &c. $= \frac{1}{18}$, by *Probl.* 1. that Series being suppos'd to be Infinite; whence, in the last Case, or when the Series of Squares is infinite, the Fractions adhering to 1 are evanish'd, because they are become $\frac{1}{2} + \frac{1}{18} - \frac{1}{2} - \frac{1}{18} = 0$.

As to the last part of this Demonstration, I *observe*, that it is superfluous; for the Argument ought to be concluded immediately from this, That the Fractions adhering to 1 decrease in one Constant Ratio: For, in this Case, if we suppose the Series *infinite*, these Fractions must decrease to nothing, this being the very Supposition upon which the Rule is founded, by which we find the Sum of an *Infinite decreasing Series:* And this Rule being used in the Argument to prove that these Fractions do at last evanish, it's manifest that the Thing to be concluded is already suppos'd.

2°. For *Cubes*.

		Sums.	Products.		
	1 . 2 . 3 . 4 : &c.				
	1 . 8 . 27 . 64 : &c.				
For 4 Terms.		100	: 256	::	$1 + \frac{1}{2} + \frac{1}{16} : 4$
For 8 Terms.		1296	: 4096	::	$1 + \frac{1}{4} + \frac{1}{64} : 4$

Again; in the *Cubes* take the first 4 Terms, and then 8 Terms, and so on doubling the Number; and hereby 'tis found that the Ratio of the Sum to the Product is always greater than 1 : 4. but approaching Infinitely near to it, because the two Fractions adhering to 1 (in the lesser Term of the Ratio) do decrease in a constant Ratio, *viz.* $\frac{1}{2}$ in the first, and $\frac{1}{4}$ in the second Fraction.

Sturmy concludes the Argument here the same Way as in the Case of *Squares*, to which the same Observation already made is also applicable.

So far then you see an Agreement in this Form of the *Theorem*, and the preceeding *viz.* that in the *Squares* the Ratio of the Sum and Product decreases faster, because $\frac{1}{18}$ is less than $\frac{1}{16}$, and the Ratios of these decreasing Fractions equal in both; just as in the preceding Form the Ratio also decreas'd faster in the *Squares* than *Cubes*, and by the same Ratios.

III. But again; Tho' I have found no better Way of demonstrating the *Theorem* (in either of the two Views of it explain'd) as to its universal extension to all the different Powers, yet as to the Demonstration of the first two particular Cases (which are the most useful) viz. *Squares* and *Cubes*, 1 being the first Term, I shall here shew you a new and easie

easie Demonstration, deduced directly *à priori* from the Canons given in *Ch.* 2, for the summing the *Squares* and *Cubes* of the Arithmetical Progression.

A Direct Demonstration *of the preceding* Theorem *for* Squares *and* Cubes, *supposing the Series to begin with* 1.

(1°) For *Squares*. The Sum of the *Squares* of an Arithmetical Progression 1. 2. 3. 4, supposing the Number of Terms to be n, is $\frac{2n^3 + 3n^2 + n}{6}$ (*Prob.*3, *Ch.*2) and in the Arithmetical Progression the last Term is always the Number of Terms, therefore the last Term of the Series of *Squares* is n^2; which multiply'd by the Number of Terms n, the Product is n^3: Wherefore the Ratio of the Sum to the Product is $\frac{2n^3 + 3n^2 + n}{6} : n^3 :: 2n^3 + 3n^2 + n : 6n^3 :: 2n^2 + 3n + 1 : 6n^2$ (by dividing each Member by n) but $\frac{2n^2 + 3n + 1}{6n^2} = \frac{2n}{6n^2} + \frac{3n}{6n^2} + \frac{1}{6n^2} = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$ $\left(\text{for } \frac{2n^2}{6n^2} = \frac{1}{3} \text{ and } \frac{3n}{6n^2} = \frac{1}{2n}\right)$. Again; it's obvious that the greater n is, the less will these Fractions $\frac{1}{2n}, \frac{1}{6n^2}$ be; and that they will decrease so, as to become infinitly little, or less than any assignable Quantity: Therefore the Ratio of the Sum to the Product, tho' it's always greater than $\frac{1}{3}$ (being for the first two Terms $\frac{1}{3} + \frac{1}{4} + \frac{1}{24}$.) yet the Fractions adhering to $\frac{1}{3}$ decreasing (as now demonstrated) so as to become infinitely small, the Ratio approaches infinitely near to $\frac{1}{3}$.

(2°) For *Cubes*. The Sum of the *Cubes* of the Arithmetical Progression is $\frac{n^4 + 2n^3 + n^2}{4}$ (*Probl.* 3, *Ch.* 2.) and the Product of the last Term n^3, by the Number of Terms, n is n^4: So that the Ratio of the Sum to the Product is $\frac{n^4 + 2n^3 + n^2}{4} : n^4 :: n^4 + 2n^3 + n^2 : 4n^4 :: n^2 + 2n + 1 : 4n^2$. And $\frac{n^2 + 2n + 1}{4n^2} = \frac{n^2}{4n^2} + \frac{2n}{4n^2} + \frac{1}{4n^2} = \frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}$. And because the greater that n is, the less will $\frac{1}{2n}$ and $\frac{1}{4n^2}$ be; therefore, tho' the Ratio of the Sum to the Product is always greater than $\frac{1}{4}$ (being for two Terms, $\frac{1}{4} + \frac{1}{4} + \frac{1}{16}$) yet the Fractions adhering to the $\frac{1}{4}$ decreasing infinitely, the Ratio approaches infinitely near to $\frac{1}{4}$.

THEOREM XI.

Take the Series of the *Squares* of the natural Progression 1. 2. 3. &c. *viz.* 1. 4. 9. 16. &c. multiply any Term of this Series by the number of Terms from 1; from the Product substract the sum of all the preceeding lesser *Squares* (which Sum will always be a lesser Number than that Product) the Ratio of the Difference to the Product is always greater than 2 : 3. But approaching infinitely near to it, the farther the Series is carried, or the greater the *Square* assumed is.

Observe.

Observe, Since the Number of Terms from 1 to any Term in the Series of *Squares* is the Root of that *Square*: therefore the Product of the *Square* by the Number of Terms is the *Cube* of the same Root; and therefore the *Theorem* may be propos'd thus: The Ratio of the Difference betwixt the *Cube* of any Integral Number and the Sum of the *Squares* of all Numbers less than that, to that *Cube*, is greater than 2 : 3, but approaching infinitely near to it, as we chuse that *Cube* greater and greater.

Exa. 1. The *Cube* of 6 is 216, and the *Squares* of 1. 2. 3. 4. 5. are 1. 4. 9. 16. 25, whose Sum is 55, then is $216 - 55 = 161$. And $\frac{161}{216} = \frac{2}{3} + \frac{51}{648}$; as you'll find by taking $\frac{2}{3}$ out of $\frac{161}{216}$.

2. The *Cube* of 7 is 343, and the *Squares* of 1. 2. 3. 4. 5. 6. are 1. 4. 9. 16. 25. 36, whose Sum is 91; then $343 - 91 = 252$, and $\frac{252}{343} = \frac{2}{3} + \frac{70}{1029}$, and this Fraction $\frac{70}{1029}$ is less than $\frac{51}{648}$

DEMON. 1°. Let any Number be represented by $n + 1$, its *Cube* will be $n^3 + n^2 + 3n + 1$; and if we take all the Numbers lesser than $n + 1$, they make the Arithmetical Progression 1. 2. 3, &c. whose last Term (and number of Terms) is n; and the Sum of their *Square* is $\frac{2n^3 + 3n^2 + n}{6}$, which is manifestly a lesser Number than the *Cube* of $n + 1$, or $n^3 + 3n^2 + 3n + 1$.

2°. Take the Difference propos'd, *viz.* $n^3 + 3n^2 + 3n + 1 - \frac{2n^3 + 3n^2 + n}{6}$
$$= \frac{6n^3 + 18n^2 + 18n + 6}{6} - \frac{2n^3 + 3n^2 + n}{6} = \frac{6n^3 + 18n^2 + 18n + 6 - 2n^3 - 3n^2 - n}{6}$$
$= \frac{4n^3 + 15n^2 + 17n + 6}{6}$ Now compare this Difference to $\overline{n+1}^3$, or, $n^3 + 3n^2 + 3n + 1$, the Ratio is plainly $4n^3 + 15n^2 + 17n + 6 : 6n^3 + 18n^2 + 18n + 6$, or, $\frac{4n^3 + 15n^2 + 17n + 6}{6n^3 + 18n^2 + 18n + 6}$, which is greater than $\frac{2}{3}$ by this Fraction $\frac{9n^2 + 15n + 6}{18n^3 + 54n^2 + 54n + 18}$, as you'll find by substracting $\frac{2}{3}$ out of it, by the common Rules. Now since in every Step the Ratio of the Difference to the *Cube* will be the same way express'd, 'tis plain it will always be greater than $\frac{2}{3}$; but if the Fraction $\frac{9n^2 + 15n + 6}{18n^3 + 54n^2 + 54n + 18}$, adhering to it, grows infinitely little, or less than any assignable Fraction, then the Ratio approaches infinitely near to $\frac{2}{3}$. What remains then to be demonstrated, is only this Infinite decrease of the Fraction adhering to the $\frac{2}{3}$, which is thus done: Were the adhering Fraction $\frac{9n^2}{18n^3}$, its infinite decrease is easily shewn, for it is equal to $\frac{1}{2n}$, by dividing Numerator and Denominator equally by $9n^2$, but n is gradually taken equal to 1. 2. 3. 4, &c. and therefore $\frac{1}{2n}$ is gradually $\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{6} \cdot \frac{1}{8}$, &c. which manifestly decreases infinitely, or so as to become less than any assignable Fraction.

Again;

Again; $\frac{9n^2+15n+6}{18n^3+54n^2+54n+18}$ is a Fraction less than $\frac{9n^2}{18n^3}$ or $\frac{1}{2n}$, as is easily seen by the Comparison; wherefore if $\frac{1}{2n}$ does decrease infinitely as n encreases, the other which is less than $\frac{1}{2n}$ must also decrease infinitely.

COROLL. The Sum of the *Squares* of the Series 1 . 2 . 3. &c. carried to any Number of Terms, is to the Difference betwixt the Cube of the Number of Terms, or last Term, and the Sum of all the *Squares*, except the last, in a Ratio approaching infinitely near to 1 : 2. but still greater: For the Ratio of the Sum of the Squares to the Cube of the Number of Terms approaches infinitely near to $\frac{1}{3}$, but still greater (by *Theorem* 9.) And by the present *Theorem* the Ratio of the Difference mention'd, to that Cube, approaches infinitely near to $\frac{2}{3}$; therefore the Ratio of the Sum of the Squares to the Difference approaches infinitely near to $\frac{1}{2}$

SCHOL. This *Theorem* is propos'd by some Authors (particularly *Sturmius*) in a very different manner, which is to this purpose, *viz.* If any Series (of Integers) begins with a square Number, and decreases by growing Differences, which are the Series of odd Numbers, 1. 3. 5, &c. the Sum of the Series carried down to 0 (in which 'twill always end) is to the Product of that *Square* (which is the greatest Term) by the Number of Terms, in a Ratio always greater than $\frac{2}{3}$, but approaching infinitely near to it, as we take that *Square* greater.

I shall first shew the Coincidence of this and the preceding Proposition, and then give you *Sturmius*'s demonstration.

For the first, take the odd Series 1. 3. 5, &c, the Sum of it to any Number of Terms is the *Square* of the Number of Terms: or the Series of its Sums taken always from the beginning, makes the Series of square Numbers. (*Corol.* 4, *Probl.* 5, *Ch* 2, *B.* 4.)

1 .	3 .	5 .	7 .	9 .	&c.	odd Numbers.
1 .	4 .	9 .	16 .	25 .	&c.	square Numbers.
1 .	2 .	3 .	4 .	5 .	&c.	Roots.

Wherefore it is plain, that if from any square Number we take successively as many Terms of the Series of odd Numbers from 1, as the Root of that *Square* expresses, when the last Substraction is made, there remains nothing; and that *Square*, with the several Remainders, is the Series propos'd Thus, beginning with 9, it is 9, 8, 5; beginning with 25, it is 25, 24, 21, 16, 9: Universally, if it begin with n^2 it is n^2; n^2-1; n^2-1-3; $n^2-1-3-5$; &c. which again is the same as n^2; n^2-1; n^2-4; n^2-9; &c. the Series being carried to as many Terms as the Root n expresses; And the *Square* substracted from n^2 in the last Term, being that next lesser than n^2, because 'tis the Sum of a Number of Terms of the odd Series less by 1 than n.

Whence the Coincidence of the two Propositions is evident; for the Number of Terms being n, therefore the Sum of this Series is equal to the Difference of n^3 and $1+4+9$ &c. carried to a Number of Terms equal to $n-1$, and n^2 multiply'd by the Number of Terms n, is n^3.

2°. The Demonstration that *Sturmy* gives us of this *Theorem* is only by Induction, in the same manner as in the preceding *Theorems*, thus: If there are three Terms, 9, 8, 5, whose Sum is 22, then is $22 : 27\ (= 3 \times 9) :: 2 + \frac{4}{9} : 3$, which is a greater Ratio than that of $2 : 3$. Again; $\frac{4}{9}$ being $= \frac{1}{2} - \frac{1}{18}$, he expresses it thus, $2 + \frac{1}{2} - \frac{1}{18} : 3$. If there are six Terms, $36 ; 35 : 32 : 27 : 20 : 11$; their Sum is 161; and, $161 : 216\ (= 6 \times 36) : 2 + \frac{19}{72} : 3$; Or, as, $2 + \frac{1}{4} - \frac{1}{72} : 3$. So he goes on examining more Cases, taking at every Step double the number of Terms of the last Step; and finds that the Fraction adhering to 2 proceeds in a continued Geometrical Progression decreasing, thus, $\frac{1}{2} - \frac{1}{18}, \frac{1}{4} - \frac{1}{72}, \frac{1}{8} - \frac{1}{288}$: the first part decreasing in the Ratio of 2 to 1, and the other in the Ratio of 4 to 1; whence he concludes the Argument in the same manner as in *Theorem* X; to which is applicable the same Observation I made upon that.

Observe: As to this last Method of proposing the *Theorem*, That it is accommodated to a particular Use which *Sturmy* makes of it in Geometry: The Reason I chose the other Way being, That in this Shape I found the direct Demonstration I have given of it.

CHAP. IV.

Of Infinite Decimals.

WHAT a *Decimal Fraction* is, and its Notation; also what a *Circulating Decimal* is, with the whole Operations about *Determinate Decimals*, has all been already taught: But, that the whole Doctrine of *Infinite Decimals* may be found here together, some of these Definitions must be repeated.

DEFINITIONS.

I. A Decimal Fraction may be call'd *Finite* or *Determinate*, when it has certain and determinate Numbers for its Numerator and Denominator; *i. e.* when the Numerator and Denominator have each a certain limited Number of Figures, as, $.3 = \frac{3}{10}$, $.046 = \frac{46}{1000}$.

II. A Decimal Fraction may be call'd *Infinite* or *Indeterminate* when the Number of Places is *Indeterminate*, and encreasing without End; whereby the Numerator and Denominator are conceiv'd to be themselves infinitely great. So in this *Exa.* .347 &c. if we suppose that there ought to be more and more Figures, *in infinitum*, annex'd to the right hand of those here set down, thereby encreasing both Numerator and Denominator infinitely, or without end, we do hereby form the Idea of an Infinite and Indeterminate Decimal.

III. *Infinite Decimals* are of two kinds, which we may distinguish by the General Denominations of *Certain* and *Incertain*,

A *Certain* Infinite Decimal is ſuch whoſe Numerator runs into Infinity by a continual repetition of one or more Figures; as in theſe Examples, .44, &c. .033, &c. .455, &c. wherein the ſame Figures, 4, 3, 5 is conſtantly repeated: Alſo .356356, &c. where 356 is repeated; and .07236464, &c. where 64 are conſtantly repeated. Such Decimals are alſo particularly call'd *Repeating* or *Circulating Decimals*, from this continual repetition or circulation of the ſame Figures in the Numerator. *Obſerve* alſo, that the Figure or Figures repeated may very conveniently and properly be call'd the *Repetend*.

Incertain Decimals are ſuch whoſe Numerator goes on for ever without a conſtant circulation of Figures.

SCHOLIUMS.

1*ſt*. The eſſential Difference betwixt theſe two kinds of *Infinite Decimals* is this; that the *Certain* have a determinate, finite, and certain Value; *i. e.* that there is a certain determinate Vulgar Fraction, which expreſſes the true and compleat Value of that Infinite Decimal (as ſhall be demonſtrated) whereas the *Incertain* have no ſuch finite and aſſignable Value: And this is the reaſon of theſe Names.

Now 'tis owing to this Certain finite Value of a Circulating Decimal that they occur in Practice; for they are no other thing than the Reſult or Effect of reducing ſome Vulgar Fraction to a Decimal: Or if they are brought into a Queſtion by meer Suppoſition, yet they are reducible to a Vulgar finite Fraction; as all Vulgar Fractions are reducible into a Decimal, either Finite or Circulating (as will be explain'd afterwards). It is this Property alſo which makes theſe Decimals, tho' they cannot be limited in that Decimal form, capable of ſuch management in Practice, as that no Error can happen from the impoſſibility of finiſhing and determining it in form.

2*d*. As to the *Incertain* Decimals, *obſerve*, that tho' they have no determinate value, yet their value is not Infinite; for notwithſtanding it encreaſes without end, yet (as ſhall be ſhewn) they are ſo limited in general, as that tho' the whole Infinity could be actually exhauſted, yet the value of the Fraction cannot exceed ſome finite aſſignable Fraction, as on the other hand it cannot come ſhort of ſome ſuch Fraction; wherefore we may juſtly ſay, they have a finite, tho' not an aſſignable value; for if their value were aſſignable, they would circulate contrary to Suppoſition. But then alſo *obſerve*, that becauſe their value is not aſſignable, there is no poſſibility of ſupplying their Defects perfectly, ſo that we muſt be content to do it nearly, or by way of Approximation.

Again *obſerve*, that tho' no *Incertain* Decimal can ever ariſe from any finite aſſign'd Fraction, yet they are not all ſo imaginary, but that they do in ſome manner, and in ſome Caſes, neceſſarily occur in Practice of *Arithmetick*; as in the Extraction of Surd Roots.

Alſo, ſometimes where we might have a Circulating Decimal, it may be convenient to take it imperfect; as when in the reduction of a Vulgar Fraction to a Decimal, the Diviſion has gone far on without coming to a Circulation; then we may chuſe to take it with a convenient number of Places, and conſider it as *Incertain*.

The *laſt Remark* I make here is, That ſome Decimals are Incertain as to their value, becauſe they do not circulate; yet, in another reſpect, they may be ſaid to be Certain; for we may conceive a Numerator conſiſting of certain known Figures, ſucceeding one another in a certain order for ever, yet ſo as there be no Circulation. *Exa.* .34 344 3444, *&c.* ſuppoſing that after 3 you have 4 repeated once more than in the preceding Step; or this, .34 345 3456, *&c.* beginning ſtill at 3, and after it taking the natural ſeries, in order to one place more in every ſtep. But now, tho' ſuch Decimals ſhould in any Caſe occur, (which yet ſeem to be meerly imaginary) this Certainty of the Figures of the Numerator does not make the Value certain; and ſo we know no better how to manage them than if

there were no such Certainty. *Again*; As to these Incertain Decimals which express Surd Roots, tho' they are not always of this kind, (and I don't know if any of them was ever found to be so, or if they can be) yet they have a Certainty of another kind, which is, that the Progress of the Fraction depends upon a certain Law or Rule, whereby the Extraction is perform'd (as has been explain'd in its place); And, because of this, they are not purely imaginary and supposititious, yet, as to the Practice, they have no more Accuracy than the rest, except in the Case of raising them to the Reciprocal Power express'd by the Denominator of the Root, when 'tis known of what Number they express the Root, and in some Comparisons of them among themselves; but then these Comparisons are manag'd not by means of the Decimal Expressions, but of the Powers themselves with their Surd or Radical Indexes; as explain'd in *Book* III.

IV. *Circulating Decimals* (or *Circulates*, as they may be conveniently call'd) are distinguish'd into *Pure* and *Mix'd*. 1°. *Pure*, when there are no significant Figures in the Numerator, but what belong to the Repetend; *i. e.* when there are no Figures, or none but 0's, betwixt the Point and the Repetend; as, .34 34 *&c*, .0044 *&c*, .046 046 *&c*, .003 03 *&c*.

2°. *Mix'd*, when there are significant Figures betwixt the Point and the Repetend; as, .344 *&c*, .3048 48 *&c*, .0467272 *&c*. Which I call *Mix'd* because they consist of two parts, a *Finite* and a *Circulate*. So the first *Exa.* is plainly the Sum of these, .3 + .044 *&c*, the second is .30 + .0048 48 *&c*, the third is .046 + .0007272 *&c*.

And here also *Observe*, that as Decimals are more generally distinguish'd into *Simple* and *Mix'd*, so, when we name a *mix'd* Decimal, it signifies a Decimal with an Integer; but a *mix'd* Circulate more strictly regards only the Decimal part: Yet, to avoid too many Distinctions, when a whole Number is join'd with a Circulate, whether *pure* or *mix'd*, we may call it a *mix'd Circulate*; as this, 46.33 *&c*. or this, 258.0044 *&c*. And as any Integer may be reduced to a Fractional Expression with a Decimal Denominator, so the finite part of all mix'd Circulates may be express'd fractionally, by taking for Numerator all the Figures as they stand, from the highest place of the Integral part to the Repetend, and for the Numerator, 1 with as many 0's as there are Places betwixt the Point and the Repetend, thus, $46.032\,32\ \&c. = \frac{460}{10} + .032\ \&c.$ also, $374.2358\,58\ \&c. = \frac{37423}{100} + .0058\ \&c.$ which in the proper Decimal form are 374.23 + .0058 *&c*. for tho' 374.23 has, in this form, the Integral and Fractional Parts distinguish'd, yet they express decimally an Improper Fraction, when all the Figures, neglecting the Point, are made Numerator; as has been explain'd in its place.

V. *Circulates* whose Repetends consist of the same number of Figures, and begin also at the same place after the Point, may be call'd *Like* or *Similar Circulates*, whether they be both *pure* or *mix'd*, or one *pure*, and the other *mix'd*: So these are *Similar*, .33 *&c*. and .77 *&c*. and these, 2.3456 56 *&c*. .004242 *&c*.

Scholiums.

1. If the Repetend be twice written down with an *&c*. after it, this will clearly shew that there is a repetition, and what the Repetend is: But this we may do more conveniently, by setting a Point over the first and last Figure of the Repetend once written down: Thus instead of .033 *&c*. write $.0\dot{3}$; And for .4 376 376 *&c*. write $.4\dot{3}7\dot{6}$; and so of others.

Again;

2*d*. Because of the different Views in which a *Circulate* may be taken, 'twill be convenient to call the Repetend which is absolutely the first in any Circulate, *The Given Repetend*; so here $.3434$ &c. 34 is the Given Repetend. But as the same Fraction may be consider'd in another view, *viz.* as a mix'd Circulate equal to $.3 + .0\dot{4}\dot{3}$ (as you'll find presently explain'd) so the Repetend it has in this view may be call'd for distinction the *New Repetend*: Also the Finite Part preceding the Given Repetend is the Given Finite Part, and that preceding the New Repetend is the New Finite Part.

3*d*. The Circulation of a Decimal may begin in the Integral Part; as, $\dot{3}.\dot{3}$ or $\dot{4}.2\dot{3}$, or $\dot{2}4.56\dot{7}$; Now if there are no Figures but what belong to the Repetend, it's in that respect a *pure* Circulate: But as we have limited that Name to the Fractional Part by it self, we shall leave this other kind of Circulate to the Class of *mix'd* Circulates, as a particular Species of it; so that we must reckon the first Period of the Repetend, as it belongs to the Fractional part, to be that which begins first after the Point, thus, $\dot{4}.2\dot{3} = 4.23\dot{4}2\dot{3}$, or rather $4.\dot{2}3\dot{4}$; see *Theor.* 3.

THEOREM 1.

Any Finite Decimal may be consider'd as Infinite, by annexing 0's without end on the right hand of the Numerator, making 0 the Repetend; thus, $.34 = .3400$, &c. or $.34\dot{0}$.

DEMON. The Numerator and Denominator of the Given Fraction being equally multiply'd by the 0's join'd *in infinitum*, the value of the Fraction is still the same.

THEOREM 2.

Any *Pure Circulate* may be consider'd as *Mix'd*, and keep still the same Repetend; by taking the given Repetend once or oftner written down for a Finite Part; and considering the same Repetend as circulating after that for the Infinite Part; thus, $.\dot{3}\dot{4} = .34 + .00\dot{3}\dot{4} = .3434 + .0000\dot{3}\dot{4}$.

DEMON. The Reason is obvious, since as far as an Infinite Fraction is continued, so far the Value is finite and determinate; and the remaining part is still infinite, tho' of a less value than the given Infinite, because of what's determin'd and taken away.

THEOREM 3.

If any *Circulate* has a Repetend of more than one Figure, it may be transform'd into another Circulate having a Repetend of the same number of Figures, and also the same Figures, but in another order, *i.e.* by beginning a new Repetend from any Figure after the first of the given Repetend; and that taken either in the first or second, or any other Period of the given Repetend, leaving all the Figures on the left of this new Repetend to the Finite part; whereby if the given Circulate was *pure*, it will in some cases become *mix'd*; or if it was *mix'd*, the Finite part becomes always greater, and the Infinite less; thus, $.\dot{3}\dot{4} = .3\dot{4}\dot{3} = .34\dot{3}\dot{4}$; $.4\dot{5}6\dot{7} = .45\dot{6}7\dot{5} = .456\dot{7}5\dot{6}$, $.\dot{0}04\dot{2} = .0\dot{0}42\dot{0} = .00\dot{4}20\dot{0}$.

DEMON. In the first Example, since 34 is suppos'd to be repeated for ever, if .3 is taken away, there must remain $.0\dot{4}\dot{3}$; or, if 34 is taken away, there remains $.0\dot{3}\dot{4}$; Since 3 succeeds 4, and 4 succeeds 3 for ever. The same Reason is obvious in every Case; which you'll also find afterwards further confirm'd.

COROLL.

COROLL. Any Circulate may be transform'd into another, whose Repetend begins at any distance after the given Finite Part.

Schol. If the Repetend of a *pure Circulate* has 0's in the first places, on the left hand; then, whether it begins immediately after the Point, or have 0's betwixt it and the Point, it's manifest that the changing of the Repetend, in the manner here explain'd, will not make it a *Mix'd Circulate*, if the new Repetend begins at the first significant Figure, or at any of these 0's, in the first Period of the given Repetend; but if it begin at any Figure after the first significant Figure of the first Period, or at any Figure in any of the other Periods, it will be a *Mix'd Circulate*; thus, $.\dot{0}04\dot{6} = .\dot{0}046\dot{0}$ (a *Pure Circulate*) $= .00\dot{4}60\dot{0}$ (a *Pure Circulate*) $= .004\dot{6}00\dot{4}$ (a *Mix'd Circulate*).

In any other kind of *pure Circulates* a new Repetend will certainly make it a *Mix'd Circulate*.

THEOREM 4.

Any *Circulate* may be transform'd into another having a greater Repetend, *i. e.* one having more Places; by taking the given Repetend, or any of equal number of Places into which it is transform'd by the last, as oft as we please, and considering all that as a New Repetend, thus, $.\dot{4} = .\dot{4}\dot{4} = .\dot{4}4\dot{4}$; Also $.0\dot{4}\dot{2} = .0\dot{4}24\dot{2}$; And $.0\dot{3}6\dot{4} = .0\dot{3}6436\dot{4} = .03\dot{6}43643\dot{6}$

The Reason of this is obvious.

Observe; When we speak of the Repetend of a Circulate without distinction, it's always to be understood of the least Repetend.

THEOREM 5.

Any two or more Circulates may be made Similar, by making all the Repetends begin where that one of them begins which stands farthest from the Point (by the Method explain'd in *Theor.* 3, and *Coroll.*) And, to make them end together, let each of them have as many Places as the number of Units express'd by the least common Multiple of the several Numbers of Places in all the given Repetends (and, to find that least common Multiple, see *Probl.* 5, *Ch.* 1, *B.* 5.) So, in both the annex'd Examples, that least common Multiple is 6.

Ex. 1. $.4\dot{3}\dot{6} = .43\dot{6}3636\dot{3}$
$.04\dot{7} = .04\dot{7}7777\dot{7}$
$.29\dot{3}4\dot{1} = .29\dot{3}4134\dot{1}$

Ex. 2. $.426\dot{7} = .426\dot{7}7777\dot{7}$
$4.9\dot{3}\dot{2} = 4.9\dot{3}2323\dot{2}$
$26.\dot{3}2\dot{8} = 26.328\dot{3}2832\dot{8}$

Demon. The Reason of that Part concerning the Beginning of the new Repetends is plain from *Theor.* 3. And as to their ending together, it's plain, that if they are all repeated so oft as that the number of Places taken in each is a common Multiple of the several numbers of Places in each given Repetend; Then, as that may be all taken for a new Repetend, (by *Theor.* 4.) so it will make them all end together, and be consequently Similar. And, lastly, the Reason why we take their least common Multiple, is to have the Expression as short and neat as possible.

THEO-

THEOREM 6.

Every Circulate has a finite aſſignable Value, thus:

Part 1ſt. If it's a *Pure* Circulate, it's equal to a Vulgar Fraction whoſe Numerator is the Repetend, and its Denominator a Number expreſs'd by as many 9's as there are Places in the Repetend, with as many 0's on the right hand as there are 0's betwixt the Point and Repetend.

Exa. (1) $.\dot{3} = \frac{3}{9}$ (2d) $.0\dot{4} = \frac{4}{90}$ (3d) $.\dot{4}\dot{6} = \frac{46}{99}$ (4th) $.00\dot{3}7\dot{2} = \frac{372}{99900}$ (5th) $.\dot{0}04\dot{6} = \frac{0046}{9999}$ or $\frac{46}{9999}$

Univerſally. Let R expreſs the Repetend, and a the number of 0's betwixt the Point and Repetend, the Sum is $\frac{R}{9, \&c.}$ or $\frac{R}{9, \&c. \times 10^a}$ (10^a expreſſing that power of 10 whoſe Index is a.)

Part 2d. If it's a *Mix'd Circulate*, find the Sum of the Circulating Part, and add it to the Finite Part: which total Sum being expreſs'd all together as a ſimple Fraction, will have for Denominator that of the Finite Value of the Circulating Part; and for Numerator the Sum of theſe two Numbers, *viz.* the Repetend, and the Product of the Numerator of the Finite Part (expreſs'd fractionally) by the ſame Denominator, without the 0's, if any belong to it; *i. e.* by a Number of 9's as many as there are Places in the Repetend.

Exa. (1) $.46\dot{3} = \frac{46 \times 9 + 3}{900}$ (2d) $46.0\dot{2}\dot{7} = \frac{460 \times 99 + 27}{990}$

(3d) $8.327\dot{4}6\dot{1} = \frac{8327 \times 999 + 461}{999000}$ (4th) $3.\dot{4}\dot{6} = \frac{3 \times 99 + 46}{99}$

Univerſally. Let A be the Numerator of the Finite Part, and 10^a its Denominator, the Sum is $\frac{A \times 9 \&c. + R}{9 \&c. \times 10^a} = \frac{B}{9 \&c. \times 10^a}$ (taking $B = A \times 9 \&c. + R$.)

And *Obſerve*, That as the multiplying by 9 &c. is a very eaſy Operation, [See *Caſe* 6, § 2d, *Ch.* 5, *B* 1.] ſo the Multiplication, and Addition of R to the Product, may be done all at once very eaſily, thus: Subtract the firſt right-hand Figure of the Numerator A from the firſt of the Repetend R; and ſo on in Order thus; In the preceding *Exa.* 1, the Operation is 6 from 3 I cannot take, but from 13, and 7 remains; then 5 from 6, and 1 remains; laſtly, 0 from 4, and 4 remains; and the reſult is $417 = 46 \times 9 + 3$. In *Ex.* 2d it is $7 - 0 = 7$. $12 - 6 = 6$. $10 - 5 = 5$. $6 - 1 = 5$. $4 - 0 = 4$. the reſult being $45567 = 460 \times 99 + 27$. In *Exa.* 3d it is $11 - 7 = 4$. $6 - 3 = 3$. $4 - 3 = 1$. $17 - 8 = 9$. $2 - 1 = 1$. $83 - 0 = 83$. the reſult being $8319134 = 8327 \times 999 + 461$. In *Exam.* 4th it is $6 - 3 = 3$. $34 - 0 = 34$. making $343 = 3 \times 99 + 46$. But had this laſt Example been $9.\dot{4}\dot{6}$, it were $16 - 9 = 7$. $94 - 1 = 93$, making $937 = 9 \times 99 + 46$. Theſe Examples ſufficiently illuſtrate the Practice. And, to make it clearer, do theſe Examples at large; firſt multiplying by 9 &c. by the Method of the Rule refer'd to, then add the Repetend.

DEMON. For the Firſt Part. Every *Pure Circulate* is, from the nature of a Decimal Fraction, a Series of Decreaſing Fractions whoſe Numerators are all the ſame, *viz.* the given Repetend, and their Denominators are a Geometrical Series encreaſing in the conſtant Ratio expreſs'd by 1, with as many 0's as (*i. e.* whoſe Ratio is a Power of 10, having for its Index) the number of Places in the Repetend (taking here the Ratio as the Quote of the greater Term divided by the leſſer) thus, $.33$ &c. $= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000}$ &c. the Ratio of the Denominators being $1 : 10$; Alſo $.004646$, &c. $= \frac{46}{100000} +$

+ 46

$+ \frac{45}{10000000}$ &c. the Ratio of the Denominators being $1 : 100$. Again; $.0404$ &c. $= \frac{04}{100} + \frac{04}{10000}$ &c. or $\frac{4}{100} + \frac{4}{10000}$ &c. But Fractions having a common Numerator are in the Ratio of their Denominators reciprocally; wherefore the several Terms or Finite Decimals, of which a Circulate is compos'd, make an Infinite decreasing Geometrical Series, whose common Ratio is the Ratio of their Denominators, which may be express'd universally 10^m, supposing as many o's as the Repetend has Figures, or m to be equal to the number of Places in the Repetend. Again: Let R represent the Repetend or common Numerator of this Series of Fractions, and 10^n the Denominator of the first Fraction, which therefore is $\frac{R}{10^n}$; then, by the Rules of Infinite Series, the Sum is $\frac{R}{10^n} + 10^m \times \overline{10^m - 1}$ [for l being the greatest $l \times l^2$, and r the Ratio, the Sum is $rl \div \overline{r-1}$]. Now in the Dividend $\frac{R}{10^n} \times 10^m$, the Multiplier 10^m being the Ratio, it's manifest it cannot have more Places than 10^n, the Denominator of the greatest Extreme; but it may have fewer, or the same Number. If it have the same, *i. e.* if $n = m$, then is $\frac{R}{10^n} \times 10^m$ (or 10^n) $= R$; but if n is greater than m, 'tis evident that 10^n must have as many more o's as the number of o's from the Point to the Repetend. Therefore in this Case the Product $\frac{R}{10^n} + 10^m$ may be simply express'd $\frac{R}{10^{n-m}}$ ($n-m$ expressing the number of o's from the Point to the Repetend). And if we take $a = n - m$, it is $\frac{R}{10^a}$. Then for the Divisor $10^m - 1$, it's plainly $= 9$ &c. taking as many 9's as there are o's in 10^m (for $10 - 1 = 9$; $100 - 1 = 99$, and so on). From all which it is clear, that the Sum is universally $\frac{R}{9\&c}$ if there is no o betwixt the Point and Repetend; but if there is, then the Number of them being $= a$, the Sum is $\frac{R}{9\,\&c. \times 10^a}$

For the 2d Part; Let A express the Numerator of the Finite Part, and 10^a its Denominator, which Part is therefore $\frac{A}{10^a}$; then the Circulating Part being $\frac{R}{9\,\&c. \times 10^a}$, the Sum is $\frac{A}{10^a} + \frac{R}{9\,\&c. \times 10^a}$, which by due reduction is $\frac{A \times 9\,\&c. + R}{9\,\&c. \times 10^a} = \frac{B}{9\,\&c. \times 10^a}$ (taking $B = A \times 9$ &c. $+ R$.)

COROLLARIES.

1. If the Repetend of any Circulate is 9, the Value or Sum of that Series is an Unit of the Place next that Repetend on the left hand, so, $.\dot{9} = 1$, $.0\dot{9} = .1$ $.00\dot{9} = .01$, and so on. The Reason is plain from the *Theorem*; for $.\dot{9} = \frac{9}{9} = 1$, $.0\dot{9} = \frac{9}{90} = \frac{1}{10}$, $.00\dot{9} = \frac{9}{900} = \frac{1}{100}$

2. If a *Mix'd Circulate* is such that it has no other Figures than what belong to the Repetend, which therefore begins in some Integral Place, it's turn'd into a Vulgar Fraction by making the Denominator as many Places of 9's as the Repetend has Places; and the Numerator is that Repetend with as many o's on the right as there are Integral Places

Places in the given Circulate: So, $\dot{3}.\dot{4} = \frac{340}{99}$; $34.\dot{3}\dot{4} = \frac{3400}{99}$; $34\dot{3}.\dot{4} = \frac{34000}{99}$, and so on. The Reason is plain; for if in any of these Examples you suppose the Point set on the Left of all the Figures, it becomes a *Pure* Circulate, whose Finite Value has for its Numerator the Repetend, and as many 9's for its Denominator: Wherefore if the Point is set forward where it was at first, that removing it forward is in effect multiplying it by 1 with as many 0's as the number of Places on the Left of the Point in the given Position; so that as many 0's must be set on the Right hand of the Repetend, to make the Numerator of the Finite Fraction sought.

3*d*. If the Denominator of a Vulgar Fraction consists of 9's, or 9's with 0's on the right hand, the Numerator not having more Figures than the 9's in the Denominator, (after equal 0's are taken away from Numerator and Denominator) that Fraction resolves into a *Pure* Circulate, whose Repetend is the Numerator, with as many 0's on the left as the difference of the number of Places in the Numerator, and 9's in the Denominator; betwixt which Repetend and the Point there must be set as many 0's as stand after the 9's in the Denominator; so that if the Denominator is 9 &c. without 0's, the Repetend begins immediately after the Point.

Exa. (1.) $\frac{34}{99} = .\dot{3}\dot{4}$; (2.) $\frac{26}{999} = .\dot{0}2\dot{6}$; (3.) $\frac{32}{9900} = .00\dot{3}\dot{2}$;

(4.) $\frac{567}{9999900} = .000\dot{0}56\dot{7}$; (5.) $\frac{30}{9900} = \frac{3}{990} = .00\dot{3}$;

The Truth of this *Corollary* appears from its being plainly the Reverse of the *Theorem*; for such Circulates being form'd, as here directed, their Finite Value will, by the *Theorem*, necessarily become the suppos'd Fraction.

4*th*. Suppose a Vulgar Fraction as in the last, but let its Numerator have more Places than the 9's in the Denominator; that Fraction will be a *Mix'd* Circulate: More particularly if the Figures which the Numerator has more (on the right hand) than the number of 9's in the Denominator, be any of them a significant Figure, or other than 0, the Circulate must be sought by actual Division: But if these Figures be all 0's, the Circulate has no other Figures but what belong to the Repetend, which begins in some Integral place: And, to find this Circulate, suppose these 0's last mention'd to be taken away, then it becomes an Example of *Coroll.* 3; by which find its Circulate, and multiply this by 1 with as many 0's as were taken away, *i.e.* remove the Point as many places to the right hand.

Exa. (1.) $\frac{340}{99} = \dot{3}.\dot{4}$ (for $\frac{34}{99} = .\dot{3}\dot{4}$; and this multiply'd by 10 is $\dot{3}.\dot{4}$

(2.) $\frac{24000}{999} = \dot{2}4.\dot{0}$ (for $\frac{240}{999} = .\dot{2}4\dot{0}$; wwich multiply'd by 100, gives $\dot{2}4.\dot{0}$) (3.) $\frac{24000}{99} = 24\dot{2}.\dot{4}$

S C H O L I U M S.

(1.) If the Numerator of a Vulgar Fraction consist of the same Figures compleatly repeated, the Denominator having as many 9's as the Figures in the Numerator, that Fraction is the same as if it had but one Period of the Figures repeated, for its Numerator, and as many 9's for its Denominator, (with the 0's belonging to the given Denominator, if there were any) thus; $\frac{2424}{9999} = \frac{24}{99}$; for by *Cor.* 3. $\frac{2424}{9999} = .\dot{2}42\dot{4} = .\dot{2}\dot{4} = \frac{24}{99}$ Or thus; $24 : 99 :: 2400 : 9900$; therefore $24 : 99 :: 2424 : 9999$, from the Property of Proportionals. Hence $\frac{24}{99} = \frac{2424}{9999}$. And $\frac{2424}{99990} = \frac{24}{990}$. This is plain from the other.

Wherefore, if such a Fraction occurs, reduce it first to the Case of *Corol.* 3, and by that find the Circulate sought.

(2d.) If a Vulgar Fraction has a Repeating Numerator, the Repetend having as many Places as the 9's in the Denominator; or with some 0's after so many Places in all the Periods of the Repetend; or in them all but the last on the right; that Fraction is Equal to the Sum of two or more others, each of which will turn to a Circulate by the Rules of *Coroll.* 3d and 4th, whose Sum is therefore the Circulate sought. Thus,

$\frac{32\ 32}{99} = \frac{3200}{99} + \frac{32}{99} = 32.\dot{3}\dot{2}, + .\dot{3}\dot{2}$. (Exa. 2.) $\frac{250\ 250}{99} = \frac{250000}{99} + \frac{250}{99} = 25\,25.\dot{2}\dot{5}, + \dot{2}.\dot{5}$. (3.) $\frac{45\ 45}{990} = \frac{4500}{990} \left(= \frac{450}{99}\right) + \frac{45}{990} = \dot{4}.\dot{5}, + .0\dot{4}\dot{5}$

5th. From this *Theorem* we also learn, that no *Surd* Root can possibly be a *Circulate*; for Surds have no finite assignable Value, as has been demonstrated in its place, but Circulates have: Wherefore *all* Surds are necessarily Infinite Decimals of the *Incertain* Kind.

THEOREM 7.

Every Vulgar (finite) Fraction is reducible either to a Determinate Decimal, or to a *Circulate*.

DEMON. In the reduction of a Vulgar to a Decimal Fraction, after a Decimal Point in the Quote, we set as many 0's, less by 1, as are necessary to make the Numerator equal at least to the Denominator; and then the division begins, by which the Numerator of the Decimal is found; the Operation being continued by setting 0's to the Remainders successively, and at every Step finding a new Figure in the Quote: But now in Division, how great soever the Dividend be, or however many Figures the Quote contains, the Remainder must always be less than the Divisor: Therefore we can never make so many Steps in this Division as the Divisor expresses, till either we find 0 remaining, or two Remainders the same: For otherwise it would follow, that there are as many Numbers less than the Divisor as the Divisor it self expresses; which is manifestly absurd. Now in the reduction of a Vulgar Fraction, if the Division comes to 0 Remainder, the Decimal is plainly determin'd: But if two Remainders in the Work are found equal, then 'tis certain there must be a Circulation; for the Work will go on for ever, as it has done before betwixt these two equal Remainders, because the Figure to be prefix'd in the next Step is 0, which was also prefix'd to that preceding, and must be to all the succeeding.

SCHOLIUMS.

1. If it happens that there is a Remainder equal to the Given Numerator; then it's plain that all the Figures already set in the Quote will continually circulate, and so be the Repetend, having the same number of Places as that Repetend would have, which would be found by carrying on the Division till two Remainders are found equal; for it's plain that this would happen after you have made as many more Steps in the Work as the number of 0's which make the Numerator equal to the Denominator; as the annex'd Example shews.

$\frac{1}{21}$

$\frac{1}{21} = .\dot{0}4761\dot{9}$ $(= .0\dot{4}761904\dot{})$

Operation.

```
21) 100 (.04761904
    84
    ---
     160
     147
     ---
      130
      126
      ---
        40
        21
        ---
        190
        189
        ---
         100
          84
         ---
          16
```

Again; If a Remainder occurs equal to the Numerator with any o's on the right, then also you have already the Repetend, which is all the Figures set in the Quote after the Point, excluding as many o's next the Point as are in number equal to these o's on the right of this Remainder. The Reason is manifest.

2d. If an Improper Fraction is given, it will also resolve into an Improper Decimal, either Determinate or Circulate; the fractional part being the Resolution of the fractional part of the given Improper Fraction, and the integral part the same in both.

COROLL. The Repetend in any Circulate can never have more Places of Figures than the Number express'd by the Denominator, less by 1, of that Vulgar Fraction in its least Terms, which is equal in value to the Circulate; *i. e.* which, being reduced, will turn into it: But it may have fewer, as one Example shews, thus; $\frac{7}{13} = .\dot{5}3846\dot{1}$; And this is limited to a Vulgar Fraction in its least Terms, because it's plain that the same Fraction, in whatever Terms, being the same or equivalent Quote, must reduce into the very same Decimal, and consequently if it's a Circulate, the Repetend must be limited by the Denominator of that Fraction in its least Terms.

THEOREM 8.

Part I. If the Denominator of a Vulgar Fraction, in its lowest Terms, has in its composition no Primes but 2 or 5, that Fraction will reduce into a Determinate Decimal, whose Denominator is 1 with as many o's as are express'd by the Index of the highest Power of 2 or 5 (whichever of them has the highest) in the composition of the given Denominator.

Exa. $\frac{3}{40} = .075$, whose Denominator is 1000; and $40 = 2 \times 2 \times 2 \times 5$; so that 2 has the highest Power in the composition of 40, its Index being 3, the number of o's in 1000.

DEMON. Let the Vulgar Fraction be $\frac{N}{D}$, then because D has in it no Prime but 2 and 5, it may be thus represented; $D = 2^n \times 5^m$ (n, m being either equal or different) so that the Fraction is $\frac{N}{2^n \times 5^m}$. Now suppose, according to the Rule of reducing a Vulgar to a Decimal, that N is multiply'd by some Decimal Denominator, or Power of 10, thus, $N \times 10^r$; if this Index r is less than n or m, then $2^n \times 5^m$ cannot measure $N \times 10^r$ (*Theor.* 10, *Ch.* 1.); But if r is equal to n or m, whichever of them is the greater, Then $2^n\, 5^m$ must measure $N \times 10^r$; for $10^r = 2^r \times 5^r$; wherefore 10^r is the Denominator of a Decimal equal to $\frac{N}{D}$.

Part II. If the Denominator D of a Vulgar Fraction $\frac{N}{D}$, in its least Terms, is any other Prime, or has in its composition any other Prime than 2 or 5, (tho' it has these also) that Fraction must resolve into a Circulate. And, the number of o's necessary to finish the Reduction, and discover the first Period of the Repetend, is equal to the number of Places in the Denominator of the finite Value of that Circulate, taken in the Expression of *Theor. 6.*

Exa. $\frac{2}{3} = .66$ &c. or $.\dot{6}$; $\frac{5}{13} = .\dot{3}8461\dot{5}$

D E M O N. The Denominator has in it no Prime which is in the composition of the Numerator (because the Fraction is in lowest Terms) and it has some Prime other than 2 or 5, which is therefore in no Power of 10, wherefore it cannot measure the Product of that Numerator by any Power of 10 [*Theor.* 10, *Ch.* 1.] and so cannot make a Determinate Decimal, consequently must turn into a Circulate [*Theor.* 7.]

Again; $\frac{N}{D} = \frac{B}{9\,\&c.}$ or $\frac{B}{9\,\&c. \times 10^a}$ by *Theor. 6, Part* 2d; in which 9 &c. hath as many Places as the Repetend of the Circulate, and 10^a as many as (i. e. a is equal to) the number of Places betwixt the Point and Repetend: Wherefore tis plain that the Places of this Denominator $9\,\&c. \times 10^a$ are precisely as many as the o's used in order to finish the first Repetend; Because for every such o there is some Figure placed in the Quote after the Point.

S C H O L. If a Fraction is not in its least Terms, and the Denominator have in it Primes neither 2 nor 5; yet if the same Primes, in the same or a greater Power, be also in the Numerator; Then, because these Primes are out of the Denominator when the Fraction is reduced to least Terms, the Fraction becomes a finite Decimal. Also if any Prime in the Denominator, not 2 nor 5, is not, or is in a lower degree, in the Numerator, that Fraction becomes a Circulate; because, being in lowest Terms, such Primes are all out of the Numerator.

L E M M A.

Let D be any Number in whose composition there is neither 2 nor 5; I say, there is some Number express'd by 9's, as, 9, 99, 999, &c. which is a Multiple of D; *i. e.* take the least Number of 9's, which, written one after another, makes a Number not less than D (which will necessarily consist of as many Places of 9's as there are Places in D); divide that by D, and to the remainder prefix 9, and then again divide; go so on, to every remainder prefixing 9, and dividing by D, there will at last be no remainder; so that the 9's used, written one after another, is a Multiple of D.

Exa. $3 \times 7 = 21$, and $999999 \div 21 = 47619$

D E M O N. $\frac{1}{D}$ is a Fraction in its least Terms; and D having neither 2 nor 5 in its composition, this Fraction must reduce to a Circulate [*Theor.* 8.] whose finite value, according to *Theor.* 6, is $\frac{B}{9\,\&c.}$ or $\frac{B}{9\,\&c. \times 10^a}$ wherefore these are $::1, 1 : B :: D : 9$ &c. or $9\,\&c. \times 10^a$; but 1 measures B, therefore D must measure 9 &c. or $9\,\&c. \times 10^a$. If it's the first Case, the thing propos'd is prov'd; if it's the other, then, because 10^a has no Primes but 2, 5, D, which has neither 2 nor 5, is Prime to 10^a, and consequently it must measure 9 &c. [*Theor.* 6, *Ch.* 1.]

COROL-

COROLLARIES.

1st. If we take a Number of 9's written successively one after another, and whose Number is a Multiple of the least Number of that kind which D measures, D will also measure that assumed Number.

Exa. If D measure 99, it will also measure 9999 or 999999.

2d. If a Number D has neither 2 nor 5 in its composition, then there is some Number express'd by a Number of 3's written successively one after another, as, 3, 33, 333, *&c.* and also a Number express'd by 1's, as, 1, 11, 111, *&c.* which is a Multiple of D; For either D has in its composition some Power of 3, or not: If it has not, then take 99 &c. $=9\times 11$ &c. a Multiple of D; and because D, 9, are Prime to one another, but D measures 99 &c. $=9\times 11$ &c. consequently D measures 11 &c. And hence again it must measure 33 &c. $=3\times 11$ &c. Again; If D has any Power of 3, let $D=3^n\times a$; And if we take $3^{n+2}\times a$, this measures some Number 99 &c. $=3^2\times 11$ &c. And dividing them equally by 3^2, it follows that $3^n\times a$ measures 11 &c. and therefore also it must measure 3×11 &c. $=33$ &c.

THEOREM 9.

If a proper Fraction in its least Terms has a Denominator which is not measurable by 2 nor 5, it will reduce into a *Pure* Circulate whose Repetend begins immediately after the Point, and has as many Places as the least Number of 9's, which written one after another is a Multiple of the Denominator; which is also the number of o's necessary to be used, in order to find or bring out the first Period of the Repetend.

Exa. $\frac{5}{21}=.\dot{2}3809\dot{5}$; and 999999 is the least Number of 9's which is a Multiple of 21.

DEMON. $\frac{N}{D}$ being the given Vulgar Fraction, D measures some Number 99 &c. (*per* Lem.) And, supposing 99 &c. the least Number of this kind that D measures, take these $::1$, $D:99$ &c. $::N:R$; then since D measures 99 &c. N must measure R, therefore R is an Integer; for else $N\times q$ (q being an Integer, the Quote of $R\div N$) would be a *Mix'd* Number, *i. e.* the Product of two Integers, a *Mix'd* Number, which is impossible. Again; $D:N::99$ &c.$:R$, and $\frac{N}{D}=\frac{R}{99\,\&c.}$ But by *Corol.* 3, *Theor.* 6, $\frac{R}{99\,\&c.}$ is equal to a *Pure* Circulate whose Repetend begins immediately after the Point, and is R with as many o's on the left as the difference of the number of Places in R and 99 &c, so that it has as many Places as this 99 &c. whence the number of o's necessary in the reduction to finish the first Period of the Repetend is plainly equal to that number of 9's. Nor can this Repetend possibly contain in it a lesser Repetend beginning immediately after the Point; for supposing that lesser Repetend to be A, and the Denominator 9 &c. then is $\frac{N}{D}=\frac{R}{99\,\&c.}=\frac{A}{9\,\&c.}$ (See *Schol.* 1, *Theor.* 6.) and N, D being incommensurable, D measures 9 &c, which having fewer 9's than the other 99 &c. this other is not the least which D measures, contrary to Supposition: Wherefore 99 &c. the least number of 9's that is a Multiple of D, is the number of Places in the least Repetend of a *Pure* Circulate to which $\frac{N}{D}$ resolves; which is also the least number of o's necessary in the reduction to finish the first Period of the Repetend.

SCHOL.

SCHOL. If an Improper Vulgar Fraction is given, take out the Integral Part, and the *Theorem* applies to the remaining Fraction: Or that Improper Vulgar becomes an Improper Decimal or Mixt Circulate, whose Fractional Part, taken by it self, is a Pure Circulate: So that every Vulgar Fraction (proper or improper) whose Denominator is not measurable by 2 nor 5, becomes a Circulate whose Repetend begins immediately after the Point.

THEOREM 10.

If the Denominator of a Vulgar Fraction $\frac{N}{D}$, in its lowest Terms, has in its composition any Powers of 2 or 5 with other Primes, take out all these Powers of 2 and 5, and take the Result or Product of all the remaining Primes, *i.e.* divide D by 2 and 5 as oft as possible without a Remainder, and mark the last Quote; the given Fraction will reduce to a Circulate, *Pure* or *Mixt*, whose Repetend has as many Places as the least number of 9's, which is a Multiple of that last Quote; and it begins after so many Decimal Places as are express'd by the Index of the highest Power of 2 or 5, whichever of them has the highest Power involv'd in D.

Exa. $\frac{13}{420} = .03\dot{0}9523\dot{8}$; and $420 = 2 \times 2 \times 5 \times 3 \times 7$; out of which all the 2's and 5's being taken, there remains $3 \times 7 = 21$, and the least number of 9's, which is a Multiple of this, is 999999

DEMON. $\frac{N}{D}$ resolves into a Circulate, because D has in it some other Prime than 2 or 5, [*Theor.* 8.] but again D being suppos'd to have in it some Power of 2 or 5, or of both, may be represented thus; $D = A \times 2^n$, or $A \times 5^m$, or $A \times 2^n \times 5^m$ [which last may represent all the Cases; for if the Index n or m is o, that Factor is expung'd] so that A is a Number which has in it no power of 2 or 5; Then is $\frac{N}{D} = \frac{N}{A \times 2^n \times 5^m} = = \frac{N}{A} \times \frac{1}{2^n \times 5^m}$; but $\frac{1}{2^n \times 5^m}$ being a Fraction whose Denominator has no Prime but 2 or 5, it is (by *Theor.* 8.) resolvable into a Determinate Decimal whose Denominator is 10^r (r being $= n$ or m, which soever of them is the greater) and may therefore be express'd thus, $\frac{a}{10^r}$: Hence $\frac{N}{D} = \frac{N}{A} \times \frac{a}{10^r} = \frac{Na}{A} \div 10^r$. Again; $\frac{Na}{A}$ may be an Improper Fraction, but cannot be an Integer; for, if it is, then $\frac{Na}{A} \div 10^r \left(= \frac{N}{D}\right)$ is necessarily an Integer, or a determinate Decimal; either of which is impossible, because $\frac{N}{D}$ is suppos'd to be such that it resolves into a Circulate: Wherefore A being a Number not measurable either by 2 or 5, $\frac{Na}{A}$, if it's a proper Fraction, resolves into a *Pure* Circulate, whose Repetend begins immediately after the Point (by *Theor.* 9). And if it's an improper Fraction, it resolves into a *Mixt* Circulate, whose Finite Part is an Integer, and the other Part a *Pure* Circulate; and therefore the complete Quote is a *Mixt* Circulate whose Repetend begins immediately after the Point. Now this Decimal Quote being divided by 10^r (as $\frac{Na}{A} \div 10^r$ directs) it's plain the Figures of it will not be chang'd; only the Decimal Point will be remov'd as many places to the left hand as there are Units in

in r; the Effect of which is plainly this, that whereas the Repetend began immediately after the Point, it begins now after so many places as r expresses; which is the thing to be shewn. *Observe* also, that as $\frac{N\ a}{A}$ makes a *Pure* or *Mixt* Circulate, so does $\frac{N}{D}$; the reason of which is manifest; for in the first Case the Point is remov'd by setting only o's on the left hand of the Repetend; and in the other Case the Integral part supplys all or some of those places of o's before the Repetend, and consequently makes a *Mixt* Circulate.

PROBLEM I.

To add Circulates.

RULE. Make them all *Similar*, (by *Theor.* 5.) then take the Sum of the Repetends upon a separate Paper, and divide it by a Number consisting all of 9's, as many as the number of Places in the Repetend; the remainder of the division is the Repetend of the Sum, to be set under the Figures added, with o's on the left hand if it has not as many Places as the Repetends: The Quote is to be carried to the next Column, and the rest of the Addition done by the common Rules.

Exam. 1.	*Ex.* 2.	*Ex.* 2. reduced.	*Ex.* 3.	*Ex.* 3. reduced.
$.45\dot{3}$	$3.0\dot{4}$	$3.04\dot{4}\dot{4}$	$67.3\dot{4}\dot{5}$	$67.3\dot{4}545454\dot{5}$
$.06\dot{8}$	$8.45\dot{6}$	6.4566	$8.\dot{6}2\dot{1}$	9.621621621
$.32\dot{7}$	$23.\dot{3}\dot{8}$	23.3738	$.\dot{2}\dot{4}$	$.242424242$
$.94\dot{6}$	$.2\dot{4}\dot{8}$	$.2484$	$.\dot{8}$	$.888888888$
$1.79\dot{6}$		$33.12\dot{3}\dot{4}$		$78.09\dot{8}38929\dot{8}$

Ex. 4.	*Ex.* 4. reduced.	*Ex.* 5.	*Ex.* 5. reduced.
267.3456	$267.3456\dot{0}\dot{0}$	$.47836$	$.4783\dot{6}00000\dot{0}$
$33.\dot{8}$	39.888888	$.8\dot{7}2\dot{5}$	$.87257257257$
$.6\dot{7}\dot{2}$	$.672727$	$.\dot{3}\dot{9}$	$.39393939393$
	$307.9072\dot{1}6$		$1.74487\dot{1}9665\dot{1}$

Explanation of the Examples.

In *Exa.* 1. the Repetends are all upon one Figure in the same place, and their Sum is 24, which contains two 9's, and 6 remaining; and so 6 is set in that Sum as the Repetend, and 2 carried to the next Column. In *Ex.* 2d reduced to Similar Circulates whose Repetends have two Figures, the Sum of these Repetends is 232, which divided by 99, the Quote is 2, and 34 remains, so that 34 is the Repetend of the Sum, and 2 is carried to the next Column. In *Ex.* 3d reduced to Similars, the Sum of the Repetends is 2389296, which divided by 999999, the Quote is 2, and 389298 remains, which is the Repetend of the Sum, and 2 carried to the next Column. In *Ex.* 4th & 5th there is a finite Decimal, which is also reduced to the form of a Circulate by o's annex'd to it, which, *observe*, ... they do not alter the Sum. And in such Examples the

the Similar Repetends will always begin after the last Figure of that Finite Decimal which has the greatest number of Decimal Places.

Observe also, That if the Repetend of the Sum consists of the same Figure repeated, the true Repetend is but that one Figure; as in the following Example, the Sum, according to the preceding Rule, comes out $.82\dot{2}$, which is the same thing as $.8\dot{2}$

Example.
$$\begin{array}{r} .3\dot{4}\dot{9} \\ \underline{.4\dot{7}\dot{5}} \\ .8\dot{2}\dot{2} = .8\dot{2} \end{array}$$

DEMON. By *Theorem* 6, the Finite Value of a *Pure* Circulate is a Fraction whose Numerator is the Repetend, and its Denominator a Number of as many Places of 9's, with as many 0's on the right as there are 0's betwixt the Point and Repetend. Now let the Similar Repetends of several Circulates be added, their Sum is a Numerator to the common Denominator, and this Fraction is the Value of the Sum of these Circulates. Call the Numerator, or Sum of the Repetends, s, and the Sum sought is $\frac{s}{99\,\&c. \times 10\,\&c.} = \frac{s}{99\,\&c.}$ of $\frac{1}{10\,\&c.}$, so that the Sum sought is the Fraction $\frac{s}{99\,\&c.}$ refer'd to an Unit of the Value of the Place next the Repetend on the left hand: Consequently as oft as the Denominator 99 &c. is contain'd in the Numerator s, that Fraction is equal to so many Units of the Value of that next place; and the Remainder is the Numerator of a Fraction having the same common Denominator 99 &c. and to be refer'd to an Unit of the same Place or Value: Wherefore 'tis evident that the remainder of the Division (of $s \div 99$ &c) being placed as a Repetend in the same places as the Repetend added (supplying what places it wants with 0's on the left hand) and the Quote being carried to the next place, the rest of the places added in common form, we have the true Sum sought in all Cases, whether of *Pure* or *Mix'd Circulates*.

PROBLEM 2.

To subtract Circulates.

RULE. Make the Subtrahend and Subtractor *Similar Circulates*, and subtract as they were Finite Decimals: Then, if the Repetend of the Subtractor is a lesser Number than that of the Subtrahend, the Figures in the remainder that stand under the given Repetends (*i. e.* that are their Difference) is the Repetend of the Difference sought; But if the Repetend of the Subtractor is greatest, subtract 1 from the Repetend of the remainder, and the Figures that stand, after this 1 is subtracted, under the given Repetends, make the Repetend of the difference.

Exa. 1.	*Ex.* 2.	*Ex.* 3.	*Ex.* 4.
8.467	$24.3\dot{8}\dot{4}$	$.4\dot{2}\dot{7}$	$4.\dot{3}\dot{7}$
$.73\dot{5}$	$9.0\dot{7}\dot{2}$	$.0\dot{3}\dot{4}$	$.\dot{1}\dot{7}$
$7.73\dot{2}$	$15.3\dot{1}\dot{2}$	$.3\dot{9}\dot{2}$	$4.\dot{2}\dot{0}$

Ex. 5.

Ex. 5.	Ex. 6.	Ex. 7.	Ex. 8.
$3.5\dot{3}\dot{3}$	$.7\dot{4}\dot{2}$	$3.85\dot{6}\dot{4}$	$.46\dot{3}\dot{7}$
$2.4\dot{1}\dot{4}$	$.4\dot{1}\dot{8}$	$.03\dot{8}\dot{2}$	$.37\dot{3}\dot{7}$
$1.1\dot{2}\dot{2}$	$.3\dot{2}\dot{4}$	$3.81\dot{8}\dot{1}$	$.09\dot{3}\dot{3}$
$1.1\dot{2}$		or $3.\dot{8}\dot{1}$	$.09$

These *Examples* are so easily compared with the Rule, I shall not insist upon it.

DEMON. If two pure Circulates are Similar, and if the lesser is to be subtracted from the greater, the Reason of the Rule is manifest. But in mixt Circulates the Repetend of the Subtractor may be greater than that of the Subtrahend; and in this Case to follow common Rules, we should add the common Denominator of the Finite Value of the Circulating Parts, with relation to an Unit of the place next the Repetend, the common Denominator is *99, &c.* consisting of as many *9's* as the places of the Repetend (as has been already explained;) But by Subtracting in the common way, 'tis plain we do add 10, *&c.* the 0's being as many as the Places of the Repetend; now it's evident, that if instead of *99, &c.* we add 100, *&c.* (this having as many 0's as the other has *9's*) we have added 1 too much; and therefore 1 is to be taken from the remainder according to the Rule; the rest is obvious. So in *Ex.* 3. the Repetend having two Places, the Finite Values of the Circulate Parts are $\frac{27}{99}$ of $.\dot{1}$ and $\frac{34}{99}$ of $.1$; but 34 cannot be taken from 27, therefore I Subtract in common Form, whereby I do add 100 to the Repetend 27, which makes the remainder *93*: But because I should only have added *99*, I take one from the remainder and it is *92*; then because in Subtracting the Circulate Parts, 1 was borrowed from the next Place (for we considered the Circulate Parts as Fractions referred to an Unit of that next Place) therefore 1 is added to the next Place of the Subtracter, and so the Work is carried on.

PROBLEM 3.

To Multiply Circulates.

RULE. Express Circulates by their Finite Values, and then Multiply by the Rule of Vulgar Fractions, reducing and compleating the Answer as the Question requires: And particularly, carry on the Division of the Product of the Numerators by that of the Denominators till 0 remain; or till you come at a Repetend. But if this does not soon happen, then it may be left off at any place you please: But if you are content to have the Product in a Vulgar Fraction, you have it already compleatly in the Product of the two Finite Values of the given Numbers.

The *Reason* of this Rule is manifest; because it's reduced to that of Vulgar Fractions, which is demonstrated in its place; see the following Examples.

Exam. 1. To Multiply $8.4\dot{7}$ by .68, having reduced the first to its Finite Value it is $\frac{763}{90}$, which Multiplied by .68 or $\frac{68}{100}$ produces $\frac{51884}{9000}$ which being reduced to a Decimal is $5.764\dot{8}$ as in the Margin. For the Divisor being

 9000

```
 763
  68      9000) 51.884
 ----             5.7648
6104
4578
-----
51884
```

9000, I first take off three places from the Dividend, which is dividing it by 1000, then I divide this by 9, which gives for a quote 5.764 and 8 remains, to which 0 being prefixt, the next Figure in the quote is 8, and 8 again remains, therefore 8 is repeated.

Exam. 2. To Multiply $7.68\dot{4}$ by $.4\dot{5}$. Being reduced to their Finite Values they are $\frac{6916}{900}$ and $\frac{41}{90}$ whose Product is $\frac{283556}{90 \times 900}$, which being reduced to a Decimal is $3.50\dot{0}691358024\dot{}$ All that needs be said as to the Operation is, that in the Division of $31.506\dot{2}$ by 9, when we have got the quote so far as 3.5006 then to every succeeding Remainder the Repetend 2 is prefixt; and so the Work is carried on, till there is a Circulation as marked in the Example.

```
 6916
   41      9| 283.556
 ----      9|  31.5062
 6916       |       3.50069135802
27664
------
283556
```

Exam. 3. To Multiply $65.7\dot{2}\dot{3}$ by 4.6; the first reduced to its Finite Value is $\frac{65066}{990}$ which Multiplied by 4.6 or $\frac{46}{10}$ produces $\frac{2993036}{990 \times 10}$ equal to this Decimal $302.32\dot{6}\dot{8}$. which is found thus: for the two 0's in the Denominator 990×10 I take off two Places in the Numerator, and then divide by 99; which gives for a quote 302.32, and 68 remains, which being $\frac{68}{99}$ of an Unit of the Value of the last place of the quote, is therefore to be placed after it as a Repetend.

```
 65066    99| 29930.36
    46      |    2 2 6 8
 -----      | -----------
390396      |  302.3268
260264
-------
2993036
```

Observe; the dividing by 99 is here done by *Case* 5. § 2. *Chap* 6. *Book* 1. with this difference only, that I have here placed the Figures of the Operation, and also the quote below the Dividend; and every quote Figure under the first, and not the last Figure of the Dividend, as is there done.

Exam. 4. To Multiply $74.03\dot{6}\dot{7}$ by $4.7\dot{5}$; their Finite Values are $\frac{73624}{9900}$ and $\frac{428}{90}$ whose Product is $\frac{31511072}{9900 \times 90}$ which reduced is $35.365\dot{9}618406285072951\dot{7}\dot{3}$

```
73624 × 428 = 31511072
    then
 9| 31511.072.
99|  3501.230222222222222222222
  |       3659518306 8407 9417395
  |          6   4     5     5
  --------------------------------
   35.365961840628507295173
```

For having taken three Places from the Product of the Numerators for the three 0's in the Denominator; I proceed to divide first by 9 which quotes $3501.230\dot{2}$ and then I divide by 99, which gives the quote mentioned; for it's plain, that having brought the Division so far as is here set down, the last remainder is 95, the same with a preceeding remainder belonging to the fifth step of the Division: Wherefore all the Figures in the quote, from the sixth which is 9, do Circulate.

SCHOL.

SCHOLIUM. As this *Rule* is universal, so it is easily kept in mind, if you but remember the Rule for finding the *Finite* Value of Circulates; nor is it much more tedious than the Multiplication of Finite Decimals, considering how easily the Finite Value of a Circulate is found; and how easy it is to divide by their Denominators, which consist all of *9*'s, or with o's; as the preceding Examples shew.

There are other Rules for this Multiplication, in some things different from the general Rule, but little or nothing easier or shorter in the Operation; and therefore I might reasonably pass them over. Yet that you may know the different ways of managing Circulates, and chuse as you like best, I shall here also explain the Multiplication of Circulates in two particular Cases, in order to which; mind that we call that given Number the Multiplier which has fewest significant Figures.

Case 1. The Multiplicand being a Circulate, and the Multiplier an Integer or Finite Decimal, *Simple* or *Mixt*.

RULE. Multiply by each Figure of the Multiplier, Thus; take first the Product of that Repetend (of the Multiplicand) and divide it by a Number consisting all of *9*'s, as many as the Number of Places of the Repetend: Write down the remainder in the Product, and carry the quote to the Product of the next Place, and go on with the other Places in common Form: And observe that this remainder is a Repetend in every partial Product, and if it has not as many Places as the Divisor, or Repetend of the Multiplicand, you must supply the Defect with o's on the left; and in this State set it in the Product as the Repetend. When you have thus got all the partial Products for every Figure of the Multiplier; make all the Repetends similar, which is done by drawing them all out as far as the first; then add them by *Probl.* 1. the Sum is the Product sought, in which set the Decimal Point according to the common Rule.

Ex. 1.

$$\begin{array}{r} 8.4\dot{7} \\ .68 \\ \hline 678\dot{2} \\ 5086\dot{6} \\ \hline 5.764\dot{8} \end{array}$$

Ex. 2.

$$\begin{array}{r} 65.7\dot{2}\dot{3} \\ 4.6 \\ \hline 3943\dot{3}\dot{9} \\ 26289\dot{2}\dot{9} \\ \hline 302.32\dot{6}\dot{8} \end{array}$$

In this *Ex.* 2d. $23 \times 6 = 138$ which divided by *99*, quotes 1 and *39* remains; therefore *39* is the Repetend of the Product, and 1 is carried to the next Place, or to the Product 6×7, and so that Line is carried on.

Again $23 \times 4 = 92$ which divided by *99*, the quote is o, and *92* remains; which is therefore the Repetend of the Product, the rest of which is found by the common Rule. But to make this similar to the other, it's reduced to *29*; then in summing the two partial Products, $39 + 29 = 68$ being less than *99*, is the Repetend of the Sum, and o carried to the next Column.

Ex. 3.

$$\begin{array}{r} 52.7\dot{6}7\dot{8} \\ .43 \\ \hline 1583\dot{0}3\dot{6} \\ 21107\dot{1}4\dot{7} \\ \hline 22.690\dot{1}8\dot{3} \end{array}$$

Here $678 \times 3 = 2034$; and this divided by *999* quotes 2, and 36 remains, therefore the Repetend of the Product is $\dot{0}3\dot{6}$, and the quote 2 is carried to the next Place. Again, $678 \times 4 = 2712$ which divided by *999* quotes 2, and 714 remains, which is the Repetend of the Product: The rest of the Work is obvious.

Case 2. The Multiplier being a Circulate, whatever the Multiplicand is. *Rule*. Take the Finite Value of the Multiplier, and by its Numerator multiply the Multiplicand, by the Method of *Case* 1. then divide the Product by the Denominator.

Exa.

$$\begin{array}{r} 7.68\dot{4} \\ 41 \\ \hline 768\dot{4} \\ 30737\dot{7} \\ \hline 315.06\dot{2} \\ \hline \end{array} \qquad 90)\,315.36\dot{2} \qquad 3.50069135802\dot{4}$$

To Multiply $7.68\dot{4}$ by $.4\dot{5} = \frac{41}{90}$, the Product is $3.50069135802\dot{4}$ compare this with the *Ex*. 2. to the general Rule, it is the same Example, and the Answer the same. And the Reason and Method of Operation being also obvious, I shall insist no more upon it.

Problem 4.

To divide Circulates.

Rule. Express Circulates by their Finite Values, and then apply the Rule of Vulgar Fractions.

Example. To divide $22.690\dot{1}8\dot{3}$ by $52.7\dot{6}7\dot{8}$; being reduced to their Finite Values they are $\frac{22667493}{999000}$ and $\frac{527151}{9990}$; and the first being divided by the other quotes $\frac{22667493}{52715100}$ (for the two Denominators having 9990 as a common Factor, the quote is reduced to this) whose Value in a Decimal is .43.

Scholium. As Multiplication was explained in two particular Cases differing from the general Rule, so may Division; thus,

Case 1. The Dividend being Circulate, but not the Divisor; *Rule*. Divide as they were Finite Decimals, carrying on the Operation by applying the Repetend so oft till either the Quote circulate; or till you have a sufficient Number of Places: But because in many Cases the Circulation of the Quote will not happen till after a very long Operation, if you would have a compleat Quote, you must take it by the preceding general Rule in a Vulgar Fraction.

Exa.

$$\begin{array}{l} .68)\,5.764\dot{8}\,(8.4\dot{7} \\ \quad\; 5\,44 \\ \quad\; \overline{\;\;324} \\ \quad\;\;\; 272 \\ \quad\;\;\; \overline{\;\;528} \\ \quad\;\;\;\; 476 \\ \quad\;\;\;\; \overline{\;\;52} \end{array}$$

Exa.

$$\begin{array}{l} 8)\,46.5287\,(5.816091 \\ \quad 40 \\ \quad \overline{\;\;65} \\ \quad\;\; 64 \\ \quad\;\; \overline{\;\;12} \\ \quad\;\;\; 8 \\ \quad\;\;\; \overline{\;\;48} \\ \quad\;\;\; 58 \\ \quad\;\;\; \overline{0072} \\ \quad\;\;\;\;\; 72 \\ \quad\;\;\;\;\; \overline{008} \\ \quad\;\;\;\;\;\;\; 8 \\ \quad\;\;\;\;\;\;\; \overline{8} \\ \quad\;\;\;\;\;\;\; \overline{0} \end{array}$$

In

In *Exa.* 1. the Repetend is 8; For the Quote being brought to 8.47, the Remainder is 52, which is the same as the preceding Remainder; to which the same Repetend 8 being prefix'd, the same Quote 7 must continually come out.

In *Exa.* 2d, after one Period of the Repetend is employ'd, we come at this Quote 58160; but to come at a Circulation in the Quote, we must proceed two Steps farther, by employing again the same Figures of the Repetend in order: and after the two first, *viz.* 2 and 8, are used a second time, we have the same Remainder 0 which was upon the third preceding Step, and therefore the same Figures will repeat again, and so the Quote is $5.81\dot{6}09\dot{1}$

Case 2d. If the Divisor is a Circulate, whatever the Dividend is; take the Finite Value of the Divisor; and by its Denominator multiply the Dividend. by *Probl.* 3d, (either by the general Rule or the particular Case 1); then divide the Product by the Numerator (according to the preceding Case, if that is Circulate) and you have the Quote sought.

Exa. 1. To divide $5.764\dot{8}$ by $8.4\dot{7} = \frac{763}{90}$. First I multiply $5.764\dot{8}$ by 90, the Product is 51.884, which divided by 763 quotes .68.

Exa. 2. To divide $3.50\dot{0}69135802\dot{4}$ by $7.68\dot{4} = \frac{6916}{900}$, I multiply $3.50\dot{0}69135802\dot{4}$ by 900, the Product is $3150.\dot{6}2$, which divided by 6916, quotes $.4\dot{5}$

CHAP. V.

Of Logarithms.

DEFINITION.

Logarithms are Numbers so contriv'd and adapted to other Numbers that the Sums and Differences of the former correspond to, and shew, the Products and Quotes of the other, and also their Powers and Roots.

SCHOL. 1. This Definition expresses in general the Design and Use of the Numbers call'd *Logarithms*; but, for the more strict and etymological Sense of the Word *Logarithm*, and other Definitions deduced from it more immediately, they will be better understood after we have explain'd the Foundation of their Contrivance; which you have in this

LEMMA.

Take any Geometrical Progression of Numbers beginning with 1, whose second Term call a, the Series is $1 : a^1 : a^2 : a^3 : a^4 :$ &c. $: a^n$, whereof every Term after 1 is some Power of the second Term a, their Indexes being a Series in Arithmetical Progression, which express the Distances of the several Terms after 1.

From the nature of this Geometrical Series, and what has been explain'd in *Book* 3, *Theor.* 6, 7, 8, these Consequences are manifest, *viz.*

COROLLARIES.

1*st.* The Product or Quote of any two Terms is also a Term of the Series whose Index (or Distance after 1) is the Sum or Difference of the Indexes of these two Terms. *Exam.* $a^3 \times a^4 = a^7$. $a^7 \div a^3 = a^4$. Universally, $a^n \times a^m = a^{n+m}$, and $a^n \div a^m = a^{n-m}$.

2*d.* Any

2*d*. Any Power of any of these Terms is a Term of the Series whose Index is the Product of the Index of that Power by the Index or Distance of that Term from 1. *Exam.* The Square of a^3 is a^6; Universally; the n Power of a^m is a^{nm}. *Reversely*: If the Index of any Term is Multiple of any Number, then, being divided by that Number, the Quote is the Index of a Term in the Series, which is such a Root of the Term whose Index is divided as the Divisor denominates. *Exa.* The Cube Root of a^6 is a^2. Universally; the n Root of a^m is $a^{m \div n}$.

3*d*. If from the double of any Index, or the Sum of any two Indexes, be subtracted the Index of another Term, the Difference is the Index of a Term in the Series which is a third or fourth :: l to the 2d or 3d Terms whose Indexes are given. *Exa.* 1st, $a^2 : a^5 :: a^5 : a^8$; where $8 = 5 + 5 - 2$. *Ex.* 2d, $a^2 : a^5 :: a^7 : a^{10}$; where $10 = 5 + 7 - 2$. Universally; $a^n : a^m : a^{2m-n}$ are $\div l$; for by common Rules a third to $a^n : a^m$ is $a^n \times a^m \div a^n$; but $a^m \times a^m = a^{2m}$, and $a^{2m} \div a^n = a^{2m-n}$. Again; $a^n : a^m :: a^r : a^{m+r-n}$; for $a^m \times a^r = a^{m+r}$, and $a^{m+r} \div a^n = a^{m+r-n}$.

SCHOL. 2. Here then we have the *Fundamental* Grounds of the *Invention* of *Logarithms*: For 'tis obvious that the Indexes or Distances of the several Terms of a Geometrical Series from the first Term 1, are Numbers answering to the preceding Definition of *Logarithms*, for those Numbers that make the Geometrical Series; which I shall more particularly explain and apply: But first *Observe*, that from this Foundation is deduced the common Definition of *Logarithms*, viz. *Numbers in Arithmetical Progression answering to others in Geometrical Progression.* Again, *observe*, that in the more strict sense of the Word it signifies *a Number of Ratios*: And, to understand the reason of its application here, consider, That in a Geometrical Series the Ratio of the Extremes is compos'd of as many equal Ratios as the number of Terms less 1, or as the Number expressing the Distance of the one Extreme after the other; which Distance is therefore call'd the *Logarithm* of the Ratio of the one Extreme to the other; and if one Extreme is 1, it's call'd simply the *Logarithm* of that other; but it strictly signifies the Logarithm of the Ratio of 1 to that other.

Let us now suppose a to be any Number, as 2; the Geometrical Series from 1 : 2 is $1 : 2^1 : 2^2 : 2^3$: &c. or 1 : 2 : 4 : 8 : 16 : &c. And if this Series is carried to any length, and the several Terms be dispos'd orderly in a Table, and against them be set their Indexes or Distances from 1, (*viz.* their *Logarithms*) setting 0 against 1, because it's not distant from it self, as in the following Table; then from what's explain'd it is evident that we can, by means of these Indexes or Logarithms, find the Product or Quote of any two Terms of the Geometrical Series, without actual Multiplication or Division; also any Power or Rational Root of any Term; and, lastly, a third or fourth Proportional to any two or three of them, supposing the Table carried to a sufficient extent, as in the following Examples; the Rules of which are contain'd in the Consequences to the preceding *Lemma*, and which 'twill be useful to repeat here in somewhat a different form, with a direct Regard to the Practice and Use of Logarithmick Tables, under the Title of

The Fundamental General RULES *for the Use and Practice of* LOGARITHMS.

I. Add the Logarithms of any two Numbers, the Sum is a Logarithm, against which in the Table stands the Product of these two Numbers:

Exa. $8 \times 32 = 256$, whose Logarithm is $8 = 3 + 5$, the Logarithms of 8 & 32.

II. Take the Difference of the Logarithms of any two Numbers, it is that Logarithm against which stands the Quote of the greater divided by the lesser of these two Numbers. *Exa.*

Exa. $2048 \div 128 = 16$, whose Logarithm is $4 = 11 - 7$, the Logarithms of 2048 & 128.

III. Multiply the Logarithm of any Number by any Number, the Product is a Logarithm against which stands that Power of the Number whose Logarithm is multiply'd, denominated by the Multiplier, *viz.* the n Power, if the Multiplier is n.

Exa. $16^3 = 4096$, whose Logarithm is $12 = 4 \times 3$; the Logarithm of 16 (*viz.* 4) multiply'd by 3, the Index of the Power sought.

IV. Divide the Logarithm of any Number by any Number, and if there is no Remainder the Quote is a Logarithm against which stands that Root of the Number whose Logarithm is divided, denominated by the Divisor, *viz.* the n Root of the Divisor is n.

Exa. $4096^{\frac{1}{3}} = 16$, whose Logarithm is $4 = 12 \div 3$, the Logarithm of 4096 (*viz.* 12) divided by the Divisor 3.

V. From the Double of the Logarithm of any Number, or the Sum of the Logarithms of any two Numbers, subtract the Logarithm of any Number, the Difference is a Logarithm against which stands a Number that is a third or fourth Proportional to these two or three given Numbers.

Exa. 1) $4 : 16 : 64$ are $\div l$, whose Logarithms are $2 . 4 . 6$ $(= 4 + 4 - 2.)$

Exa. 2) $8 : 32 :: 256 : 1024$, whose Logarithms are $3 . 5 . 8 . 10$ $(= 5 + 8 - 3.)$

TABLE *whence these* Examples *are taken.*

N°.	Log.	N°.	Log.	N°.	Log.
1	0	64	6	4096	12
2	1	128	7	8192	13
4	2	256	8	16384	14
8	3	512	9	32768	15
16	4	1024	10	65536	16
32	5	2048	11	131072	17

SCHOL. 3. Any Arithmetical Progression beginning with o may be apply'd as Logarithms to any Geometrical one from 1: For tho' they will not be Logarithms according to the stricter meaning of the Word, (*i. e.* the number of Ratios from 1) yet they will answer the other Definitions, and the preceding Fundamental Rules: Which will be manifest from the common Properties of Arithmetical and Geometrical Progressions: For any three or more Terms, taken in either Series, are continuedly proportional in their kind; and any four Terms are proportional, whereof the first and second are as far distant as the third and fourth. Whence the Correspondence of the Arithmetical, as Logarithms to the Geometrical, according to the preceding Rules, is plain.

Geom. $1 : A : B : C : D : E : F :$ &c.

Arithm. $0 . a . b . c . d . e . f .$ &c.

Thus, $B \times C = E$ for $1 : B :: C : E$. Also $b + c = e$ for $0 . b : c . e$. Again, $F \div D = B$ for $1 : B :: D : F$. Also $f - d = b$ for $0 . b : d . f$.

The Rule for Powers is deduced thus; $B^2 = D$ for $1 : B :: B : D$; also $2b = d$ for $0 . b : b . d$. Again, $B^3 = F$, for $1 : B : D : F$ are $\div l$; then $3 \times b = f$, for $0 . b . d . f$ are $\div l$. And so it goes on thro' all the rest of the Powers, And the Rule for Roots is but the Reverse of this.

For the finding a 3d or 4th Proportional, the Rule must also still be good, because the Rules for Multiplication and Division are good.

SCHOL.

SCHOL. 4. Any Arithmetical Progression whatever may be apply'd as Logarithms to a Series geometrical from 1; but if the Logarithm of 1 be any other than 0, the preceding Rules will not answer, and instead of them we must put these:

1°. From the Sum of the Logarithms of two Numbers take the Logarithm of Unity, the difference is the Logarithm of the Product.

2°. To the difference of the Logarithm of two Numbers add the Logarithm of 1, the Sum is the Logarithm of their Quote.

The *Reason* of these Rules is plain from the Proportionality of the Terms with 1; for 1 is to the Multiplier as the Multiplicand is to the Product; or, 1 and the Multiplier are at the same distance as the Multiplicand and Product; So also the Divisor and Dividend are proportional with, or at the same distance as the Quote and Unity; but the corresponding Terms in the Arithmetical Series are also arithmetically proportional; whence the Rules are clear. So, in the preceding Example, suppose the Logarithm of 1, (*viz.* 0) to represent any Number; then, as $B \times C = E$, so $b + c - 0 = e$; and as $F \div D = B$, so $f - d + 0 = b$.

3°. The Rule for finding a 3d or 4th Proportional is the same in all Suppositions, and the Reason the same.

4°. For finding the Power of any Number, multiply its Logarithm by the Index of the Power, (viz. n) and from the Product take the Product of the Logarithm of 1, multiply'd by $n - 1$; thus $B^2 = D$, whose Logarithm is $d = 2b - 0$ (because $0 . b : b . d.$) Again, $B^3 = F$, for $1 : B :: D (= B^2) : F$; then $0 . b : d . f$, also $d = 2b - 0$; Whence $0 . b : 2b - 0 . f$ and $3b - 2 \times 0 = f$, the Logarithm of F, when 0 is the Logarithm of 1, and b the Logarithm of B. The Reasoning will proceed in this manner for ever. Or, the Demonstration of this Rule may be made *Universally*, thus; Let any Number in the Geometrical Series be A, then $1 : A :: A^n : A^{n+1}$; And if the Logarithm of 1 is 0, and that of A is a, then, according to this Rule, the Logarithm of A^n is $na - \overline{n-1}\,a$. Now if this Rule is good in one Case as the n Power, it's good in the next as the $n + 1$ Power; For $1 : A$ and $A^n : A^{n+1}$ standing at the same distance in the Geometrical Series, so do their corresponding Terms in the other Series: Call l the Term corresponding to A^{n+1}, then is l at the same distance from $na - \overline{n-1.a}$ (the suppos'd Logarithm or Term corresponding to A^n) as a is from 0; *that is*, $0 \cdot a : na - \overline{n-1} . l$ $(= \overline{n+1}\,a - na$, by the common Rules) which is therefore the Logarithm of A^{n+1} according to this Rule. But the Rule is shewn to be true for the *Square* and *Cube*, and consequently 'tis true for all above.

5°. To find the Root of any Number; to the Logarithm of the Power add the Product of the Logarithm of Unity by $n - 1$, and divide the Sum by n, the Quote (being integral) is the Logarithm of the Root. The *Reason* of this is contain'd in the last Rule; For if l is the Logarithm of the Root, and ll that of the Power whose Index is n, then is $ll = nl - n - 10$. Hence $ll + n - 10 = nl$, and $\frac{ll + \overline{n-1} \times 0}{n} = l$, which is the Rule.

SCHOL. 5. A Geometrical Series may not only encrease from Unity, but also decrease, thus, $\frac{1}{8} : \frac{1}{4} : \frac{1}{2} : 1 : 2 : 4 : 8$; Universally, $\frac{1}{a^4} : \frac{1}{a^3} : \frac{1}{a^2} : \frac{1}{a}$; $a : a^2 : a^3 : a^4$ &c. And to these Terms decreasing from Unity their Distances are also Logarithms, with this Variation only in the Practice, That we must consider them as negative Numbers, or Numbers less than 0, because their distance from 1 is upon the opposite side to that of the Integral Terms: And hence we must apply the *Rules* with a Regard to this, *i. e.* by

observing

obſerving the Rules of the Addition, *&c.* of negative Numbers, as explain'd in *Book* I.

Suppoſe $a = 2$, then the Series decreaſing is $\frac{1}{2} : \frac{1}{4} : \frac{1}{8}$, *&c.* which reduced to Decimal Fractions, are $.5 : .25 : .125 : 0625$, *&c.* And the whole Series encreaſing and decreaſing with their Logarithms, ſtand as in the following Scheme.

And theſe Logarithms being thus taken poſitive upon the one ſide of 0, and negative on the other, make ſtill a Series Arithmetical whereſoever you begin it, and ſo muſt anſwer to the Rules.

Exa. $16 \times .125 = 2.000 = 2$, whoſe Logarithm is $1 = -3 + 4$ or $4 - 3$, the Sum of the Logarithms of .125 and 16

Ex. 2. $4 \times .0625 = .2500 = .25$, whoſe Logarithm is $-2 = -4 + 2$, the Sum of the Logarithms of .0625 and 4

But *Obſerve* that the Logarithms of ſuch a mix'd Series may be made all poſitive to any Term of the Fractional part, by making 0 the Logarithm of any of the Fractions, and then the Logarithms of all above are poſitive; or by applying any Series of poſitive Numbers in Arithmetical Progreſſion as Logarithms; for theſe will anſwer according to the Rules explain'd in *Scholium* 4, and for the Reaſons there alſo demonſtrated. Thus, Suppoſe 0 the Logarithm of .0625, then the Logarithms of the reſt above it are as in this Scheme: And $16 \times .125 = 2$, whoſe Logarithm is $5 = 8 + 1 - 4$ according to *Rule* 1, *Schol.* 4; 8, 1, 4 being here the Logarithms of 16, .125, 1

.0625	.125	.25	.5	1	2	4	8	16
-4	-3	-2	-1	0	1	2	3	4

SCHOL. 6. Whatever Arithmetical Progreſſion we apply to a Geometrical one, they are Logarithms only to that Series to which we apply them, and anſwer the Ends propos'd only for theſe particular Numbers; ſo that if we have Logarithms adapted only to particular Geometrical Series, they would be of little Uſe. The great End and Deſign of Logarithms is, the Eaſe and Expedition of Calculations, by ſaving the Laborious Work of *Multiplication*, *Diviſion*, and *Extraction of Roots*: But this End could never be compleatly reach'd, unleſs Logarithms could be adapted to the whole Syſtem of Numbers, 1 . 2 . 3 . 4 . *&c.* And as here lay the Excellence and Merit of the Contrivance, ſo alſo the Difficulty; for the Natural Syſtem of Numbers, 1 . 2 . 3 . 4 . *&c.* being an Arithmetical, and not a Geometrical Series, ſeems rather fit to be made Logarithms, than to have Logarithms apply'd to it: Yet this Difficulty the Excellent Genius of the Renown'd Author, and Unrival'd Inventor of them, [the Lord *Neper*] conquer'd. And in order to underſtand his Method of Conſtructing theſe Logarithms, *Conſider*;

Tho' the Whole Natural Syſtem of Numbers, 1 . 2 . 3 . 4 . *&c.* makes not One Geometrical Series, and cannot by any Means be brought within one ſuch Series of Determinate Numbers, yet they may be brought near to it, within any aſſignable degree of Approximation; which may be conceiv'd in general, thus: Suppoſe a Fraction infinitely ſmall repreſented by x, and a Series Geometrical ariſing from 1 in the Ratio of 1 to $1 + x$, this Series is repreſented by $1 : \overline{1+x}^1 : \overline{1+x}^2 : \overline{1+x}^3 : \overline{1+x}^4$: &c. ſome of whoſe Terms muſt coincide, infinitely near, with the natural Numbers 2 . 3 . 4 . *&c.* becauſe among Numbers that ariſe by infinitely-ſmall Increments [as it is in this Caſe, ſince the common Ratio is infinitely near to a Ratio of Equality] ſome of them muſt exceed, or come ſhort of, any determinate Number by an infinitely little Exceſs or Defect: Wherefore if in the places of the Terms of this Series, that do approach infinitely near to any of the natural Numbers, we ſuppoſe theſe natural Numbers themſelves, then the Series will be a Geometrical Progreſſion to an exactneſs, which I call *Indefinite* (and not *Perfect*) becauſe the approximation of its Terms to the natural Numbers can never end, but goes on *in infinitum.*

Now, as this Imagin'd Geometrical Series comprehends infinitely near the whole System of Numbers 1 . 2 . 3 : &c. so their Indexes comprehend a complete System of Logarithms for the natural System of Numbers, extended to any length we please; and do answer to all the preceding Definitions and Rules: For, tho' the natural System make not by themselves a Geometrical Series, yet they are conceiv'd as a part of such a Series; and so the Logarithms are the Indexes of their Distances from Unity in that Series; or, more generally, they are the corresponding Numbers of an Arithmetical Series apply'd to that Geometrical one.

But again *observe*, that since we cannot assign an infinitely-little Fraction, therefore in the actual construction of Logarithms we must be content with a determinate degree of Approximation: Whence, according as we take x, so in the Series $1 : \overline{1+x} : \overline{1+x}^2 :$ &c. the Approximation of its Terms to the natural Numbers will be in different degrees; for the lesser x is, the nearer will the Approximation be: but then the more are the Involutions of $2 + x$ necessary to come within any determinate degree of nearness to any of the natural Numbers.

Thus then we may conceive the absolute Possibility of making Logarithms to the natural System 1 . 2 . 3 : &c. to any determinate degree of exactness, *viz.* by assigning a very small Fraction for x, and actually raising a Series in the Ratio of 1 to $1 + x$, and taking for the natural Numbers such Terms of that Series as are nearest to them, and their Indexes for the Logarithms. But *observe* also, that to construct Logarithms in this manner, to such an extent of Numbers, and degree of Exactness, as would be necessary to make Logarithms of any considerable use, is next to Impossible to us, because of the almost infinite Labour and Time it would require: This, however, is an Introduction for understanding the Method of the *Noble Inventor*, who, as he (no doubt) took the Hint of Logarithms from the Consideration of the Indexes of a Geometrical Series, so, to compleat the Invention, he behov'd to lay before him the Idea of a Geometrical Progression comprehending, infinitely near, all the Terms of the Natural Series; but, that the Labour of constructing these Logarithms might not be insuperable, he went to work another Way: For,

From the Foundation already laid down in the Consideration of an Infinite Progression, this Conclusion was obvious, *viz.* That if the natural Numbers were comprehended in this Series to an infinite or indefinite degree of nearness, there must also be an infinite or indefinite number of Means betwixt any two of the natural Numbers, or such Terms of the Progression in whose places we substitute them: And upon this *Principle* he constructed his Logarithms; the Method of which I shall explain in the following *Problems*.

PROBLEM I.

Betwixt any two given Integral Numbers, an indefinite Number of Geometrical Means being suppos'd, 'tis requir'd to find one of them so approximate that it be within an assign'd difference of a given Number which lies in the natural System betwixt the former two given Numbers.

RULE. Let the given Extremes be call'd A (the lesser) and B (the greater) and the given Mean C; Then, betwixt A, B find one Geometrical Mean; and if A, B admit not a rational Mean, find the Approximate to any number of Decimal Places: Call this Mean D; and, if D is less than C, find a Mean betwixt D and B; but if D is greater than C, find a Mean betwixt D and A: Call this second Mean E; then, as E is lesser or greater than C, find a third Mean betwixt E and the first Mean D, if this is contrarily greater or lesser than C; or betwixt E and the opposite Extreme B or A, if D is of the same quality with E (*i. e.* greater or lesser than C, as E is): Call this third Mean F; and, as it is lesser or greater than C, find a fourth Mean betwixt it and that one of the preceding Means

Means which is next greater than C; but if all the preceding Means are of the same quality with F, find a Mean betwixt F and the opposite Extreme B or A. In this manner go on finding Geometrical Means approaching till you find one within any Difference you please.

Examp. To find a Mean in the Infinite Series betwixt 1 and 10, which approaches to 9 within a Difference of $\frac{1}{1000000}$

Operation. Betwixt 1 and 10 there is not a Rational Mean (for 10 is not a Square Number, and the Mean is the Square Root of 1×10 or 10) but I find one approximate to 7 Places of Decimals (*viz.* the Number in the Denominator of the Fractional Difference) which is 3.1622777 &c. which being less than 9, betwixt it and 10 I find another approximate Mean, which is 5.6234132 &c. which being also less than 9, I find betwixt it and 10 another, which is 7.4989421 &c. and this being also less than 9, I find betwixt it and 10 another, 8.6596432 &c. which being yet less than 9, I find another Mean betwixt it and 10, *viz.* 9.3057204 &c. which is greater than 9, therefore I find a Mean betwixt it and 8.6596432, (the next lesser Mean) which is 8.9768713 &c. which being less than 9, betwixt it and 9.3057204 &c. (the next greater Mean) I find a Mean 9.1398170 &c. greater than 9, wherefore betwixt it and 8.9768713 &c. (the next lesser) I find a Mean 9.0579777 &c. greater than 9. Going thus on, you'll find at the 25th Step this Mean, 8.9999998 &c. which wants of 9 this Fraction, $\frac{2}{10000000} = \frac{1}{5000000}$: which is less than the propos'd Difference, $\frac{1}{1000000}$

So much of the Operation as is here explain'd you see placed in Order, as it was wrought, in the Example of the following *Problem* 2, which will give a clearer view of it; the rest of the Steps are easily imagin'd by these: But *Observe*, that the Means being all Approximates which requir'd Decimal Places in the extraction, therefore the given Numbers 1, 10 are taken in the Operation with as many o's as the Approximate Mean ought to have, which has the same Effect in the Operation, since 1 : 10 : : 10000000 : : 100000000; so that a Mean betwixt 1, 10 is the same as betwixt 10000000 : 100000000

DEMON. All that's necessary to be said as to the reason of this Operation is, in short, this; That if at every Step of the preceding Operation we conceive the Series to be fill'd up betwixt the given Extremes, in the Ratio that every new Mean makes with the Terms betwixt which 'tis taken, we can thus carry on the Number of Means *in infinitum*; so that the Means thus found are still a part of the infinite number of Means suppos'd to lie betwixt these Extremes: Wherefore, by assuming any two Numbers of the Natural System, we can thus find Approximates to all the intermediate ones within any assign'd Difference, and such too as shall make a Geometrical Series indefinitely near.

SCHOL. 7. Let us now suppose any Series Geometrical from 1, as, 1 : 2 : 4 : &c. or 1 : 10 : 100 : &c. it is plain how, by the Method of this *Problem*, we can find Mean Terms betwixt each of these, so nearly approximate as to make of the whole one Geometrical Series, to an indefinite degree of exactness; and among which we can find Terms approaching within any Difference of any of the Intermediate Natural Numbers. But *Observe* also, That if the Construction of Logarithms requir'd the finding the Approximates to every one of these Intermediate Numbers, the Labour would still be intolerably great; which is prevented by the Consideration of the Fundamental Principles and Rules of *Logarithms*; as in the next *Problem*.

PROBLEM 2.

To Construct *or Find* LOGARITHMS *to the Natural System of Numbers* 1 . 2 . 3 . 4 . &c. *carried to any Extent.*

SOLUTION.

1. Take the Geometrical Progression, 1 : 10 : 100 : &c. to which apply as Logarithms the Arithmetical Series 0 . 1 . 2 . 3 : &c. *viz.* 0 the Logarithm of 1, 1 the Logarithm of 10, and so on; then

2. For the Logarithms of the Intermediate Numbers, the *General Rule* is this: Find, by the preceding Problem, a Mean betwixt 1 and 10, or 10 and 100, or any two adjacent Terms of the Series betwixt which the Number propos'd lies, so approximate that it be within the propos'd Limit of the Number whose Logarithm is sought; for example, so near that it want not $\frac{1}{1000000}$ [whereby, if it's taken less than that Number, 'twill necessarily have in the Integral part a Number wanting 1 of it, and the Decimal part will have 9 in 6 Places immediately after the Point: But if the Denominators of the limiting Fraction has any other Number of 0's, the approximate Mean must have as many 9's immediately after the Point.] Then, betwixt the Logarithm of 1 and 10 (or other two Terms betwixt which the Number lies) find as many Arithmetical Means in the same order as you found Geometrical Means betwixt 1 and 10; and thus you find gradually the Logarithms answering to each of these Geometrical Means, and consequently the Logarithm of the Mean approximate to the Number propos'd, which we therefore take for its Logarithm.

DEMON. That Logarithms thus found to all the Intermediate Numbers betwixt the Terms of the Series 1 : 10 : 100 : &c. are true, and must answer to the preceding Rules, is clear, because they are found by those very fundamental Principles, whereby the Logarithm of a Geometrical Mean betwixt any two Numbers is necessarily an Arithmetical Mean betwixt the Logarithms of these two Numbers: And as all the Geometrical Means thus found are part of the infinite number of Means suppos'd betwixt any two Terms of this Series, so the Arithmetical Means thus found must likeways be their Correspondents among the infinite number of Arithmetical Means lying betwixt the Logarithms of these two Numbers; and hence the Logarithms are truly found, according to the determin'd degree of approximation.

Observe also in the following Example, that because in halving the Sum of two Logarithms to find the Arithmetical Mean there will be Fractions, Therefore, either to prevent this, or to find them in Decimals, the Logarithm of 10 is made 1.000000 (instead of 1); *that is*, 1 with as many 0's as are in the Denominator of the limiting Fraction within which the approximate Mean was determin'd; whence all the other Logarithms will have as many Decimal Places. And the Logarithms thus found may be consider'd either as altogether *whole* Numbers, or as *mix'd*, for the Effect will be the same; because by considering them as *whole* Numbers, they are only the equal Products of what they are when consider'd the other way; and therefore, either way, they are correspondent Terms of an Infinite Arithmetical Progression adapted to an Infinite Geometrical one; for if they are so when consider'd as *mix'd* Numbers, or when the Logarithms of 1 : 10 : 100, &c. are 0 . 1 . 2, &c. they must be so when consider'd as Integral, or when the Logarithms of 1 : 10 : 100, &c. are 000000 . 1000000 . 2000000, &c. since an Arithmetical Progression, equally multiply'd, continues still Arithmetical.

In

In the following Scheme you see the Order of the Operation, whereby the Logarithm of 9 is found, carried to the 8th Step, which is sufficient to illustrate and make plain the rest of the Work for this or any other Example.

Order of the Operation whereby is found the Logarithm of 9.

		Numbers.	Logarithms.			Numbers.	Logarithms.
Given	*A*	1.0000000	0.0000000		*F*	8.6596432	0.9375000
1st Mean	*C*	3.1622777	0.5000000	5th	*G*	9.3057204	0.9687500
Given	*B*	10.0000000	1.0000000		*B*	10.0000000	1.0000000
	C	3.1622777	0.5000000		*G*	9.3057204	0.9687500
2d	*D*	5.6234132	0.7500000	6th	*H*	8.9768713	0.9531250
	B	10.0000000	1.0000000		*F*	8.6596432	0.9375000
	D	5.6234132	0.7500000		*H*	8.9768713	0.9531250
3d	*E*	7.4989421	0.8750000	7th	*I*	9.1398170	0.9609375
	B	10.0000000	1.0000000		*G*	9.3057204	0.9687500
	E	7.4989421	0.8750000		*I*	9.1398170	0.9609375
4th	*F*	8.6596431	0.9375000	8th	*K*	9.0579777	0.9570312
	B	10.0000000	1.0000000		*H*	8.9768713	0.9531250

Which carried on, the 25th Mean is 8.9999998 | 0.9542425.

In the same manner may we find the Logarithm of any other Number betwixt 1 and 10, or betwixt 10 and 100, &c. *Observe* also, that having found the Logarithm of any Mean betwixt 1 and 10, &c. we may either use the same Extremes 1, 10 for finding any other Mean and its Logarithm, or we may use any other two Extremes whose Logarithms are already known. Thus, having found the Logarithm of 9, To find that of 8, or 2, or any other, we may use 1, 10, or 1, 9. And if we have the Logarithms of 7 and 5, we may use these for finding that of 6; and so of other Cases: And it will be best to chuse the Extremes as near to one another as we can. The Reason of the Work is still the same, because betwixt any two Terms of the natural System we suppose an indefinite number of Means.

Again Observe, That tho' this *General Rule* is good, yet to find all the intermediate Numbers betwixt 1 and 10, 10 and 100, &c. in this manner would be an endless Labour, which is sav'd by the following

Particular RULES.

1°. Having by the *General Rule* found the Logarithm of any Number, the Logarithms of all the superior Powers of it are found by simply multiplying the Logarithm of that Number into the Series of Indexes of all the superior Powers, *viz.* 2 . 3 . 4 . &c. which produces

produces the Logarithms of its Square, Cube, &c. The Reason of this is contain'd in the Third *General Rule*: So having the Logarithm of 2, we have the Logarithms of its Powers, 4 . 8 . 16 . &c. if we multiply the Logarithm of 2 by 2, 3, 4, &c. successively.

2°. Having found by the General Rule the Logarithm of any Power, divide it by the Index of the Power, and the Rule is the Logarithm of the Root: The Reason is contain'd in the Fourth General Rule. Thus having found the Logarithm of 9, its half is the Logarithm of 3, the square Root of 9.

3°. Having found the Logarithms of any two or more Numbers, (as of 2 and 3) we have the Logarithm of their Product (6) by adding their Logarithms into one Sum; as in the First General Rule.

4°. Having the Logarithm of any Number, and of another which measures it, we have, by the Second General Rule, the Logarithm of the Quote or Number by which the lesser measures the greater: Thus, from the Logarithm of 10 take that of 2, and the remainder is the Logarithm of 5.

From these Rules therefore it's plain how, with much less Labour than applying the General Rule to every Number, we can compleat a System of Logarithms: And to go on regularly with the Application, the natural Method is obviously this, *viz.*

Apply the *General Rule* to Prime Numbers, and then by these find the Logarithms of their Powers and Products, thus; By the Logarithms of 2 and 3 we find that of 6, 12, 24, and all in that Progression: Also of 18, 54, 162, and all in that Progression: And, lastly, of 36, 216, 1296, and all in that Progression: Other Applications are easily conceiv'd by this. *Observe* also, that the Prime Numbers, 2 and 5, do not both require the *General Rule*, because we have the Logarithm of 10 assum'd, and 10 is $= 2 \times 5$. Again; it's sometimes as convenient, or rather better, to find the Logarithm of the Square of a Prime Number by the *General Rule*, and then the Logarithm of the Root by the *Particular Rules*; so we may chuse to find the Logarithm of 9 by the *General Rule*, and then its half is the Logarithm of 3.

SCHOLIUM 8. Upon the *Foundations*, and by the *Rules* now explain'd, were the *Logarithms* first calculated, which we have in our present Tables, (tho' they have been constructed anew by Methods incomparably easier). It's true indeed, that in the *First Logarithms* made by the Lord *Neper*, the Logarithms of 1 : 10 : 100 &c. were other Numbers than 0 . 1 . 2 . &c. but afterwards he chang'd them into these, which were, after his Death, further compleated and carried on upon this Foundation by Mr. *Henry Briggs*, by a Method also somewhat different from *Neper*'s, yet equally laborious.

Where-ever you find Tables of *Logarithms*, you'll find also Directions for the Use of them; And therefore since I refer you to other Books for Tables (and you have Books containing nothing but Tables, as *Ozanam*'s and *Sherwin*'s, which last are the best extant) I shall add nothing to the general Rules already delivered, which do sufficiently shew the Practice and Use of *Logarithms*: For what is more to be said as to the Use of Tables relates only to the different Methods of disposing the Numbers and *Logarithms* in the Tables, which every Book of Tables explains; but still there remains a few Articles to be explained, concerning the finding *Logarithms* for Numbers that are not contained in your Tables, or Numbers corresponding to *Logarithms*; which being the result of an Operation with *Logarithms* found in the Table, are not themselves exactly found in the Table: These things I shall explain in the following Problems; but first I'll dispatch these Observations.

1. That there may be great Variety in the Systems of *Logarithms*, which depends in general upon these two things, *viz.* The fundamental Geometrical progression, whose *Logarithms* we assume; for that may be 1 : 2 : 4 : 8, &c. or 1 : 3 : 9 : 27, &c. or 1 : 10 : 100 :

100 : 1000, *&c.* Then the different Arithmetical Progressions we may assume for the *Logarithms* of this Geometrical one: For thus *Logarithms* may be varied infinitely; yet they will not be all alike convenient. The consideration of which obliged the Inventer to change his first *Logarithms* into others, whose fundamental Progression is 1 : 10 : 100 : *&c.* and the *Logarithms* 0 . 1 . 2 . *&c.* which are those now used.

2. The great Advantage and Conveniency of the *Logarithms* now in Use is this, That the Integral part in every *Logarithm* shews how many Figures after the Place of Units the corresponding Number contains, whence that Number is called the *Index* or *Characteristick* of the *Logarithm.* Thus all the Numbers from 1 to 10 exclusive, consist of one Figure: For the *Logarithm* of 1 being 0, and that of 10 being 1, therefore the *Logarithms* of all the intermediate Numbers must be Decimal Fractions, and so have 0 for the Integral part or Characteristick. Again; the *Logarithm* of 10 being 1, and of 100 being 2; all the intermediate Numbers must have 1 for their Integral part or Index; and so on. The Benefit of this is remarkable in finding the *Logarithms* of Numbers that are in a decuple Progression, having the *Logarithm* of any one of them, (and consequently for Decimal Fractions) because the *Logarithm* of 1 being 0, that of 10 being 1, and that of 100 being 2, and so on; it follows that the *Logarithm* of the Product or Quote of any Number by 10, is had by adding 1 to, or substracting 1 from the *Logarithm* of the given Number A; because $1 : 10 :: A : 10 \times A$, whose *Logarithm* is therefore the Sum of the *Logarithms* of A and 10; and $10 \times A \div 10 = A$ whose *Logarithm* is the difference of the *Logarithms* of $10 \times A$, and of 10.

	Numbers.	Logarithms.
Exa.	674800	5.8291751
	67480	4.8291751
	6748	3.8291751
	674.8	2.8291751
	67.48	1.8291751
	6.748	0.8291751

From this it is plain that the *Logarithms* of Numbers in a decuple Progression, differ only in their Indexes, which differ gradually by 1.

A further Application of this to pure Decimal Fractions, you'l find in the following *Probl.* 5.

PROBLEM 2.

To find the *Logarithm* of an Integral Number exceeding the Limits of any Table of *Logarithms*; for *Exa.* exceeding 10,000; to which our common Tables are carried.

RULE. Take as many Figures on the left-hand of the given Number as you can find in the Table [*i.e.* 4 of them if the Limit of the Table is 10,000; or 5 if it is 100,000] and in place of the Figures cut off from the right hand, annex 0's, so this will express a Number less than the given Number: Again, to the Number express'd by the Figures taken on the left hand add 1, and on the right of the Sum annex as many 0's as the Number of Figures cut off the right hand of the given Number, and this will be a Number greater than it: Then take the difference of these two Numbers, which are the one lesser, and the other greater than the given Number; also the difference of the given Number, and the Number lesser than it (which difference consists of the Figure cut off the right hand) and make this Proportion.

As the difference of the Numbers greater and lesser than the given Number
is to the difference of the *Logarithms* [which can be found by the Tables and the preceding second Observation.]
So is the difference of the given Number, and that lesser than it
To the difference of their *Logarithms*, which therefore added to the *Logarithm* of that lesser Number, gives the *Logarithm* of the Number proposed.

Exa. To find the *Logarithm* of 123459 from a Table carried only to 10000. The two Numbers lesser and greater than 123459, taken according to the Rule are 123400, and 123500, whose *Logarithms* are 5.0913152, and 5.0916670, for the *Logarithm* of 1234 is 3.0913152; to which add 2, the *Logarithm* of 100 (because $123400 = 1234 \times 100$) the Sum 5.0913152 is the *Logarithm* of 123400. Also the *Logarithm* of 1235 is 3.0916670, and so that of 123500 is 5.0916670; and the Proportion is

from	123500	5.0916670	123459
take	123400	5.0913152	123400

As 100, is to .0003518, so is 59 to .00020756 &c. which added to 5.0913152, the Logarithm of 123400, the Sum is 5.09152276 &c. the Logarithm of 123459 nearly. But if we used a Table carried to 100000, the Logarithm sought would be 5.09152278 nearly: And still the more of the Figures of the propos'd Number we have in the Table, the nearer or more exact will the Logarithm be found.

DEMON. The *Reason* of this Rule is founded upon this; That the greater any Numbers are in respect of their Differences, the nearer those Differences are to being proportional with the Differences of their Logarithms, To understand which clearly, *Consider*, 1°. That if we take the natural System 1 . 2 . 3 . 4 . &c. the farther it is continued, the Terms are the nearer to a Ratio of Equality; for $\frac{2}{3}$ is greater than $\frac{1}{2}$, and $\frac{3}{4}$ greater than $\frac{2}{3}$; and so on: Whence, the farther from the beginning we take any two adjacent Terms, the nearer they stand together in the infinite Scale of proportionals from 1. Thus since 1 : 2 :: 3 : 6, therefore there fall as many Means betwixt 1 and 2, as betwixt 3 and 6, and consequently more than betwixt 3 and 4; and so it is thro' the whole natural System, considered now as a part of the infinite Progression from 1. But again, 2°, the *Logarithms* of these Numbers are their distances from 1 in the infinite Progression, *i. e.* the number of the intermediate Ratio's; (or they are in proportion to one another as these Numbers;) so that the difference of the *Logarithms* of any two Numbers, is the Number of intermediate Ratios in the infinite Progression betwixt these two Numbers; and from hence, with the preceding Article, it plainly follows, that the differences of the *Logarithms* of the natural System do continually decrease. 3°. The Ratios of the several Terms of the natural System do so grow, as that if we take any three adjacent Terms; the farther they are taken from the beginning, the nearer they approach to being Geometrically proportional: Thus, to 1, 2 a third proportional is 4, which exceeds 3 by 1. To 2, 3 a third $\div l$ is $4\frac{1}{2}$, which exceeds 4 by $\frac{1}{2}$. To 3 : 4 a third $\div l$ is $5\frac{1}{3}$ which exceeds 5 by $\frac{1}{3}$, and so on, the third $\div l$ to any two Numbers adjacent in the Series exceeds the third Term in the Order of the Series, by such an aliquot part of Unity as is denominated by the least of the given Numbers; consequently these excesses become less and less; for this Series does constantly decrease $\frac{1}{2} . \frac{1}{3} . \frac{1}{4}$, &c. *i. e.* every Term of the Series wants less and less of being a true third $\div l$ to the two preceding Terms; and what they want being certain determinate Fractions which grow infinitely little, or less than any assignable Fraction, therefore the Terms grow infinitely near to being $\div l$. From whence 4°. The differences of the *Logarithms* of the several Terms of the natural System decrease so as to approach infinitely near to being equal. And since the differences of the Terms of the natural System are all equal: Hence it follows, 5°. That taking any three Terms in the natural System (tho' not immediately adjacent) the

the difference of the Extremes is to the difference of the Mean and either Extreme in a Ratio which approaches nearer and nearer to the Ratio of the differences of the Logarithms of the same Terms, the farther these Terms are taken from the beginning, and the nearer they stand together at the same time, *i.e.* the greater the Numbers are with respect to their Differences. And this also shews the Reason why the Rule is more exact and true in finding the *Logarithm* sought, the more of the Figures of the given Number we have in the Table; for, by this means, the Numbers are the greater, and their Differences the lesser.

Observe; If the Number given is compos'd of two or more Numbers within the Table, then the Sum of their Logarithms is that sought: But if it is a *Prime*, we must either use the Method of this *Problem*, or *Problem* 2d: But this Problem, as 'tis easier, so 'tis exact enough for any Numbers we can have Use for, which are not in Tables, these being carried to 10000, and some to 100000.

PROBLEM 4.

To find the Number corresponding to any *Logarithm*, which being the Result of an Operation with Logarithms found in the Table, is not it self found exactly in the Table.

1°. If the Characteristick and first 4 or 5 Figures of the Fractional Part of your Logarithm is found in the Table, that's near enough for common use; and the Number found against such a Logarithm, or that one of several such that is nearest to the given Logarithm, you may take for the Number sought. But if you would have it exacter, or that you cannot find a Logarithm having so many of the Figures of the given Logarithm; Then,

2°. Take the two Logarithms in the Table which are next greater and lesser than the given one, and also their corresponding Numbers, and make this proportion:

As the Difference of the greater and lesser Logarithms
is to the Difference of their corresponding Numbers,
So is the Difference of the given and next lesser Logarithm
to the Difference of their corresponding Numbers. [Which Difference added to the Number corresponding to that lesser Logarithm, makes the Number corresponding to the given Logarithm nearly.]

DEMON. The Reason of this is in the last Problem, of which this is but the Reverse.

Exa. Given this Logarithm, .4669347; the next lesser and greater (in *Sherwin's* Tables) are .3010300, the Logarithm of 2, and .4771213, the Logarithm of 3; and the Proportion is thus form'd;

from	.4771213	3	.4669347
take	.3010300	2	.3010300

As, .1760193, is to 1, so is .1659047, to .94215 &c. Which added to 2, makes 2.94215 &c. the Number sought nearly,

But we may work another Way, and somewhat more exactly, thus: Seek among the Logarithms, two whose Fractional Parts are next lesser and greater than the given Logarithm (which is a Fraction) these are 4.4669269, the Logarithm of 29304; and 4.4669417 the Logarithm of 29305: But, taking away the Characteristicks, the Fractional Parts are Logarithms to these Numbers, 2.9304 and 2.9305 (according to what has been explain'd in the *Scholium* after *Probl.* 2.) Wherefore the Operation is

from	.4669417	2.9305	.4669347
take	.4669269	2.9304	.4669269

As, .0000148 is to .0001, so is .0000078 to .000052: But the Remainder of the Division is such as makes the nearest answer .000053; which added to 2.9304, makes the Answer or Number sought 2.930453; which is less, and more exact, than the former Answer.

From this *Example*, duly consider'd, there will be no Difficulty to solve any other; wherefore I shall only add this general Direction, *viz.* If the given Logarithm is a *mix'd* Number, we may find a Number answering the Fractional Part of the Logarithm, by this last Method, and then multiply this Number by 1 with as many 0's as there are Units in the Index of the Logarithm; the Product is the Number sought. The *Reason* is obvious from what is explain'd.

3°. If the given Logarithm has a greater Characteristick than any in your Table, seek a Number answering the greatest Index in your Table with the Fractional Part of your Logarithm, by the Method of one of the preceding Articles; Then multiply that Number by 1 with as many 0's as the Number by which the given Characteristick exceeds the greatest in your Table, and that is the Number sought.

But, in some Cases, we can find the Number sought more exactly, thus; Seek a Logarithm whose Fractional Part is nearest to that of the given Logarithm, without regarding the Index; then take the Number corresponding to that Logarithm, and divide it by 1 with as many 0's as the number of Units by which the Index of that Logarithm exceeds the given Index. See an Example of this in the following *Rule* for reducing a Vulgar to a Decimal Fraction.

PROBLEM 5.

To find the LOGARITHM *of a* FRACTION; *and Reversely, a* FRACTION *from its* LOGARITHM.

There are various Ways of finding and expressing the Logarithms of Fractions.

1st Method. Subtract the Logarithm of the Denominator out of that of the Numerator, the Remainder is the Logarithm of the Fraction: But, *Observe*, if it is a proper Fraction, then, because the Numerator is less than the Denominator, so is its Logarithm less than the other's Logarithm, and consequently the Subtraction is impossible: Wherefore we must subtract the Logarithm of the Numerator out of that of the Denominator, and the Remainder, taken negatively, is the Logarithm sought.

Exa. 1. The Logarithm of $\frac{463}{24}$ $\left(= 19\frac{7}{24}\right)$ is 1.2853698; for the Logarithm of 463 is 2.6655810; and that of 24 is 1.3802112, which exceeds the other by 1.2853698.

Exa. 2. The Logarithm of $\frac{74}{863}$ is -1.0667791; for the Logarithm of 74 is 1.8692317, and that of 863 is 2.9360108; and their Difference is 1.0667791

DEMON. Let $\frac{A}{B}$ express any Fraction, then is $B : A :: 1 : \frac{A}{B}$, whose Logarithms are therefore in Arithmetical Proportion; *that is*, Log. $A -$ Log. $B =$ Log. $\frac{A}{B} -$ Log. 1; But the Log. of 1 is $= 0$, therefore Log. $A -$ Log. $B =$ Log. $\frac{A}{B}$; And there-

therefore, if A is less than B, the Log. of $\frac{A}{B}$ is their Difference taken negatively: By which is shewn, that the corresponding Number is below Unity as far as the Reciprocal of that Number is above Unity, in the infinite Series of Proportionals, the Logarithm of that Reciprocal being the same Logarithm taken positively. So, the Logarithm of $\frac{74}{863}$ being -1.0667791, the Logarithm of $\frac{863}{74}$ is 1.0667791; For as $\frac{74}{863} : 1 : \frac{863}{74}$ are $\div l$, so $-1.0667791 : 0 : 1.0667791$ are $\div l$.

For the *Reverse* of this Problem, *viz.* If a Negative Logarithm is given to find its corresponding Fraction, find a corresponding Number to the Logarithm, consider'd as positive, and by that Number divide 1, the Quote is the Fraction sought. The *Reason* is plain; for if two Numbers or Logarithms consist of the same Figures, but the one Positive, and the other Negative, their Sum is 0; Also the Product of two reciprocal Fractions is 1; or 1 divided by any Number, makes a Quote, which multiply'd by that Number produces 1; wherefore if 1 is divided by the Number corresponding to any Logarithm taken positively, the Quote is the Number corresponding to the same Logarithm taken negatively.

But the finding the *Fraction* from the *Logarithm* is not so convenient by this Method as by the following.

2d Method. Subtract the Logarithm of the Denominator from that of the Numerator; and if it's a proper Fraction, when you come to the Characteristick or Integral part, subtract that of the Numerator from that of the Denominator (after adding to it the 1 borrow'd in the preceding place, if there was 1 borrow'd) the Fractional part of this Remainder is taken positively, and the Integral part negatively; and the negative Sign set over it, to shew that this part only is negative.

Exa. The Logarithm of $\frac{74}{863}$ is $\bar{2}.9332209$; as below, in the Margin.

Log. of			
	74	is	1.8692317
	863	is	2.9360108
	$\frac{74}{863}$	is	$\bar{2}.9332209$

DEMON. This is in Effect the same Logarithm as was found by the former Method; for we may take it thus, $.9332209 - 2$, which being resolv'd by Subtraction, is -1.0667791, *viz.* the difference of 2 and $.9332209$ taken negatively. That the Methods must coincide in all Cases, I shall demonstrate *Universally*, thus:

Let any proper Fraction be $\frac{N}{M}$; the Logarithm of N be call'd $A+B$ (A the Characteristick, and B the Fraction); the Logarithm of M be $C+D$ (C the Characteristick, and D the Fraction); Then, by the 1st Method, the Log. of $\frac{N}{M}$ is $A+B-C-D$. or $C+D-A-B$, taken negatively. For the 2d Method, we shall first suppose that D does not exceed B; then the Log. of $\frac{N}{M}$ is by that Method $B-D-\overline{C-A}$; for we first subtract D from B, then we take A from C, and take this Remainder negatively, which makes $B-D-\overline{C-A}=B-D-C+A=A+B-C-D$, as before. *Again*; If D is greater than B, in this case 1 must be taken from A and added

$$\begin{aligned}\text{Log. of } N &= A+B = A+1-1+B\\ \text{of } M &= C+D.\end{aligned}$$

$$\begin{aligned}\text{Log. of } \frac{N}{M} &= A+B-C-D. \text{ by 1st Method.}\\ &\left.\begin{aligned}&= B-D-\overline{C-A}=\\ &= B+1-D-\overline{C+1-A}\end{aligned}\right\}\text{2d Method.}\end{aligned}$$

added to B; so that it is $A + B = \overline{A - 1} + \overline{1 + B}$. Now, if we first subtract D from $1 + B$, the Remainder is $B + 1 - D$; and subtracting $A - 1$ (because 1 was before taken from it and added to B) from C, or A from $C + 1$, which is the same thing, the Remainder is $C - A + 1 = C + 1 - A$; which being taken negatively, or subtracted from the former part, (*viz.* $B + 1 - D$) it is $B + 1 - D - \overline{C + 1 - A} = B + 1 - D - C - 1 + A = A + B - C - D$, as before.

S C H O L. As the preceding *Rule* is General, relating to all Fractions, so it comprehends *Decimal* Fractions; and because the Denominator of every Decimal is in this Series 10 . 100 . 1000 . &c. whose Logarithms are pure Integers in this Series 1 . 2 . 3 . &c. Therefore it's evident, that to find the Logarithm of a Decimal Fraction, having found the Logarithm of the Numerator, the Fractional part of it is the Fractional part of the Logarithm sought; and for the Index, apply with a negative Sign the Difference of the Indexes of the Logarithm of the Numerator, and the Logarithm of the Denominator: So the Logarithm of 64 being 1 . 8061800, the Logarithm of .064 must be $\bar{2}$.8061800; for the Denominator is 1000 whose Logarithm is 3, then $1 - 3 = \bar{2}$: Hence again it is plain, That the Index of the Logarithm of a Decimal Fraction shews in what Place after the Point the first Figure on the left hand of the Numerator stands; and that Distance, therefore, does shew, reciprocally, what the Characterisick of the Logarithm is. And if we take any Decimal, pure or mix'd, *i. e.* a Decimal Fraction, *proper* or *improper*, the General Rule for finding the Logarithm is plainly this, *viz.* Find the Logarithm of the Numerator (*i. e.* of the Number express'd by all the Figures, in order as they stand, neglecting the Point) the Fractional part of that Logarithm is the Fractional part of the Logarithm sought; And for the Index, 'tis the Number which expresses the Distance of the last Figure on the left hand, after the Place of Units of the Integral part, if it's a *mix'd* Decimal; or the Distance of the last Figure on the left hand after the Decimal Point, if it's a *pure* Fraction.

Observe also, That if we take the same Number, and multiply it continually by 10, also divide it continually by 10, whereby we form a Geometrical Progression in the Ratio of 1 to 10, the Logarithms of this Series will have all the same Fractional part, and differ only in their Characteristicks, which will be an Arithmetical Progression differing only by 1: So that when in the descending Series we are come to a Term whose Logarithm has 1 for its Index, the Index of the next must be 0; and all after that are the preceding Indexes, 1, 2, 3, &c. taken negatively, as in the prefix'd Scheme: Which is all but the application of what has been explain'd, particularly of what has been said in the Consequences last deduced from this Method of finding the Logarithm of a Fraction compar'd with what was formerly observ'd in the General *Scholium* 8.

Numb.	Logarithms
25400	4 . 4048337
2540	3 . 4048337
254	2 . 4048337
25.4	1 . 4048337
2.54	0 . 4048337
.254	$\bar{1}$. 4048337
.0254	$\bar{2}$. 4048337
.00254	$\bar{3}$. 4048337

For the *Reverse* of this Method, (*viz.* finding a Fraction from its Logarithm) the *Rule* is, Take the Logarithm as altogether positive, and find the corresponding Number; to which apply a Decimal Point on the left hand of it, so that it's first Figure on the left stand in such a place from the Point as is express'd by the Index of the Logarithm; And this is the Fraction sought, reduced to a Decimal Fraction.

Exam.

Exam. The given Logarithm $\bar{2}.8001800$; the correſponding Number to the Log. 2.8001800 is 640; and, to adjuſt it to the negative Index $\bar{2}$, it is .0640 = .064:

Obſerve: If the correſponding Number firſt found be a mix'd Decimal, the reſt of the Work is the ſame: for we neglect the Point in that firſt Decimal: So, if the given Logarithm is $\bar{2}.6707096$. the correſponding Number to 2.6707096 is 468.5; and, becauſe of the negative Index, it is .04685

The *Reaſon* of this Rule is evident from what has been explain'd in the preceding *Scholium*.

COROLL. From this we have learnt the following

RULE *for reducing a* Vulgar *to a* Decimal *Fraction.*

Find the Logarithm of the *Vulgar* Fraction by this *2d Method*, and then find its correſponding *Decimal* Fraction by the *Reverſe*.

2d Method. Subtract the Logarithm of the Denominator from that of the Numerator, borrowing (*viz.* 1 from the next place) wherever the Figure of the Subtractor exceeds its Correſpondent in the Subtrahend, even to the very laſt place, tho' the Subtractor be a greater Number than the Subtrahend. The Remainder, or Number thus found, is the Logarithm ſought, and is all poſitive.

Exa. The Logarithm of $\frac{74}{863}$ is 8.9332209, as in the Margin.

Log. of 74	=	1.8692317
of 863	=	2.9360108
		8.9332209

DEMON. The Foundation of this Method, (which gives the Logarithm in a Syſtem different from the common one) is this, *viz.* That as a decuple Progreſſion may be taken both aſcending and deſcending from 1; thus, .001, .01, .1, 1, 10, 100, 1000, &c. So we may apply 0 as the Logarithm of any of theſe Terms we pleaſe, whereby the Logarithms of the reſt will be, in the aſcending Series, 1.2.3.&c. and for the deſcending Series, they will be −1, −2, −3, &c. *Now* ſuppoſe we chuſe 1 in the 10th place of Decimals, *viz.* .0000000001, from whence to begin the Logarithms (*i. e.* whoſe Logarithm we make 0) then is 1 the Logarithm of 1 in the 9th place, or .000000001; and ſo on till we come to 1 whoſe Logarithm is 10; Whence the Logarithm of 10 will be 11, the Logarithm of 100 is 12, and ſo on; So that the Logarithms of all Numbers above .0000000001 are poſitive; the Indexes of all from .0000000001 to .000000001 being 0; from .000000001 to .00000001 being 1; and ſo on. But the Logarithms of all Numbers below .0000000001 are negative; or their Indexes at leaſt, according as they are taken by the preceding 1*ſt* or 2*d Method*.

Numbers.	Logarithms.
.000000000001	−2
.00000000001	−1
.0000000001	0
.000000001	1
.00000001	2
.0000001	3
.000001	4
.00001	5
.0001	6
.001	7
.01	8
.1	9
1.	10
10.	11
100.	12

Hence again, The Logarithm of 1 being 10, every Index from 10 upwards belongs to the Logarithm of a Number integral or mix'd: And if we take 10 from any Index greater, the remainder ſhews in what Place after that of Units (of the integral part) the firſt Figure on the left hand of the correſponding Number ſtands; But if the Index is leſs than 10, then, as it belongs to a Decimal Fraction, ſo, if it's taken from 10, the Remainder ſhews in what

what Place after the Point the first Figure on the left of the Numerator stands. And here 'tis to be observ'd, that all *pure* Decimals, whose first Figure on the left of the Numerator stands in any given Place after the Point, belong all to the same Class, or have the same Index in their Logarithm, because whatever number of Figures follow after that Place, they cannot make the whole equal to an Unit in the preceding Place on the left. So all Decimals, whose first Figure of the Numerator, on the left, stands in the first Place after the Point, (as .3, or .47, or .5067) are Intermediates betwixt .1 and 1, and so have 9 for the Index of their Logarithm. All whose first Figure stands in the second Place are intermediate betwixt .01 and .1, and so the Index of their Logarithm is 8; and so on: And this is to be understood, tho' there were an infinite number of Figures, from any Place after the Point, unless that infinite number were all 9, for then the Value of that Infinity is an Unit in the preceding Place on the left, and ought rather to be written thus, .0999 &c = .1; *Thus* then we have a new System of Logarithms, differing from the common ones only in their Indexes, which exceed these, in every Logarithm, by an equal excess of 10; and consequently the Logarithms of Numbers in decuple Progression differ as in the common Logarithms, *viz.* only in their Indexes, which differ gradually by 1.

Now, for the *Reason* of the preceding Rule, consider, that to find the *Logarithm* of a Fraction (which does not fall below .0000000001) as $\frac{74}{863}$, we must argue thus; $863 : 74 :: 1 : \frac{74}{863}$; wherefore the *Logarithm* of the Fraction is the Remainder after the *Logarithm* of 863 is taken from the Sum of the *Logarithms* of 74 and 1: But, by what's now explain'd, the Indexes of these *Logarithms* must, in this new System, be more by 10 than they are in the common form; and, because the *Logarithm* of 1 (*viz.*10) is to be added to that of 74, therefore the Index in the Subtrahend will be 21 (for 11 is now the Index of the *Logarithm* of 74) and that in the Subtractor will be 12, whence that in the Remainder is 8: But because there is 20 added to the Index of the common Logarithm of 74, (which is but 1) and 10 to that of 863 (which is but 2) 'tis the same thing in Effect if we take the Index of the common Logarithm of 863, and add only 10 to that of 74; which is the thing the Rule prescribes, and is therefore just. The like Reason is obvious in all Cases.

Observe again, that 0 may be also apply'd as the *Logarithm* of a Decimal whose Numerator is 1 in the 100th Place after the Point; and then the *Logarithm* of 1 in the 99th Place after the Point will be 1; and so on to 1 Integral, whose *Logarithm* will be 100, and that of 10 will be 101, and so on upwards: Wherefore to the Indexes found in the common form there will be 100 added; and, consequently, if the common Index of the Number wants more than 10 of the Index of the Denominator, we must add 100; Or, it is the same thing if we add 10 gradually from one Place to another, according as the Rule directs.

As to the *Reverse* of this Method, *viz.* From the *Logarithm* of a Fraction to find the Fraction; *Consider*, that as the Index of the *Logarithm* of a *pure* Fraction, which falls not below 1 in the 10th or 100th, &c. Place of Decimals, is some Number below 10 or 100, &c. So, if we take the *Difference* betwixt the Index of the *Logarithm* and the Logarithm of 1, *viz.* 10, or 100, &c. and apply that as the Index to the Fractional part of the *Logarithm*, then find the corresponding Number to that *Logarithm*, and qualifie it again by setting a Decimal Point on the left hand of it, so that the first Figure on the left stand as far from the Point as the Index expresses, we shall have thus the Decimal Fraction sought.

SCHOL.

SCHOLIUM. For the further application of this Method, *observe*, that as the Logarithms of Fractions are taken in a System different (at least in their Indexes) from the common ones; so in all Operations with such Logarithms, regard must be had to the Logarithm of Unity, which is now 10, or 100, &c. For *Exa.* If two Fractions are to be Multiplied, and if they are such that their Logarithms are both taken out of the System, wherein the Logarithm of 1 is 10, then take the difference betwixt 10 (the Logarithm of Unity) and the Index of the Sum of these Logarithms; and if that Sum was greater than 10, the difference shews how many places to the left of the 1 whose Logarithm is 0 (*viz.* .0000000001) the first significant Figure on the left of the Decimal sought must stand; and consequently taking this difference from 10, the new difference shews how many places from the point this first Figure stands; but if 10 is greater than the Index of the Sum, the difference is to be taken negatively, and shews that the first Figure on the left of the Decimal stands so many places to the right after the 1 whose Logarithm is 0; so that adding 10 to that remainder, the Sum shews in what place after the point the Decimal begins. *Observe* also in this last case, that you'll find the same Number by subtracting the first remainder from 20. *Exam.* If the Index of the Sum of the Logarithms of two Fractions (taken out this way) is 14, then the corresponding Decimal Fraction begins in the fourth place after the point; but if the Index is 8, the Decimal begins in the 12th place.

Again; tho' it is a certain general Rule, that the Logarithms of two Numbers which are to be Multiplied together, are to be taken out of the same System of Logarithms; yet that is not always necessary, because other Circumstances save it; as an Example or two will explain.

Suppose the Logarithm of one Fraction is 3.5768340 (belonging to that System where in 10 is the Logarithm of 1,) and of another 12.3742067 (belonging to that System wherein 100 is the Logarithm of 1;) if we reduce them to one System, the first is 93.5768340, and then their Sum is 105.9510407; from which take 100 (the Logarithm of 1) the remainder is 5.9510407, whose Index 5 taken from 100, the difference 95 shews at what place after the point the correspondent Decimal begins. But suppose we do not reduce the first Logarithms, adding them as they are, the Sum is 15.9510407, from which if we take 10 (the Logarithm of 1 in the one System) the remainder is 5.9510407 as before: The *Reason* is obvious; for as here one of the Indexes is 90 less than in the other Method, so there is 90 less subtracted from the Sum, which must give the same difference.

If an Integer and Fraction are to be Multiplied, then if we consider the Integer Fraction ways, making 1 the Denominator, taking its Logarithms out of any System, it will be the same as when we take it out of the common System as a whole Number; because what we add to the common Index is taken away again by substracting the Logarithm of the Denominator 1; and therefore from the Sum of the Logarithms of an Integer, taken out of the common System, and the Logarithm of a Fraction taken by this third Method, we are to take the Logarithm of 1 (as it is in the System out of which the Logarithm of that Fraction was taken) the remainder is the Logarithm of the Product; the difference of whose Index and 10, if it's greater than 10, or their Sum if its less, shews in what place from the Point the Decimal begins.

Exa. Suppose the Logarithm of any Integer is 4.3720679 (taken out of the common System, wherein the Logarithm of 1 is 0) and the Logarithm of some Fraction is 3.2578006, (taken out of the System where 10 is the Logarithm of 1) the Sum of these Logarithms is 11.6298685 from which take 10 (the Logarithm of 1) the remainder is 1.6298685 whose Index taken from 10 the remainder is 9, shewing that the corresponding Decimal begins in the ninth place after the Point.

But at last *observe* as to both this and the preceeding Method, that if Fractions are given Terms in a Question which is to be solved by Multiplication or Division, we need

not,

not, in order to the Operation by Logarithms, take out the Logarithms of these Fractions separately; but, considering the Method of Working with the given Numbers, we shall need only to take out the Logarithms of the several Terms as whole Numbers, and apply them to one another by Addition, so as to have at last but one Simple subtraction; and then if a greater Logarithm is to be taken out of a lesser (which gives the Logarithm of a Fraction) we may do it either by the preceding second or third Method, and then find the corresponding Fraction in the manner directed: The following Questions will illustrate this: For

I shall finish this Chapter with a few Examples in the Practice of Logarithms for a further illustration of the Rules; where you'l also find some further useful Instructions

In Multiplication.

Exa. 1. To Multiply 39674 by 4685, I find the Sum of their Logarithms; which having a greater Index than any in the Tables, I find the nearest Logarithm to 8.2692156 which is 4.2692093, whose corresponding Number is 18587; and because the Index of the Logarithm whose corresponding Number I want is 8, I multiply the Number found by 10000 and it is 185870000, which is less than the true Product, this being 185872620. If we we had Tables carried to a greater extent than 101000 (which is the extent of *Sherwin*'s Tables) then we should find the Product true to more Places.

Logarithms.	
of 4685	$= 3 \,.\, 6707096$
of 39674	$= 4 \,.\, 5985060$
of 185870000	$= 8 \,.\, 2692156$

Exa. 2. To find the Product of 268 by $\frac{57}{342}$ I add the Logarithm of 268, to that of 57 and from the Sum take the Logarithm of 342, and the Number corresponding to the remainder nearest is 44.667; the true Product being 44.666, *&c.* having 6 circulating *in Infinitum*; so that the Product found is a very little more than true. I have also wrought it another way, to shew the correspondence of both; thus, I take the Logarithm of $\frac{57}{342}$ by the second Method of the preceding Problem, which is $\overline{1}.2218488$ which added to the Logarithm of 268, the Sum is the same Logarithm as was found by the other Method.

Logarithm.	
of 268	$= 2 \,.\, 4281348$
57	$= 1 \,.\, 7558749$
Sum	$4 \,.\, 1840097$
of 342	$= 2 \,.\, 5340261$
of 44 . 667	$= 1 \,.\, 6499836$
of $\frac{57}{342}$	$= \overline{1} \,.\, 2218488$
of 268	$= 2 \,.\, 4281348$
Sum	$1 \,.\, 6499836$

Exa. 3. To Multiply $\frac{23}{478}$ by $\frac{7}{59}$ I add the Logarithms of 7 and 23, also those of 59 and 478; and subtracting this Sum from that, as directed in the second Method of the preceding Problem; the remainder is the Logarithm sought, *viz.* $\overline{3} \,.\, 7565459$ whose correspondíng Number I find, thus, I seek a Logarithm whose Fraction is nearest to .7565459, and this I find to be .7565448, and the corresponding Number is 57088, which qualified according to the Index $\overline{3}$, is .0057088; which exceeds by a little the true Product, for this is .0057008, *&c.*

Logarithms.	
of 7	$= 0 \,.\, 8450980$
23	$= 1 \,.\, 3617278$
7×23	$= 2 \,.\, 2068258$
59	$= 1 \,.\, 7708520$
478	$= 2 \,.\, 6794279$
59×478	$= 4 \,.\, 4502799$
$\frac{7 \times 23}{59 \times 478}$	$= \overline{3} \,.\, 7565459$, diff.

whose Value reduced is this Decimal .0057088.

In Division.

Exa. 1. To Divide 4762 by 24, I subtract the Logarithm of this from the Logarithm of that, the remainder is 2.2975782 to which the nearest Logarithm in the Table (setting aside the Index) is .2975636 which is the Fractional part of the Logarithm of 19841; but applying the Index 2, the corresponding Number is 198.41.

	Logarithms.
of 4762	= 3 . 6777894
24	= 1 . 3802112
198.41	= 2 . 2975782, diff.

Exa. 2. To Divide 74568 by 4.37; having found the Logarithm of 437, whose Fractional part is .6404814, the Index due to make it the Logarithm of 4.37 is 0; therefore taking the Logarithm of 4.37 from that of 74568 the remainder is a Logarithm, whose nearest in the Table has for its corresponding Number 17103, which exceeds the true Quote, for this is 17063.615, &c.

	Logarithm.
of 74568	= 4 . 8725525
4.37	= 0 . 6404814
17103	= 4 . 2320711

Exa. 3. To divide 5670 by $\frac{37}{456}$; because the Quote is equal to $5670 \times 456 \div 37$, therefore I add the Logarithm of 5670 to that of 456, and from the Sum take the Logarithm of 37; the Logarithm in the Table which is nearest to the remainder, has for its correspondent Number 69879, the true Quore being 69878.918, &c. wanting very little of the former.

	Logarithms.
of 5670	= 3 . 7535831
456	= 2 . 6589648
Sum	6 . 4125479
of 37	= 1 . 5682017
of 69879	= 4 . 8443462

To Divide a Fraction by a Fraction is multiplying the Dividend by the Reciprocal of the Divisor, and the Operation by Logarithms is the same therefore as in Multiplication.

In finding Proportionals.

Exa. 1. To find a 3d in Geometrical Proportion to these 14 : 359, I take the Logarithm of 359, and from the double of this Logarithm take the Logarithm of 14, the remainder is a Logarithm whose nearest in the Table has for its correspondent Number 9205.8, which is the 3d Proportional sought nearly, this being 9205.78, &c.

	Logarithms.
of 359	= 2 . 5550944
its double	5 . 1101888
of 14	= 1 . 1461280
of 9205.8	= 3 . 9640608 diff.

Exa. 2. To find a fourth Proportional to these 24 : 367 . 29 : : 5348 . 6 from the Sum of the Logarithms of the second and third Terms, I take the Logarithm of the first 24; the nearest Logarithm to the remainder has for its correspondent Number 81854, which exceeds the true fourth a little, this being 81853.63, &c.

	Logarithms.
of 5348 . 6	= 3 . 7282401
of 367 . 29	= 2 . 5650091
Sum	6 . 2932492
of 24	= 1 . 3802112
of 81854	= 4 . 9130380

Exa. 3. To find a fourth Proportional to these $\frac{13}{14} : \frac{35}{37} :: \frac{5}{9}$: By the common Rules this is had by this Operation, *viz.* $5 \times 25 \times 14 \div 9 \times 37 \times 13$, wherefore having added together the Logarithms of 5, 25, 14, also the Logarithms of 9, 37, 13, I take this Sum from the other: And seeking a Logarithm in the Table whose Fractional part is nearest to 6066504, its corresponding Number is 40425, and because the Index of my Logarithm is $\bar{1}$, therefore the Number sought is .40425, which is the true fourth Proportional, true in all these Figures.

Logarithms.	
of 5 =	0 . 6989700
of 25 =	1 . 3979400
of 14 =	1 . 1461280
Sum	3 . 2430380
of 9 =	0 . 9542425
of 37 =	1 . 5682017
of 13 =	1 . 1139434
Sum	3 . 6363876
of .40425 =	$\bar{1}$. 6066504 diff.

Observe. If in *Multiplication, Division*, or finding a third or fourth Proportional, any of the Terms is a mixt Number, either reduce it to an improper Fraction, or the Fractional part to a Decimal, and then proceed.

For Involution.

This being no other than Multiplication; the practice of it by Logarithms is the same also as that for Multiplication.

For Extraction of Roots.

Exa. 1 To extract the Square Root of 1156: I find its Logarithm to be 3.0629578, whose half is 1.5314789, and the correspondent Number is 34, which is exactly the Square Root of 1156.

Exa. 2. To find the fifth Root of 32768; I find its Logarithm 4.5154499, and the fifth part of this is .9030899, &c. the nearest Logarithm to which in the Table is .9030900 whose correspondent is 8, the true fifth Root of 32768.

Exa. 3. To find the Cube Root of 13839, I take its Logarithm which is 4.1411047, whose third part is 1.3803682, &c. The Logarithm whose Fractional part is nearest to this is .3803741; and its corresponding Number is 24009, but because of the Index 1, it is 24.009, which is an excessive Root, for the Cube of this is 13839.578, &c. The Integral part of which Root 24, is the Root of the greatest Integral Cube, which is contained in 13839.

Observe. If the given Number whose Root is sought is greater than any Number in your Table, use this Method; take a Number lesser, which is a Power of the proposed Order, by which divide the given Number; if the Integral Quote is a Number within your Table (and if it is not, you must chuse another Divisor of the same kind that will bring the Quote within the Table) seek the proposed Root of the Quote, and multiply it by the Root of the Divisor, the Product will be the Root sought, or near to it.

Exa. 4. To find the Cube Root of 262144 because it's greater than can be found in the Table, I divide it by 8, (the Cube of 2) which gives for a Quote precisely 32768, whose Logarithm is 4.5154499, and the third part of this is 1.5051499, and the Number corresponding to that Logarithm which is the nearest to this in the Table, is 32; which multiplied by 2 (the Cube Root of the Divisor 8) produces 64, the true Cube Root sought.

The *Reason* of this Rule you have in *B* 3, *Theor.* 2d. For if any Number is a Power, as A^n and if it is divided by a Similar Power B^n, the Quote is a Similar Power, whose Root is the Quote of the Roots of the Dividend and Divisor. So $A^n \div B^n = \overline{A \div B}^n$ are $A \div B \times B = A$, that is, the *n* Root of the Quote of $A^n \div B^n$ multiplied by *B*, the

the Root of the Divisor B^n, produces A, the n Root of the Dividend A^1. If the Dividend is not a Rational Power, or the Divisor is not an aliquot part of it; you can only expect to find a Root nearly true: But as Involution is easier than Evolution, having found such a Root as your Logarithms will give, prove it by actual Involution, and by an Allowance for what it errs, and one or two Trials, you may bring it near enough for common Applications: And in the Extraction of high Roots, where the common Rules prove very tedious, *This* will with much less Trouble bring out a Root sufficiently near.

APPENDIX, *shewing the* Reason *of the* RULES *given for finding the number of Terms in a Geometrical Progression.*

See *Probl.* 4, 6, 9, *Chap.* III, *Book* IV.

In *Problem* 4th having the Extremes a, l, and Ratio r, to find the number of Terms n. The Rule given by Logarithms is this; $n - 1 = \frac{\text{Log. } l - \text{Log. } a}{\text{Log. } r}$; the *Demonstration* of which is this, $l = ar^{n-1}$ (*Cor.* 6, *Probl.* 3, *Chap.* III.) whence $r^{n-1} = \frac{l}{a}$; And consequently, Log. $r^{n-1} =$ Log. $\frac{l}{a}$. But, by *Prob.* 5. preceding, Log. $\frac{l}{a} =$ Log. $l -$ Log. a; and by the 3d Fundamental Rule of *Logarithms*, Log. $r^{n-1} = \overline{n-1} \times$ Log. r; Wherefore $\overline{n-1} \times$ Log. $r =$ Log. $l -$ Log. a; and, lastly, $n - 1 = \frac{\text{Log. } l - \text{Log. } a}{\text{Log. } r}$

In *Problem* 6th the Rule is $n - 1 = \frac{\text{Log. } l - \text{Log. } a}{\text{Log. } \overline{s-a} - \text{Log. } \overline{s-l}}$; the Reason of which is this; By the preceding part of that Problem it is shewn, that $r = \frac{s-a}{s-l}$; Whence, Log. $r =$ Log. $\frac{s-a}{s-l} =$ Log. $\overline{s-a} -$ Log. $\overline{s-l}$, which being put in the preceding Rule for Log. r, makes the present Rule.

In *Problem* 9 we have this Rule; $n - 1 = \frac{\text{Log. } l - \text{Log. } \overline{rl + s - rs}}{\text{Log. } r}$. The Reason of which is this; 'Tis there shewn, that $a = rl + s - rs$; so that this Rule is only putting Log. $\overline{rl + s - rs}$ for Log. A, in the Rule of *Problem* 4.

We have also this Rule, $n - 1 = \frac{\text{Log. } \overline{rs + a - s} - \text{Log. } r - \text{Log. } a}{\text{Log. } r}$; the Reason of which is, that $l = \frac{rs + a - s}{r}$, whence Log. $l =$ Log. $\frac{rs + a - s}{r} =$ Log. $\overline{rs + a - s} -$ Log. r; which is put in place of Log. l, in the Rule of *Problem* 4.

CHAP. VI.

Of the Combinations *of Numbers.*

DEFINITIONS.

I. *Combinations* of Things are, the Various Ways a number of Things may be taken and join'd together, either in respect of the Order of the Whole, or the Choice of a number of Particulars out of the Whole. But this will be more clearly understood by the Species into which *Combinations* are distinguish'd, viz. *Permutations*, *Elections*, and *Compositions*.

II. *Permutations*, or *Changes*, (or, as some call them, *Alternations*) are such Combinations of any number of Things wherein respect is had to the Order of the Whole, either as to Place or Succession, thus. (1^o) In regard of Place: Any number of Things being propos'd, the number of different Ways these Things may be dispos'd in an equal number of determin'd Places, so that they shall never be all in the same Places, is call'd their *Changes* (in respect of Place). *Exa.* Suppose 6 Things *A*, *B*, *C*, *D*, *E*, *F*, are to be dispos'd in 6 Places: This may be done various Ways, according to the different Places every one may possess, Regard being still had to the Whole; *i.e.* if any two, or more of them change Places, that makes a new *Alternation* or Order of the Whole, tho' all the rest remain unchang'd.

(2^o) In regard of *Succession*: The different Ways several Things may be taken or order'd in Succession one after another, are also call'd *Changes*, or *Alternations*, as to Order of Succession, depending upon the taking of *A*, or *B*, or any one of them, 1st or 2d, &c. And as the taking any one of them 1st or 2d, &c. may be call'd putting them in the 1st, 2d, &c. Place of the Succession, this shews the Coincidence of these two Ways of ordering Things, as to the Number of *Changes*; For they are both reducible to one Notion of Place, either as it relates to *Space*, which is more strictly call'd *Place*; or to *Time* and *Succession*; which, as to the Number of Changes, is the same; for *Places* cannot be better distinguish'd than by numbering them 1st, 2d, 3d, &c. and the Order of Succession of Things is distinguishable no other Way, than by marking which Thing is 1st, 2d, 3d, &c.

III. *Elections* or *Choices* are Combinations which regard not the Order of the Whole, but the Way of taking a particular Number out of the Whole. *Thus*, Suppose a lesser number of Things is to be taken out of a greater, and we are at liberty to take them out of any Part of the Whole; the number of Ways this may be done, so that some (one at least) shall be different in every Choice or Combination, is call'd the *Choices* of that number of Things in the other. *Exa.* If 4 Men are to be drawn out of 100, the number of Ways this can be done, so as some one of them shall be a different Man, is the *Choices* of 4 Men (or any other Things) in 100.

COROLLARIES.

1*st*. The *Choices* of 1 in any Number is equal to that Number; and any Number can be taken out of it self but once, or one Way.

2*d*. If

2*d.* If any Number N is equal to two Numbers, $A+B$, the *Choices* of A and B in N are equal: For, since the one being taken, the other is left; then as many Choices as you can take away of the one, so many you leave of the other.

3*d.* If two Numbers differ by 1, as A and $A+1$, the Choices of A in $A+1$ is equal to $A+1$; because the Choices of 1 in $A+1$ is $A+1$, and the Choices of A and 1 are equal.

IV. *Compositions* are limited Elections. Thus, conceive two or more different Setts (or Systems) of Things, containing each the same, or a different number of Things; then suppose we are to chuse out of the whole a number of Things, either equal or unequal to the number of Setts, so that we take some Part out of every Sett [if possible, *i. e.* if the number of Setts be not greater than the number to be chosen; for then we may take any Choice of a number of the Setts equal to the number to be elected] the number of Choices thus limited is call'd the *Compositions* of that number of Things out of that number of Setts. *Exa.* Suppose 16 Companies of Men, 16 Men may be drawn out of these various Ways, taking only 1 Man out of each Company, and the number of Choices we can make with this Limitation of 1 out of each Company, is the *Compositions* of 16 in the 16 Companies.

THEOREM 1.

If any Number N is resolv'd into two Parts, $A+B$, the *Changes* of N or $A+B$ are equal to the Product of the *Changes* of A, and of B; and the *Elections* of A (or B, which are equal by *Corol.* 2. to *Defin.* 2d) in $A+B$.

Exa. If the Changes of 4 be 24, and of 6 be 720, and the Elections of 4 in 10 ($=4+6$) be 210, then the Changes of 10 will be $24 \times 720 \times 210 = 2158800$.

DEMON.

$$\underbrace{\overbrace{\cdot\;\cdot\;\cdot\;\cdot}^{A=4}\;\;\overbrace{\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot}^{B=6}}_{A+B=10}$$

Conceive a number of Places equal to $A+B$ represented by Points set in a Row, as in the Margin: wherein there are distinguish'd a Number equal to A on the left, and the remainder equal to B on the right: It's certain that in every one of the *Changes* of $A+B$ Things in all these Places, some one particular Election of a number of Things equal to A must possess that individual number of Places equal to A, which lies first on the left hand. Suppose any one Election of A Things to possess these Places, it's plain they can continue there so long, and no longer, than till all their Changes in these Places be join'd with all the Changes of the remaining B Things in the remaining Places on the right hand; and these will make so many different Changes of the whole, *viz.* a Number equal to the Product of the Changes of A into the Changes of B: But then every Election of A Things out of the whole will possess these A Places, that lie first on the left hand, as often, or in as many different Changes of the whole, as the first Election did; and when every Election of A has possess'd these A Places as often as possible, (*i. e.* in as many different Changes of the whole as the Product of the Changes of A and B) then all the Changes of the whole are finish'd. Consequently they are the continual Product of the Changes of A, and of B, and the Elections of A (or B) in $A+B$: Which may be express'd in Characters, thus; $ch : A+B = ch : A, \times, ch : B, \times$ Elect. A in $\overline{A+B}$.

THEOREM 2d.

Let any Number N be equal to two others $A+B$, the number of those different Alternations of the whole N, in which the Part A will possess the same number of certain determin'd

min'd Places, is equal to the Product of the number of Alternations of A by those of B.

Exa. If the Changes of 6 are 720, of 2 if they are 2, and of 4 they are 24; then a certain Choice of 4 Things will possess a certain Choice of 4 Places, 48 ($= 24 \times 2$) times, or in 48 different Changes of the whole.

DEMON. It's plain that the Number A may possess an equal number of determin'd Places as long as while all their Alternations in these Places be join'd with all the Alternations of the remaining Things B in the remaining Places, and no longer.

Observe; If you ask how long these A Things will keep these determin'd Places without changing in them, then the Number is that of the Alternations of B.

THEOREM 3d.

If a Number n ($= a + b$) is to be elected out of a greater N, the Number of different Elections, in which a certain Choice of a Things will cast up, is equal to the number of Elections of b in $N - a$.

Exa. If the Elections of 6 in 10 are 210; and of 4 in 8 ($= 10 - 2$) they are 70; then chuse any 2 out of the whole 10, and that Choice will come up with 70 different Choices of 6 in 10.

DEMON. Take away any Choice of a from N, the Choices of b in the Remainder $N - a$ being join'd with that Choice of a, make all the Choices of $a + b$ in which this particular Choice of a is concern'd.

THEOREM 4th.

The *Elections* of any Number A in another greater than it, N, are equal to the Sum of the Elections of A and of $A - 1$ in $N - 1$.

DEMON.

$$\overbrace{\underbrace{\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot}_{N-1}}^{N} \ \&c.$$

Conceive all the Units of the Number N to be dispos'd in a Row, and one of them to be taken off, from the left hand, so that there remain $N - 1$ on the right: It's evident that the Choices of A in $N - 1$ are a part of the Choices of A in the whole N; and 'tis as plain that, having these, we want none of the Choices sought (*viz.* of A in N) but those in which the Unit taken off is concern'd (or makes one of the Units chosen). And it's again plain, that these are had by joining that Unit with all the Choices of $A - 1$ in $N - 1$, because that Unit being join'd to $A - 1$, makes the Number A; and being join'd to all the Choices of $A - 1$ in $N - 1$, makes all the Choices of A (in N) in which that Unit is concern'd: Which Number therefore being added to the Choices of A in $N - 1$, makes the whole Choices of A in N.

PROBLEM 1st.

To find how many *Alternations* or *Changes* any Number of different Things is capable of,

RULE. Take the Natural Series of Numbers from 1 (*viz.* 1, 2, 3, &c.) up to the given Number; multiply them together, the last Product is the Answer.

Exa. 1. The Changes of 3 Things are $6 = 1 \times 2 \times 3$, represented as in the Margin, by 3 Letters, A, B, C.

Exa. 2d.

A, B, C	*B, C, A*
A, C, B	*C, A, B*
B, A, C	*C, B, A*

Exa. 2d. The Changes of 8 Things are, $40320 = 1\times2\times3\times4\times5\times6\times7\times8$, So that if there are 8 Men in a Company, they may change Places so, that the Order of the whole shall be varied 40,320 different Ways.

DEMON. If the Rule is true in any one Case, (*i. e.* of a Number N, it's true of the next above, or of $N+1$ Things; and consequently of all above: But 'tis true of 2 Things, A, B, whose Changes are only $2 = 1\times2$, for they can be order'd only thus, $A\ B$ or $B\ A$; therefore 'tis true of all Numbers. What remains then to be demonstrated is this; That if the Rule is true of N Things, it's true also of $N+1$ Things, which is demonstrated thus:

By *Theorem* 1, the Changes of $N+1$ are $=$ *ch*: $N\times$, *ch*: 1, $\times$ elect: N in $\overline{N+1}$. But the Changes of 1 are only 1; *that is*, one Thing can be taken but one Way: And the Elections of N in $N+1$ are $N+1$ (*Corol.* 3, *Defin.* 2) therefore the Changes of $N+1$ are $=$ *ch*: $N\times\overline{N+1}$. But the Changes of N, according to the Rule, are $N\times\overline{N+1}\times\overline{N-2}\times$ &c. $\times 1$, (or $1\times2\times3\times$&c. N); and if this is right, then the Changes of $N+1$ are $\overline{N+1}\times N\times\overline{N-1}$ &c. $\times 1$. (or $1\times2\times3$, &c. $\times N+1$.) which being also according to the Rule, this is therefore right.

Or, this Article may be demonstrated independently of *Theor.* 1, from the nature of Alternations only, *thus*; In every *Change* of $N+1$ Things, some one Thing must possess the 1st Place, and there it may continue till the remaining N Things change Places as oft as possible; each of which Changes join'd with that One in the 1st Place, makes so many different Changes of the Whole: And, since any One of the Whole may possess the 1st Place as oft, it follows that the Changes of N Things multiply'd by the Whole number of Things, $N+1$, gives all the different Changes of $N+1$. But *ch*: $N = N\times\overline{N-1}$ &c. $\times 1$. Therefore *ch*: $N+1 = \overline{N+1}\times N\times\overline{N-1}$, &c. $\times 1$; which is the Rule.

SCHOL. 1. We have learnt how to find the Number of Changes of any Number of Things; but if it should also be requir'd actually to take them all out, or represent them, for Example, by Letters; there is one Certain Method of proceeding, by which we can go thro' the whole with the greatest Ease and Distinctness, so as to run no hazard (or the least possible) of omitting any Change, or taking any one oftner than once. This Method will be made clear by a few Examples.

Exa. 1. For 2 Things A, B, the Changes are these 2, $A\ B$, $B\ A$.

Exa. 2d. For 3 Things, A, B, C, the Changes are 6, which you see already taken out; only, to save superfluous writing, they may be order'd as in the Margin; where, because every Letter possesses the 1st Place twice, *viz.* till the remaining two have chang'd twice, therefore I write that Letter down but once in the 1st Place, supposing it to belong to the 1st Place of the next Change, which is left not fill'd up.

A B C	*B A C*	*C A B*
C B	*C A*	*B A*

Exa. 3d. For 4 Things, *a, b, c, d,* the Changes are 24, as they are here represented; where every Letter possesses the 1st Place 6 times, *viz.* till the remaining 3 have chang'd 6 times, whose *Changes* are order'd the same way as in the preceding Example.

a b c d	*b a c d*	*c a b d*	*d a b c*
d c	*d c*	*d b*	*c b*
c b d	*c a d*	*b a d*	*b a c*
d b	*d a*	*d a*	*c a*
d b c	*d a c*	*d a b*	*c a b*
c b	*c a*	*b a*	*b a*

Exa. 4*th.*

Exa. 4th. For 5 Things, a, b, c, d, e, their Changes are 120 ($= 24 \times 5$) taken as here represented. Where *Observe*, that because the Changes of 4 are 24, so, in taking out these of 5, every Letter must possess the 1st Place 24 times, *i. e.* till the remaining 4 Letters make 24 Changes; which are taken out according to the Method of *Exa. 4th.*

a b c d e	b a c d e	
e d	e d	
d c e	d c e	
e c	e c	
e c d	e c d	
d c	d c	
c b d e	c a d e	
e d	e d	
d b e	d a e	
e b	e a	
e b d	e a d	&c.
d b	d a	
d b c e	d d c e	
e c	e c	
c b e	c a e	
e c	e a	
e b c	e a c	&c.
c b	c a	
e b c d	a c d	
d c	d c	
c b d	c a d	
d b	d a	
d b c	d a c	
c b	c a	

I have carried the Work no farther here than to 48 Changes, *viz.* while the two first Letters, a, b, possess the 1st Place each 24 times; the rest are easily conceiv'd by these.

By these *Examples* the Method for any other Number may be easily understood, one depending always upon the preceding: So, if there were 6 Things whose Changes are $720 = 120 \times 6$ each of them must possess the 1st Place 120 times, *viz.* till the remaining 5 make their 120 Changes. And, *Observe*, that as every Letter has the 1st Place as oft as the Changes of the remaining Number, so, while it possesses the 1st Place, the Letter next it in the 1st Change of those wherein it has the 1st Place possesses that next (or 2d) Place as oft as the number of Changes of the remaining Letters after this one; and then the next Letter is advanc'd into that 2d Place; and so on, till they are all successively in the 2d Place. The same is to be observ'd of the 3d and 4th, *&c.* Places: Then, when all the Letters after the 1st have possess'd the 2d Place, a new Letter is advanc'd into the 1st Place, and so the Changes proceed with that Letter in the 1st Place as they did before.

Or, if we trace the Order from the right hand to the left, then *observe* that the two Letters on the right hand (in the 1st Order of the given Letters) having chang'd twice, a new Letter is advanc'd into the 3d Place (counting now from right to left) and what was last in the 3d Place is put in the 2d Place; and this new Letter in the 3d Place is there till the 2 on its right hand change twice, and then the 1st Letter (on the right) is advanc'd to the 3d Place: Then, when the first 3 Letters have thus possess'd the 3d Place, each of them twice, *i. e.* as oft as the Changes of the remaining 2, a new Letter is advanc'd into the 4th Place (by making the Letters in the 3d and 4th Places change) and there it continues till the remaining 3 Letters make all their Changes: And so on till all the Letters are advanc'd to all the Places from the right to the left Hand. An attentive Consideration of the preceding Examples will make all this very clear.

But *Observe* again, that the Number of Changes grow so fast upon the Series of Numbers, that the *Changes* of a small number of Things can never be all represented. For *Example*: The Changes of 10 are 3628800; and allowing a Man to take out 300 of them every Hour, it would cost him 504 Days to finish them all, tho' he works at it Night and Day, without Interruption: But if we only double the Number of Things, *i. e.* take 20, the Changes are 187,146,308,321,280,000; so great a Number, that if a Man could take out 500 of them every Hour (which yet I doubt any Man could do) it would take him upwards of 42 thousand million of Years to finish them all. For, divide the Changes by 500, the Quote is 374292616642560 Hours; which divided by 24, quotes 15595525693440 Days, which makes 42,727,330,666 Years 70 Days.

SCHOL. 2. In this *Problem* the Things to be *chang'd* are suppos'd to be so many distinct Individuals; which, tho' equal or alike in some respects, yet are distinct and different from one another in that respect upon which the Variety of *Change* depends; and so are capable

capable of a real Difference, and Variety of Order : But if two, or more, of them are the ſame, or like in that reſpect upon which the *Change* depends, ſo that they admit of no Variety among themſelves; Then the Number found by the former Rule muſt be corrected. I ſhall firſt explain this Likeneſs and Difference of Things.

Whatever Likeneſs we ſuppoſe among Things, while we conſider them as only numerically different, this is a ſufficient Foundation for all the Variety of Order or Change in the preceding Problem : But if we make the Subject of the Change any Thing which they have all in common, or which is common to any Number of them, the Caſe is different. For *Example* ; Suppoſe 3 Letters, whereof 2 of them are the ſame as to Sound, as *A*, *A*, *B*, theſe are not capable of all the Variety, in reſpect of the Order and Succeſſion of Sounds that 3 different Letters have, becauſe the two *A*'s having no Variety of Sound, admit of no Change betwixt themſelves, as two different Letters do : So the Changes here are only 3, viz. *A A B*, *A B A*, *B A A* ; whereas 3 different Letters have 6 Changes. But if we take two different Characters, as *A*, *a*, and make the Change regard only the Places of different Characters, without regard to Sound, then it's the ſame thing what Sounds they repreſent, they are 3 different Things as to Shape, and ſo have all the Variety of any 3 different Things as to Order of Place.

Take another *Example:* Suppoſe 4 Bells, whereof 2 of them have the ſame Note or Tone in Muſick; then, if we conſider the Changes theſe 4 Bells are capable of in the ſucceſſion of their Sounds, as Notes of Muſick, they have not ſo many as were they all different Notes ; becauſe the two that have the ſame Note cannot change with one another, and ſo it's no matter which of them is firſt ſtruck, it makes the ſame ſucceſſion of Notes. Indeed if we conſider 'em only as 4 Sounds, emitted from 4 diſtinct Bodies, they are in this reſpect capable of all the Variety of any 4 different Things, tho' they had all one Note ; but the Variety in this reſpect is not to be perceiv'd by the Ear, unleſs the ſpecifick difference of the Sounds be all different, and then the Changes may be ſaid to turn upon that, otherwiſe the Changes of them can only be mark'd by different Names to the 4 Bells.

PROBLEM 2d.

To find all the *Changes* of any number of Things, whereof 2 or more are the ſame, in that reſpect upon which the Change depends:

RULE. Find the Changes of the Given Number, and alſo the Changes of that Number of them which are the ſame, or like, by *Probl.* 1 ; divide the former by this, the Quote is the true number of different Changes. But if there are more than One Part of the Given Number that conſiſt of Things like among themſelves, (One Part being ſtill different Things from another) then take all the Parts (*i.e.* all their Numbers) which conſiſt of like Things among themſelves; find the Changes of each of theſe Numbers by *Probl.* 1. then multiply them continually together, and by the Product divide the Changes of the Given Number found by *Probl.* 1, the Quote is the true Number ſought.

Exa. 1. Of 6 Things, whereof 3 are the ſame, the Changes are 120 : For the Changes of 6 different Things are 720, thoſe of 3 are 6 ; then $720 — 6 = 120$.

Exa. 2. Suppoſe 8 Notes of Muſick whereof 3 are the ſame, and 2 are the ſame, but different from the former 3, and both different from the remaining 3, as, *fa*, *fa*, *fa*, *ſol*, *ſol*, *la*, *mi*, *fa*, the Variety in the ſucceſſion of theſe 8 Notes, is 3360; thus, the Changes of 8 different Notes are 40320; of 2 there are 2, and of 3 there are 6; then $2 \times 6 = 12$, and $40320 \div 12 = 3360$.

DEMON. Suppose any number of Changes $N = A + E$, by *Theor.* 1, $ch: N = ch: A \times ch: B \times$ Elect. A (or B) in N; therefore if any Number, as A, of these Things has but one Order, (as when they are like Things, or the same in that respect upon which the Variety depends) in this Case $ch: N = ch: B \times$ Elect. A (or B) in N: that is, the Number found by *Probl.* 1. is to be divided by $ch. A$, consider'd as different Things. Again; if there is another part of the given Number all like Things, as, suppose $B = C + D$, and that the Number C are like Things, then, by what's already shewn, the $ch: B$, taken by *Probl.* 1, must be divided by $ch: C$; Consequently, the $ch: N$ (by *Probl.* 1.) are to be first divided by $ch: A$, to correct the Error arising from A being like Things; and this Quote again divided by $ch: C$, to correct the Error arising from C being like Things; and so on, however many Parts are like Things: But it's the same to divide $ch: N$ continually by any Numbers one after another, or all at once, by the continual Product of these Numbers. Wherefore the Rule is true.

PROBLEM 3d.

To find the *Elections* of any lesser number of Things out of a greater number of Things all different.

RULE. Take the Series 1, 2, 3, &c. up to the Number to be elected, and multiply them continually together; then take a Series of as many Terms, decreasing by 1, from the Number out of which the Election is to be made, and multiply them continually together: Divide this Product by the former, the Quote is the Number sought.

Exa. 1. The Choices of 2 in 6 are $15 = \frac{6 \times 5}{1 \times 2} = \frac{30}{2}$

Exa. 2. The Choices of 4 in 9 are $126 = \frac{9 \times 8 \times 7 \times 6}{1 \times 2 \times 3 \times 4} = \frac{3024}{24}$

Universally, the Choices of A in B are express'd thus;

$$\frac{B \times \overline{B - 1} \times \overline{B - 2} \times \overline{B - 3}\ .\ \&c. \times \overline{B - A - 1}}{1 \times 2 \times 3 \times 4 \times .\ \&c. \times A}$$

DEMON. Suppose $B = A + D$, then, by *Theor.* 1, $ch: B = ch: A \times ch: D \times$ Elect. A (in B); Hence Elections A (in B) $= ch: B \div \overline{ch: A \times ch: D}$. Now, by *Probl.* 1, $ch: B = B \times \overline{B - 1} \times \overline{B - 2}$ &c. $\times 1$; And 'tis evident, that in this Series there must be one Term equal to D (since D is less than B) And therefore this Series may be thus express'd, $ch: B = B \times \overline{B - 1}$, &c. $\times D \times \overline{D - 1}$ &c. $\times 1$. But since $B = A + D$ and $D = B - A$, therefore the Term of this Series next above D or $B - A$ is $B - A + 1 = \overline{B - A - 1}$; therefore the Series may be also thus express'd, $ch: B = B \times \overline{B - 1}$ &c. $\times \overline{B - A - 1} \times D \times \overline{D - 1}$ &c. $\times 1$. Now the $ch: D = D \times \overline{D - 1}$ &c. $\times 1$, therefore $ch: B$ being divided by $ch: D$, the Quote is $= B \times \overline{B - 1}$ &c. $\times \overline{B - A - 1}$; also $ch. A = 1 \times 2 \times 3$ &c. $\times A$. And, the last Quote being divided by this, the Quote is $\frac{B \times \overline{B - 1} \text{ \&c.} \times \overline{B - A - 1}}{1 \times 2 \text{ \&c.} \times A}$, which is $= ch: B \div \overline{ch: A \times ch: D} =$ Elect. A in B. Which being according to the Rule, 'tis therefore right.

But we may also demonstrate this Rule independently of *Theor.* 1, from the Nature of *Elections* only, thus:

If the *Rule* is true in any one Case, (*i. e.* of the Elections of A in B) it's therefore true of the next Case, or of $A + 1$ in B. But the Rule is true of the Choices of 1 in any Num-

Number B, which, according to the Rules, are $\frac{B}{1} = B$, which is the true Number, (as has been observ'd in *Corol.* 1, *Definit.* 3.) Therefore the Rule is true of the Choices of 2 in B, and consequently of 3, and every other Number: What is to be demonstrated then is this; That because the Rule is true of any Number A in B, it's therefore true of $A + 1$ in B; which I thus demonstrate.

Take any one Choice of A out of B, there remain $B - A$ things; and if each Unit of this Remainder be severally combin'd with that Choice of A, we shall hereby have as many Choices of $A + 1$ as the Number $B - A$ expresses. Again; If we take every other Choice of A, and combine them with every Unit of their several Remainders, for every one we shall have as many Choices of $A + 1$ as $B - A$ expresses; *i. e.* in the whole a number of Combinations of $A + 1$, equal to the Product of the Choices of A (in B) by $B - A$; which, according to the Rule, is $\frac{B \times \overline{B-1}\ \&c. \times \overline{B-A+1}}{1 \times 2\ \&c. \times A} \times \overline{B-A} =$

$= \frac{B \times \overline{B-1}\ \&c. \times \overline{B-A-1} \times \overline{B-A}}{1 \times 2\ \&c. \times A}$. But now these Combinations of $A + 1$ that we have thus suppos'd are not all different; and to find how many of them are so, take any one Election of $A + 1$, and call it the first; then conceive all the Elections of A that are in this first Election of $A + 1$ (which are so many of the Elections of A in B) to be combin'd with each Unit of their Remainders in B, these make so many of the preceding Combinations of $A + 1$; and it's plain, that with each of these Elections of A (in this first Election of $A + 1$) so join'd with each Unit of their Remainders in B, there will arise one Combination of $A + 1$, Coincident with this first Election of $A + 1$; for each Election of A, in this Election of $A + 1$, being join'd with the remaining Unit in the same Election of $A + 1$, coincides with it: Wherefore as many Elections of A as are in this first Election of $A + 1$ (which are in Number $A + 1$) so many of the preceding Combinations of $A + 1$ are coincident, and therefore not different Elections of $A + 1$.

But the same is, for the like reason, true of every other really-different Election of $A + 1$ in B; so that for every really-different Election of $A + 1$ in B, there are $A + 1$ Combinations of $A + 1$ Things taken in the preceding Work, which are coincident Elections of $A + 1$ Things; wherefore we must divide that first Number of Combinations of $A + 1$ Things by $A + 1$ (which is done by multiplying the Divisor of that Operation by $A + 1$) the Quote is the true Number of the different Elections of $A + 1$ in B, which is this, $\frac{B \times \overline{B-1}\ \&c. \times \overline{B-A-1} \times \overline{B-A}}{1 \times 2\ \&c. \times A \times \overline{A+1}}$, exactly according to the Rule.

In the following *Scholium* you see yet another Way of demonstrating this Rule.

SCHOL. 1. The Elections of any Number in any greater may be found and dispos'd in a Table (where they may be afterwards had by Inspection) which may be carried on *in infinitum*; whereof you have here a Specimen; the Construction of which is obvious, every Column being made of the Sums of the preceding so far, or of the Sum of the preceding Terms of the same and the preceding Column. The Numbers to be elected stand on the head of the Table, the first Column being the Numbers out of which the Elections are to be made, and the several Numbers of these Columns shew the Elections of the Number on the head in the corresponding Number of the first Column.

Exa. The Elections of 7 in 10 are 120, found in the Column under 7, and against 10 in the first Column.

Table *for the Elections of* Numbers.

Numbers to be elected.

Numbers out of which the Elections are made.

Numbers of Elections.

1	2	3	4	5	6	7	8	9
1								
2	1							
3	3	1						
4	6	4	1					
5	10	10	5	1				
6	15	20	15	6	1			
7	21	35	35	21	7	1		
8	28	56	70	56	28	8	1	
9	36	84	126	126	84	36	9	1
10	45	120	210	252	210	120	45	10
11	55	165	330	462	462	330	165	55
12	66	220	495	792	924	792	495	220

The Construction and Use of this Table being thus explain'd, I shall next *demonstrate*, that it contains the true number of Elections, according to the *Rule* for using it, thus:

If 1, a, b, c, *&c.* represent the numbers of Choices of any Number n, in the several Terms of the progression of Numbers from n upwards, then the sum of this Series of *Choices*, viz. 1, $1+a$, $1+a+b$, *&c.* are the several Choices of $n+1$ in the several Terms of the progression of Numbers from $n+1$ upwards. This is plain from *Theor.* 2, which proves, that the Choices of a in n are the sum of the Choices of a and $a-1$ in $n-1$; *i. e.* that the Choices of $n+1$ in any number of the Series from $n+1$ is the sum of the Choices of $n+1$ and n in the preceding lesser Number; But the Choices of n are in the Series 1, a, b, c, *&c.* and the Choices of $n+1$ in $n+1$ are 1; Therefore the Choices of $n+1$, in all the Numbers greater, are in the Series 1, $1+a$, $1+a+b$, &c.

$$\begin{array}{ccccccc} n & : & n+1 & : & n+2 & : & n+3 \\ \hline 1 & : & a & : & b & : & c \quad \text{\&c.} \\ 1 & : & 1+a & : & 1+a+b & & \end{array}$$

But again; The Choices of 1, in any Number, are equal to that Number; *i. e.* are the natural Series 1, 2, 3, 4, *&c.* consequently the Choices of 2 in the Series of Numbers from 2, are the Sums of the preceding Series; and the Choices of 3 in the Series of Numbers from 3, are the Sums of the last Series of Choices: Which makes exactly the preceding Table of *Elections.*

Observe now, That this Table of Elections is the same as the Table of *Triangular* Numbers explain'd in *Chap.* 2, § 2. (only differently dispos'd) where it is shewn, that the a Triangular of the b Order, (or the b Triangular of the a Order) taken from a Series of Units, is $1\times\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}$ &c. $\times\frac{n-\overline{a-2}=b}{a-1}$ (See the *Schol.* after *Probl.* 1, *Ch.* 2.) Now, in this Table of *Elections*, if the Numbers to be elected are compar'd with the Numbers out of which the Election is to be made, the difference of them is always 1 less than the Place of the Number of Elections in its proper Column: Thus, the Difference of 4 and 10 is 0, and the Elections of 4 in 10 are 210, the 7th Term of the Series of Elections of 4: But the Elections of any Number in another are the same as those of its Difference from that other. Also the several Columns of this

Table

Table are the ſeveral Orders of Triangular Numbers, which becauſe as Triangular Numbers they are reckon'd from a Series of Units, the Number of the Order is 1 more than the Number on the head of the Table: And, becauſe alſo the Place of the Number of Elections, in its proper Column, is 1 more than the Difference of the Number on the head of the Table, and the Number of the firſt Column out of which it is to be elected; Therefore 'tis plain, that the Elections of any Number $a - 1$ out of another n ($=$ Elections of $n - \overline{a - 1}$ out of n) are the ſame Numbers as the $n - \overline{a - 2}$ (or b) Triangular of the a Order; which, by the Rule of Triangulars, is $1 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times$ $\times$ &c. $\times \frac{n - \overline{a-2}}{a-1}$, and which, according to the Rule of Elections, expreſſes the Elections of $a - 1$ in n; ſo that the Rule being good for Triangulars, 'tis good alſo for Elections.

Or, if we compare this Table of Elections with the Table of Coefficients of the Powers of a Binomial Root, (See *Book* 3, *Ch.* 2, § 1.) 'tis the very ſame, wanting the Column of Units: But the $n - \overline{a - 2}$, Coefficient of the n Power, is $1 \times \frac{n}{1} \times \frac{n-2}{1}$ &c. $\times$ $\times \frac{n - \overline{a-2}}{a-1}$, the ſame as the Elections of $n - \overline{a - 1}$ in n ($=$ Elections of $\overline{a - 1}$ in n) which therefore are $\frac{n}{1} \times \frac{n-1}{2} \times$ &c. $\times \frac{n - \overline{a-2}}{a-1}$, according to the Rule of Elections.

Now at laſt *Obſerve*, That as the Demonſtration of the Rules of Triangular Numbers or Coefficients is a Demonſtration of the Rule of Elections, ſo the ſame Rule being demonſtrated for Elections, from the Conſideration of Elections only, is alſo a Demonſtration of the Rule for Triangulars or Coefficients; becauſe they are the ſame Numbers under all theſe different Views.

SCHOL. 2. If 'tis requir'd to take out all the Elections of any Number out of a greater, the Certain and Regular Method of doing it will be eaſily underſtood by the following Examples.

$a\ b\ c$	$a\ d\ e$
$a\ b\ d$	$b\ c\ d$
$a\ b\ e$	$b\ c\ e$
$a\ c\ d$	$b\ d\ e$
$a\ c\ e$	$c\ d\ e$

Exa. 1*ſt.* The Elections of 3 out of 5 Things, a, b, c, d, e, are 10, as in the Margin, which are taken out thus: I take out the firſt 3 Letters as they ſtand in order, $a\ b\ c$; then I put another in the firſt place (on the right) ſucceſſively in the order of the Letters, till there is not another behind; then I put a new Letter in the ſecond place (on the right) keeping it there till I change all the Letters in order that are in the firſt place; and then I put a new Letter in the third place, keeping it there till I change all the Letters in the ſecond place as oft as poſſible, *i. e.* ſo as there remain enow behind to make out the Number; and with each of theſe in the ſecond place I change all thoſe in the firſt place; and ſo on, if there are more Things elected; as the following Example will further clear.

Exa. 2*d.* The Elections of 4 in 7 Things: a, b, c, d, e, f, g, are 35, *viz.*

a	b	c	d
			e
			f
			g
		d	e
			f
			g
		e	f
			g
		f	g
	c	d	e
			f
			g
		e	f
			g
		f	g
	d	e	f
			g
		f	g
	e	f	g
b	c	d	e
			f
			g
		e	f
			g
		f	g
	d	e	f
			g
		f	g
	e	f	g
c	d	e	f
			g
		f	g
	e	f	g
d	e	f	g

In all 35.

The Order of theſe, carefully conſider'd, is ſufficient Direction for any other Caſe. I ſhall only add this Obſervation in the laſt *Example*; That when g comes in the firſt place

place (on the right) the Letter in the second place is chang'd, and there it stands till g comes again in the first place, and then it is chang'd again, and so on till f comes in the second place; and then the Letter in the third place is chang'd, and so it stands till the same Changes as before fall upon the first and second places, *i. e.* till f, g come together in the first and second places; and then the Letter in the third place is again chang'd; and so it goes on till e comes in the third place; and then the fourth place is chang'd; and so on till such a Letter comes in the last place, that what are behind in the Order of the Letters do just make up the Number elected.

SCHOL. 3. The Things out of which an *Election* is to be made are suppos'd to be all different, else the Number found by this *Problem* must be corrected. For example; If we suppose that out of 4 Notes of Musick, 2 are to be elected; and, that 2 of the 4 are equal, or in the same degree, as *fa fa, sol, la,* then it is plain that we cannot make as many Choices of 2 Notes, that shall be all different Choices, as we could do out of 4 different Notes; for in 4 different Notes we have 6 Choices of 2; but here we have but 4, viz. *fa, fa*; *fa, sol*; *fa, la*; *sol, la*. Now, this being a Limitation upon the Circumstances of the Election, is a kind of *Composition* (which is a limited Election) and the Rule for it will be better understood after the General Rules for *Composition*; and, till these are explain'd, I refer it.

PROBLEM 4th.

To find the Sum of all the Choices of every Number that is in any Given Number of Things all different, (*i. e.* the Sum of the Choices of 1, and of 2, and of 3, &c. in any Number N) without finding the Choices of any of these particular Numbers.

RULE. Find the Sum of a Geometrical Progression proceeding from 1 in the Ratio 1 to 2, as 1 . 2 . 4 . &c. whose Number of Terms is N, the Given Number out of which the Elections are to be made; That Sum is the Number sought. Or, Find such a Power of 2 whose Index is N; subtract 1 from that Power, the remainder is the Sum or Number sought.

Exa. The Sum of the Elections of every Number that are in 12, is, $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1024 + 2048 = 4095 = 2^{12} - 1$

DEMON. Suppose only 2 Things a, b, all the Choices here are only 3, *viz.* a or b, or $a\,b$; if we join another Thing, to make the whole 3, a, b, c, then 'tis plain, that the preceding Choices in a, b, are so many of the Choices in a, b, c, and we want no more of them but these wherein c is concern'd, which are only it self, and its Combinations with all the preceding (set in the 3d Column). Again; join another, making of the whole 4 Things, a, b, c, d; the Combinations already found in a, b, c are so many of these sought, and we want only these in which d is concern'd, which are only it self, and its Combinations, with all the preceding (set in the 4th Column). The same way does the Work go on, by joining one Thing more, for ever.

a	b	c	d
	ab	ac	ad
		bc	bd
		abc	cd
			abd
			acd
			bcd
			$abcd$

But it is plain, that the Number of Combinations or Choices in the several Columns are in Geometrical progression from 1, in the Ratio 1 to 2; for the first Column has but one, and every following Column has as many Terms as the Sum of all the preceding, and one more; because the Thing on the head of the Column is join'd with all the preceding Terms: And this is the Property of a Geometrical Progression beginning with 1 in the Ratio of 1 : 2; for the Sum of such a Progression is $\frac{rl - a}{r - l}$; r being the Ratio, l the greatest,

greatest, and a the lesser extreme; but if r is 2, and $a=1$, then is $\frac{rl-a}{r-1}=2l-1$ so that every Term is 1 more than the Sum of all the preceding. Wherefore the Sum of all these Columns, *i. e.* of a Geometrical progression, as in the Rule, is the Number sought.

SCHOLIUM 1. If several of the things given are the same, or like (as explained upon the preceding *Problem*) this Rule is to be corrected; and how to do it you'll find afterwards.

2d. As we have shewn a Correspondence betwixt the Coefficients of the Powers of a Binomial Root, and the Number of Elections of one Number out of another; so if we pursue the Comparison, we learn that the Sum of all the Coefficients after the first of any one Power, is the Sum of all the Elections of every Number given, in a Number equal to the Index of that Power; for *Example*, The Coefficients of the fourth Power are 1, 4, 6, 4, 1; and the Elections of 1, 2, 3, 4 severally in 4, are 4, 6, 4, 1; and so in any other Case, as is clear by comparison of the Table of Coefficients with that of Elections, which is the very same, only wanting the Column of Units, which are Coefficients of the first Term in every Power. Wherefore it follows that the Sum of all the Coefficients, after the first of any Power is the Sum of as many Terms of a Geometrical progression proceeding from 1, in the Ratio 1 : 2. or taking in the first it is the Power of 2, whose Index is that given; as it has been also formerly shewn in Book 3. Again, taking the Rule for the Sum of the Coefficients, as it is already demonstrated, then the Rule for the Elections is demonstrated from the Correspondence; yet it was fit to demonstrate it also from the nature of Elections.

PROBLEM 5.

To find the *Compositions* of any Number in an equal Number of sets, the things being all different.

Rule. Multiply the Number of things in every set, continually into one another; the Product is the answer.

Exa. 1. Suppose four Companies, in each of which there are nine Men; to find how many ways nine Men may be chosen, one out of each Company; the answer is $6561=9\times9\times9\times9$.

Observe. In all Cases where the Number in each set is equal, the answer is always such a Power of that Number whose Index is the Number of Sets, or of other things to be chosen; which is here the same.

Exa. 2. Suppose four Companies; in one of which there are six Men, in another eight, and in each of the other two, nine; in this case the Choices (by composition) of four Men are $3888=6\times8\times9\times9$.

DEMON. Suppose only two sets, it's plain that every Unit of the one set being combined with every Unit of the other, make all the Compositions of two things in these two sets; and the Number of them is plainly the Product of the Number in the one set, by that in the other. Again, if there are three sets, the Composition of two in any two of them being combined with each Unit of the third one, makes all the Compositions of three: That is, the Compositions of two, in any two of the sets being multiplied by the Number of the remaining set, produces the Compositions of three in the three sets; which is plainly the continual Product of all the three Numbers in the three sets; and because it's no matter in what order several Numbers are continually multiplied, therefore it's no matter which two sets we had supposed to be first taken. For the same Reason the Rule is good for 4, 5, &c. sets. Wherefore universally, if a, b, c, &c. represent the Num-

Numbers in so many sets of things; the Compositions of as many things out of these Sets are, $a \times b \times c \times$, &c.

PROBLEM 6th.

To find the Compositions of any Number in a greater Number of sets, *i. e.* how to chuse a Number of Things out of a greater Number of sets, taking but 1 Thing out of 1 set (the things being all different.)

RULE. Find the Elections of the Number to be chosen (n) in the Number of sets (N) by *Probl.* 3; then find the Compositions of n, in each of these Elections; by the last *Problem*; the Sum of all these Compositions is the answer. But observe, that as the Number of Individuals in each set are supposed to be equal or not, there is a difference in applying the last part of the Rule, thus:

(1°.) If the Number of Individuals is equal in each set, call it m; then the Compositions in every Election of the sets are equal; and by *Probl.* 5. they are m^n; and this being multiplied by the Elections of n in N, gives the Number sought.

Exa. To find the choices of 3 Men out of 5 Companies, which have each 4 Men, so as to take out 1 Man out of 1 Company, the Elections of 3 in 5 are 10, and the 3d Power of 4 is 64, then $64 \times 10 = 640$ the Number sought. But as there are 20 Men in the whole 5 Companies, if the Choice were unlimited, the Number would be 1140.

(2°.) If the Numbers in each set are not equal, we must actually mark out all the Elections; and take the compositions of each separately, because they cannot be all equal; their Sum is the Number sought.

Exa. Suppose 5 Companies whose Numbers are $a = 2$, $b = 3$, $c = 4$, $d = 5$, $e = 6$, out of which are to be chosen by Composition 3 Men: The Elections of 3 in 5 are 10, which being marked out, with the Compositions of 3 in each, the total Sum is 490, as here.

		Compositions of 3 in each set.
a, b, c	$= 2, 3, 4$	24
a, b, d	$= 2, 3, 5$	30
a, b, e	$= 2, 3, 6$	36
a, c, d	$= 2, 4, 5$	40
a, c, e	$= 2, 4, 6$	48
a, d, e	$= 2, 5, 6$	60
b, c, d	$= 3, 4, 5$	60
b, c, e	$= 3, 4, 6$	72
c, d, e	$= 4, 5, 6,$	120
		490 total.

DEMON. The Reason of every part of this Rule is obvious from the nature of the thing, and the preceding Rules; and need not be further insisted upon.

PROBLEM 7th.

To find the Compositions of any Number N in a lesser Number of Sets n, of things all different, so as some part be taken out of each set.

RULE. Let the sets be represented by the Numbers in each; then (1°.) distribute the Number N into as many parts as there are Units in n, and do this as many ways as possible; after which (2°.) make out all the Alternations of the Terms or Parts in every distribution, and (3°.) compare each Alternation (taking their Terms in one order, as from left to right) with the Numbers of the several sets, (taken in the same order). These Alternations in which any Term is greater than the Number of the corresponding set are to be rejected; for this shews that these parts cannot be taken out of the sets in that order; and for every other Alternation, you are to do thus. (4°.) Find the

the Elections (by *Probl.* 3.) of every Term of the Alternation out of the Number of the correspondent Set; which multiply continually together, the Product is so many of the Choices sought: Then lastly, the Sum of all these Products, made for every Alternation of every Distribution, is the Number sought.

Exa. 1. Suppose two Companies, the one of 5, and the other of 7 Men, out of which are to be chosen 4 Men, so as a part be taken out of each: How many Choices are there? The Answer is 455; found thus,

(1°.) The sets are 5 and 7, and the Number 4 has two distributions into parts, *viz.* $1 + 3$ and $2 + 2$. (2°.) The Alternations of these 2 distributions are only these, $1 + 3$, $3 + 1$, $2 + 2$, then (3°.) Comparing these Alternations with the sets, there is none of them to be rejected; and (4°.) the Elections of their parts in the Correspondent sets being taken and Multiplied, and the Products added together, as you see done below, the total Sum is 455.

	Sets.		Sets.		Sets.		
	$\overbrace{5 : 7}$		$5 : 7$		$5 : 7$	Prod.	175
Altern.	$1 + 3$	Altern.	$3 + 1$	Altern.	$2 + 2$		70
Elect.	$5 \times 35 = 175.$	Elect.	$10 \times 7 = 70.$	Elect.	$10 \times 21 = 210$		210
							455 sum.

Exa. 2. There are three Companies of 5, 6, 7 Men; out of which 5 Men are to be chosen by Composition; the Answer is 6055, found as in the following Work.

1°. The distributions of 5 into 3 Parts are only $\begin{matrix} 1 + 1 + 3 = 5 \\ 1 + 2 + 2 = 5 \end{matrix}$

2°. The Alternations of the distribution $1 + 1 + 3$ are these 1, 1, 3 : 1, 3, 1, and 3, 1, 1; of the distribution $1 + 2 + 2$, they are 1, 2, 2 : 2, 1, 2, 2, 2, 1, each of which can be taken in order out of the sets, 5, 6, 7; and the Elections found, multiplied and added, make in all 6055.

	Sets.		Sets.		Sets.
	$5 : 6 : 7$		$5 : 6 : 7$		$5 : 6 : 7$
Altern.	$1 + 1 + 3 = 5$	Altern.	$1 + 3 + 1$	Altern.	$3 + 1 + 1$
Elect.	$5 \times 6 \times 35 = 1050.$	Elect.	$5 \times 20 \times 7 = 700$	Elect.	$10 \times 6 \times 7 = 420.$

	Sets.		Sets.		Sets.	Products	
	$5 : 6 : 7$		$5 : 6 : 7$		$5 : 6 : 7$		1050
							700
Altern.	$1 + 2 + 2$	Alt.	$2 + 1 + 2$	Alt.	$2 + 2 + 1$		420
Elect.	$5 \times 15 \times 21 = 1575.$	Ele.	$10 \times 6 \times 21 = 1260.$	Ele.	$10 \times 15 \times 7 = 1050.$		1575
							1260
							1050
						Sum	6055

Exa.

Exa. 3. Suppose 9 Men are to be chosen by Composition out of 3 Companies, of 4, 5, 6 Men.

Distributions of 9 into 3 parts.		
1 +	1 +	7
1	2	6
1	3	5
1	4	4
2	2	5
2	3	4
3	3	3

Alternations of these 7 distributions.

	Sets. 4 : 5 : 6		4 : 5 : 6		4 : 5 : 6
1st.	1 , 1 , 7	3d.	1 . 3 . 5	5th	2 . 2 . 5
	1 , 7 , 1		1 . 5 . 3		2 . 5 . 2
	7 , 1 , 1		3 . 1 . 5		5 . 2 . 2
2d.	1 , 2 , 6		3 . 5 . 1	6th	2 . 3 . 4
	1 , 6 , 2		5 . 1 . 3		2 . 4 . 3
	2 , 1 , 6		5 . 3 . 1		3 . 2 . 4
	2 , 6 , 1	4th	1 . 4 . 4		3 . 4 . 2
	6 , 1 , 2		4 . 1 . 4		4 . 2 . 3
	6 , 2 , 1		4 . 4 . 1		4 . 3 . 2
				7th	3 . 3 . 3

If each of these Series of alternations (of the several distributions of the Number 9) is compared in order with the Numbers of the Setts, 4, 5, 6 (in order to which I have set these over the Alternations) then we find that none of the 1st distribution are useful, because 7 is greater than it's Correspondent Sett; of the 2d distribution, the 2d, 4th, 5th, 6th alternations are also useless, because 6 c orresponds to the Sett 4 and 5, out of which it cannot be taken; also the 5th, 6th alternation of the third distribution; and the 3d of the 5th, are useless, all the other alternations serve: And *Observe*, That the distribution which has one part greater than the Number of any of the Setts, is useless, and therefore we need not take out it's alternations; nor the alternations of any other distribution, wherein any part is greater than the Correspondent Sett out of which it should be elected; and therefore taking only the useful distributions and alternations, the whole Work is as follows; where I have set the Elections not under, but in a line with the the Numbers elected; thus, the Elections of 1, 2, 6 (in 4 . 5 . 6) are 4, 10, 1.

Distributions.

	Setts. 4 . 5 . 6	Elections.
1st	1 . 2 . 6	4×10×1 = 40
	2 . 1 . 6	6× 5×1 = 30
2d	1 . 3 . 5	4×10×6 = 240
	1 . 5 . 3	4× 1×20 = 80
	3 . 1 . 5	4× 5×6 = 120
	3 . 5 . 1	4× 1×6 = 24
		54

	Setts. 4 . 5 . 6	Elections.
3d	1 . 4 . 4	4× 5×15 = 300
	4 . 1 . 4	1× 5×15 = 75
	4 . 4 . 1	1× 5× 6 = 30
4th	2 . 2 . 5	6×10× 6 = 360
	2 . 5 . 2	6× 1×15 = 90
5th	3 . 3 . 3	4×10×20 = 800
		1655

	Setts. 4 . 5 . 6	Elections.
7th	2 . 3 . 4	6×10×15 = 900
	2 . 4 . 3	6× 5×20 = 600
	3 . 2 . 4	4×10×15 = 600
	3 . 4 . 2	4× 5×15 = 300
	4 . 2 . 3	1×10×20 = 200
	4 . 3 . 2	1×10×15 = 150
		2750

The Sum of all these Products is 534 + 1655 + 2750 = 4939; the total Elections of 9 in the 3 Companies, whereas a free Choice out of 15, the Sum of the 3 Companies is 5005.

Observe, That tho' this Work be indeed tedious in most Cases, yet it's vastly easier than the actual marking out of all the Choices out of the Setts; which still would require the distributing the Number to be chosen, and then electing.

DEMON.

Demon. Some Part of the Number to be elected must be taken out of each Sett, which is the reason of distributing that Number into as many Parts as there are Setts, and doing this as many Ways as possible: But then the Parts of these Distributions may be apply'd variously to the Setts, which is the reason of taking all their Alternations and applying them to the Setts; for 'tis manifest that each of these different Applications or Ways of taking the Parts of the Number to be elected, among the Setts, must make a different Choice. But *again*, In each of these we are also to consider in what Part of the corresponding Sett it is taken, for that also must make a difference; so that the Elections of every Number in its corresponding Sett being found, 'tis plain that we can join any Choice of the Number taken out of one Sett with the Choice of the Numbers taken out of the other Setts; and consequently the Variety, in this respect, is the continual Product of the Numbers of Choices of the several Parts of any Distribution out of its corresponding Sett. Therefore, *lastly*, the Sum of all these Products is the Number sought

Problem 8th.

To find the *Elections* of any Number n in another Number N, of Things which are not all different, in that respect which makes the Variety of Choice.

Rule 1°. Of the Number N take the several Parts that are like Things, and reckon each of these Numbers as so many Setts; and all the remaining Things in N as another Sett: Mark each Sett by its Number, and also by some Letter, (because different Setts may have the same Number, but, being Setts of different Things, they must have another Distinction) and because all the Setts, except one, are suppos'd to consist each of like Things, (tho' different Setts consist of different Things) distinguish that one by some Mark. Then,

2°. Find all the Elections of n that can be had in that Sett which consists of Things different, and for each of the other Setts reckon but one Election: But, if the number of Things in any of these Setts is less than n, there's no Election. Then,

3°. Find and mark out all the Elections of every two Setts, of every 3, 4, &c. (as far as the Number n) by their Letters and Numbers; And,

4°. Find the Compositions of n in each of these Choices of different Setts, by *Probl.* 5, 6, and 7.

Lastly, The Sum of all these Compositions, and of the Elections in *Article* 2d, is the Number sought.

Exa. 1. Suppose 4 to be elected out of 11, whereof 5 are like Things, and 6 different: The Choices of 4 in 5 are 5; but the 5 being here all like Things, there is but one Choice; then in the 6 different Things there are 15, both which make 16. Next there is but one Combination to be made of the Setts, because there are but two of them. Now, the Distributions of 4 into 2 Parts, with their Alternations, are $1+3$, $3+1$, and $2+2$; and the Choices of these out of two Setts make in all 41, (as in the annex'd Operation) where the Sett 5 being of like Things, the Choices in it are reckon'd but 1; And to these 41 add the preceding 16, the Sum is 57, the Number sought.

a	*b*
a	*c*
a	*d*
a	*e*
a	*f*
	g

Setts.	5	.	6						
	1	.	3	Elect.	1	×	20	=	20
	3	.	1		1	×	6	=	6
	2	.	2		1	×	15	=	15
									41

But if each of the Setts contains like Things, then the Answer would be only 5; For there is 1 Choice in each Sett, and 1 for every Alternation of the Parts of 4; and if the Things to be elected are 6, then the Elections are 6; for there is no Election in the one

Sett,

Sett, and there is but 1 in the other: Then the Distributions of 6 are $1+5$, $2+4$, $3+3$, and the two former have 2 Alternations, but each of them has but 1 Election in the two Setts, which make in all 5; and this, with the former 1 in the Sett 6, is in all 6.

Exa. 2d. To find the Choices of 4 out of 18, whereof 6 are like Things, and 5 also like, and 7 different: The Answer is 201, which is found as in the following Work; for the understanding whereof nothing needs be said, but that the Sett 7, which consists of Things all different, is distinguish'd by a Cross.

$A . B . \overset{+}{C}$	$A . B$	$A . \overset{+}{C}$	$B . \overset{+}{C}$	$A . B . \overset{+}{C}$
$6 . 5 . 7$	$6 : 5$	$6 . 7$	$5 . 7$	$6 . 5 . 7$
$4 . 4 . 4$	$1+3$ Elect. $1\times1=1$	$1+3$ Elect. 1×35	$1+3$ Elect. 1×35	$1+1+2$ Elect. $1\times1\times21$
$1+1+35=37$, the	$3+1$ $1\times1=1$	$3+1$ 1×7	$3+1$ 1×7	$1+2+1$ $1\times1\times7$
sum of the Choices of	$2+2$ $1\times1=1$	$2+2$ 1×21	$2+2$ 1×21	$2+1+1$ $1\times1\times7$
4 in the several Setts.	3	63	63	35

Then $37+3+63+63+35=201$, the Number sought.

D E M O N. The *Reason* of this Rule consists all in this; That when any Number n is to be elected out of any Number N without limitation; and if the Number N, is any how distributed into Parts $A+B+C$, &c. then it's certain that every Choice of n out of the whole Sum must either be taken out of some 1 of these Parts, or out of some 2, or 3, *&c.* of them (proceeding to a Number of Parts equal to n, for we cannot distribute it into more Parts than n has Units, and find some Part of n in each Part of N.) Then, for the Manner of taking n out of these Setts or Parts of N, the Reason is contain'd in the *Problems* refer'd to.

P R O B L E M 9th.

To find the Sum of all the *Elections* of every Number in any Number N, which are not all different Things, without finding the Elections of any particular Number,

R U L E. Separate the Number N into Setts, as in the last *Problem*; then find the Sum of a Geometrical Series from 1 in the Ratio of $1:2$, whose Number of Terms is the Number of Things in that Sett which are all different Things: Next take one of the Setts which consists of Things all like, and by its Number multiply the preceding Sum, and to the Product add that preceding Sum, and also the Multiplier: This last Sum multiply by the Number of another Sett of like Things, and to the Product add the last Sum, and also the Multiplier: and thus go on, multiplying the last Sum by the Number of another Sett of like Things, adding the last Sum and Multiplier to the Product till you have gone thro' all the Setts: The Sum of all these Sums is the Answer of the Question.

Exa. To elect every Number out of 9 Things whereof 2 are like, and 3 are like, and 4 different, the Answer is 253. For, the whole Elections in the 4 different Things are (by *Probl.* 4.) $1+2+4+8=15$; Then $15\times2=30$. and $30+15+2=47$. then $47\times3=141$. and $141+47+3=191$. Lastly, $15+47+191=253$, the Number sought.

D E M O N.

DEMONSTRATION. However many different Things there are in the 1st Sett, the Elections sought in that Number are according to the Rule; which is demonstrated in *Problem* 4th. Then, if all the following Things were also different, the same Operation would go on; but if 2 or more of them are the same, the combining these with each of the preceding, will produce only the same Elections: But then taking 1, or 2, or 3, &c. of these like Things (to the greatest Number of them) what you take consider'd by it self, and also join'd with each of the preceding Elections, makes so many new Elections: And thus going thro' them all, we have the whole Election sought, according to the Rule, as the annex'd Scheme will easily shew.

a : *b* : *c* : *d* : *e* : *ee* : *f* : *ff* : *fff*
ab . *ac* . *ad* . *ae* . *cee* . *fa* . *aff* . *afff*
bc . *bd* . *be* . *bee* . *fb* . *bff* . *bfff*
abc . *ed*
abd
acd &c.
bcd
abcd

PROBLEM 10th.

To find the *Compositions* of a Number out of an equal number of Setts; When the Things in the several Setts are different (*i. e.* there is nothing in one Sett which is in any other) but the Individuals of the same Sett such, that 2 or more of them are the same or like Things, which may obtain in any one or more Setts.

RULE. Each Sett, whose Individuals are all the same Things, are to be reckon'd as if there were but one Thing in it: And if in any Sett there is any Part consisting of the same Things, reckon all that Part as it were but one Thing; then apply the Rule of *Problem* 5th.

Exa. Suppose 3 Setts of Things, one of 6 Things all different; another of 4, but which are all the same Things; another of 8 Things, whereof 5 are like, and 3 are like. To find the Compositions of 3 in these three Setts, we must reckon the 2d Sett as having but 1 Thing, and the 3d as having 2, and then the Number sought is $6 \times 1 \times 2 = 12$.

a	*b*	*m*
b	*b*	*m*
c	*b*	*m*
d	*b*	*m*
e		*n*
f		*n*
		n

The *Reason* of this Rule is obvious. So, in the preceding Example, we can join each Thing in the first Sett with *b*, and with *m*, or with *n*; nor have we any more Choice, since every *b*, *m*, and *n* are the same.

PROBLEM 11th.

To find the *Compositions* of any Number in a greater number of Setts, Circumstances being as in the last *Problem*,

RULE. Reckon all the Things that are the same, or like, in any Sett, as but one Thing, and proceed by *Probl.* 6.

PROBLEM 12th.

To find the *Compositions* of any Number N in a lesser Number n of Setts, Circumstances being as in *Probl.* 10.

RULE. Apply the first 3 Steps of the Rule of *Probl.* 7; and for the third Step (*i.e.* taking the Elections of the Parts of N out of the corresponding Setts) we must distinguish *thus*: 1°, If any Sett has all its Individuals like, there is but one Election. But,

But, 2°, if they are neither all like, nor all different, find the Elections of the corresponding Part of N in that Sett by *Probl.* 8.

The *Reason* is obvious.

SCHOL. In the Matter of *Compositions* we have hitherto suppos'd, that there is no Thing in one Sett which is the same, or like to what is in any other Sett: But now, if we suppose it otherwise, *i. e.* that there are several Things the same, or like, in several Setts, this is a new Limitation, and more difficult than the former, and indeed is such as I have not found how to bring under any General Rules: Some particular Cases have their proper Rules, that will be easily discover'd when they occur, but what has been already done being the principal Part of this Doctrine, and the Foundation of what else relates to it, I shall not insist much further on these Things, only add one Problem more, *viz.*

PROBLEM 13th.

The Individuals of several Setts of Things being the same or like Things, *i. e.* the Things that are in any one Sett, (however like or unlike among themselves) being the same, and the same in Number, as are in all the other Setts; To find the different Compositions of a number of these Things equal to the number of Setts.

RULE 1°. If any of the Things is oftner than once in the same Sett, reckon as it were but once; Then, 2°, take the Arithmerical Progression 1, 2, 3, *&c.* to such a number of Terms as the number of different Things in each Sett; take the Series of their Sums continually from the beginning, the last of them is the number of Compositions sought in two Setts. Again; Take the Sums of their Series continually from the beginning, the last of them is the number of Compositions in three Setts. Go on thus taking the Sums of every succeeding Series of Sums, and the last Term is the Number for a number of Setts equal to the number of Series from the beginning of the Work.

Thus:

1 . 2 . 3 . 4
1 . 3 . 6 . 10
1 . 4 . 10 . 20
&c.

If there are 2 Setts, and 4 Things in each, the Compositions of 2 are 10; if there are 3 Setts, the Compositions of 3 are 20; and so on.

a	*a*	*a*	
b	*b*	*b*	&c.
c	*c*	*c*	
d	*d*	*d*	

DEMON. Suppose only 2 Setts; then the first Letter *a* of the second Sett may be join'd with each of the first Sett; but the second Letter *b* is not to be join'd with the *a* of the first Sett, because that coincides with the Combination of the *a* of the second, and *b* of the first Sett; and so the *b* of the second Sett is combin'd only with the *b* of the first Sett, and with all below. For the same Reason the *c* of the second Sett is join'd only with the *c* of the first Sett, and with all below, and so on thro' the other Letters. All which joinings, taking the whole Combinations of each Letter of the second Sett with all from the same Letter downwards in the first Sett, as one Term, makes an Arithmetical Series 1, 2, 3, *&c.* to a number of Terms equal to the number of Things in the Sett; and it's plain, that the Sum of them is the total Compositions of 2 in the two Setts. Again; For the same Reason the *a* of the third Sett will join with each Composition of 2 in the preceding two Setts; but the *b* must begin only to be join'd with *bb* of the preceding Compositions, and with none in which *a* is; *i. e.* it will be in as many Compositions of 2 as are in two Setts which have one Thing fewer, which must be the next lesser Term of the second Series: And the *c* of the third Sett will in like manner be join'd with *cc*, and all the Compositions of 2 in the preceding two Setts which have neither *b* nor *a*, (*i. e.* as many as the Compositions of 2 in two Setts which have two Things fewer than the last) which must be the next lesser Term of the second Series; and so on thro' all the Letters of the third Sett: So that the total Compositions of 3 in the three Setts is the Sum of all the Terms of the second Series. The same *Reason* is manifest, however many Setts there are.

ARITH-

ARITHMETICK.

BOOK VI.

The Application of the Doctrine of Proportion, *in the Common Affairs of Life and Commerce.*

IN *Books* I, and II, the Fundamental Operations of *Arithmetick* are fully explain'd in their Abstract Practice; and for the Application, all is done with respect to *Addition* and *Subtraction* that can be reduced to General Rules; also for the more simple Applications of *Multiplication* and *Division*: But the Great and Universal Rule for the application of *Multiplication* and *Division* is founded in the Doctrine of *Proportion Geometrical*, and contain'd chiefly in *Probl.* 1, *Chap.* 3, *Book* 4. whose Use, in the Common Affairs of Life and Commerce, is of the Greatest Importance, and for the explaining of which this Book is design'd.

CHAP. I.

The Rule of Three, Golden Rule, *or Rule of* Proportion.

DEFINITION.

THIS *Rule* (which is the Foundation of most of what follows) is the Application of *Probl.* 1, *Ch.* 3, *B.* 4, call'd *The Rule of Three* from having three Numbers given to find a fourth; but more properly *The Rule of Proportion*, because by it we find a 4th Number proportional to 3 given Numbers, (*i. e.* which has the same Ratio or Proportion to one of 3 given Numbers as another of them has to the remaining one). And because of the necessary and extensive Use of it, 'tis call'd *The Golden Rule*.

But,

But, to define it with regard to Applicate Numbers, or Numbers of particular and determinate Things, it is, "The Rule by which we find a Number of any kind of "Thing (as Money, Weight, *&c.*) so proportion'd to a Given Number of the same "Things, as another Number of the same, or different Things, is to a third Number of "the last kind of Thing. For the four Numbers that are proportional must either be all apply'd to one kind of Things; or two of them must be of one kind, and the remaining two of another; because there can be no Proportion, and consequently no Comparison of Quantities, of different Species: As for example, of 3 Shillings and 4 Days; or of 6 Men and 4 Yards. But of this more fully afterwards.

All Questions that fall under this Rule may be distinguish'd into two kinds: The first contains these wherein 'tis simply and directly propos'd to find a 4th Proportional to 3 Given Numbers taken in a certain order: As, if it were propos'd to find a Sum of Money so proportion'd to 100 *l.* as 64 *l.* 10 *s.* is to 18 *l.* 6 *s.* 8 *d.* or as 40 *lb.* 8 ounces is to 6 hundred weight. The second kind contains all such Questions wherein we are left to discover, from the Nature and Circumstances of the Question, That a 4th Proportional is sought; and, consequently, how the State of the Proportion (or Comparison of the Terms) is to be made; which depends upon a clear understanding of the Nature of the Question, and of Proportion. And after the Given Terms are duly order'd, as the Proportion ought to run, what remains to be done is, to apply *Probl.* 1, *Ch.* 3, *B.* 4. But, because there is something further in the Application, and, to remove all Difficulty as much as possible, the whole Solution is reduced to the following General Rule, which contains what further Direction is necessary for solving such Questions wherein the State of the Proportion is given; in order to which, 'tis necessary to premise these Observations.

1°. In all Questions that fall under the following Rule there is a Supposition and a Demand: Two of the Given Numbers contain a Supposition, upon the Conditions whereof a Demand is made, to which the other Given Term belongs; and it is therefore said to raise the Question, because the Number sought has such a Connection with it as one of these in the Supposition has to the other. For example; If 3 Yards of Cloth cost 4 *l.* 10 *s.* (here is the Supposition) what are 7 Yards 3 quarters worth? (here is the Demand or Question rais'd upon 7 Yards 3 quarters, and the former Supposition.

2°. In the Question there will sometimes be a superfluous Term; *i. e.* which, tho' it makes a Circumstance in the Question, yet is not concern'd in the Proportion, because it is equally so in both the Supposition and Demand. This superfluous Term is always known by being twice mention'd, either directly, or by some Word that refers to it. *Exa.* If 3 Men spend 20 *l.* in 10 Days, how much at that rate will they spend in 25 Days? Here the 3 Men is a superfluous Term, the Proportion being among the other 3 given Terms, with the Number sought; so that any number of Men may be as well suppos'd as 3.

RULE. 1°. The superfluous Term (if there is one) being cast out, state the other 3 Terms, thus: Of the two Terms in the Supposition, one is like the Thing sought (*i. e.* of the same kind of Thing the same way apply'd); set that one in the 2d or middle place; the other Term of the Supposition set in the 1st place (or on the left hand of the middle); and the Term that raises the Question, or with which the Answer is connected, set in the 3d place (or on the right hand); and thus the Extremes are like other, and the middle Term like the Thing sought. Also the 1st and 2d Terms contain the Supposition, and the 3d raises the Question; so that the 3d and 4th have the same Dependance or Connection as the 1st and 2d. This done,

2°. Make all the 3 Terms simple Numbers of the lowest Denominations express'd; so that the Extremes be of one Name. Then,

3°. Repeat the Question from the Numbers thus stated and reduced, (arguing from the Supposition to the Demand) and observe whether the Number sought ought to be greater

greater or leſſer than the middle Term (which the nature of the Queſtion rightly conceiv'd will determine) and accordingly multiply the middle Term by the greater or leſſer Extreme, and divide the Product by the other, the Quote is like the middle Term, and is the compleat Anſwer if there is no Remaindet: But if there is, then,

4°. Reduce the Remainder to the Denomination next below that of the middle Term, and divide by the ſame Diviſor, the Quote is another part of the Anſwer in this new Denomination. And if there is here alſo a Remainder, reduce it to the next Denomination, and then divide. Go on thus to the loweſt Denomination; where, if there is a Remainder, it muſt be apply'd fractionwiſe to the Diviſor: And thus you will have the compleat Anſwer in a Simple or Mix'd Number.

Obſerve, If any of the Dividends is leſs than the Diviſor, reduce it to the next Denomination, and to the next again, and ſo on, till it be equal to the Diviſor.

EXAMPLES.

Queſt. 1. If 3 Yards of Cloth coſt 8 *s.* what is the Value of 15 Yards? *Anſw.* 40 *s.* or 2 *l.*

Work.

```
y.    s.    y.
3 — 18 — 15
      15
     ----
3)   120 (40 s.
```

Explanation.

3 yards and 8 *s.* contain the Suppoſition; and 8 *s.* is like the Thing ſought; therefore 8 *s.* is the middle term, and 3 *y.* on the left: Then the Demand ariſes on 15 *y.* and therefore it is on the right. Again; from the nature of the Queſtion it is plain, that 15 *y.* requires more than 3 *y* *i. e.* the Anſwer muſt be greater than the middle term; wherefore I multiply 8 *s.* by 15, the Product is 120 *s.* which divided by 3, quotes 40 *s.* without a Remainder; So 40 *s.* or 2 *l.* is the Number ſought.

Qu. 2*d.* If 4 *lb.* of Sugar coſt 2 *s.* 9 *d.* what is the Value of 18 *lb*? *Anſ.* 12 *s.* 4 *d.* 2 *f.*

Work.

```
lb.  s.  d.   lb.
4 — 2 : 9 — 18
    12
   ----
    33 d.
    18
   ----
4) 594 d. (148 d.
Rem. 2
     4
    ---
4)  8 f  (2 f.
```

Explanation.

The Suppoſition is in 4 *lb.* and 2 *s.* 9 *d.* this being like the Thing ſought, which is connected with 18 *lb.* wherefore the Terms are ſtated according to the Rule: then the middle Term being mix'd, I reduce it to *d*, and then I argue thus: If 4 *lb.* coſt 33 *d.* 18 *lb.* muſt coſt more; therefore I multiply 33 *d.* by 18, and divide the Product 594 *d.* by 4; the Quote is 148 *d*, and 2 remains, which I reduce to *f*, and divide the Product 8 *f* by 4 the Quote is 2 *f*; ſo the Anſwer is 148 *d* : 2 *f* or 12 *s.* 4 *d.* 2 *f.* becauſe 148 *d.* is by reduction 12 *s.* 4 *d.*

Qu. 3*d.* What is the Price of 50 *lb.* weight of Tobacco, when 32 *lb* : 12 *oz.* coſt 4 *l.* 10 *s*? *Anſ.* 6 *l.* 17 *s.* 4 *d.* 3 *f.*

Work.

```
lb.  oz.    l.  s.     l.
32 : 12 — 4 : 10 — 50
16           20        16
----        ----      ----
524 oz. — 90 s. — 800 oz
            800
524 )    72000 (137 s.
          524
         -----
          1960
          1572
         -----
          3880
          3668
         -----
Remainder  212
            12
         -----
524)      2544 d (4 d.
          2096
         -----
Remainder  448
             4
         -----
524 )     1792 f. (3 f. 220/524
          1572
         -----
Rem.       220
```

Explanation.

The Suppoſition being contain'd in 32 *lb* : 12 *oz.* and 4 *l.* 10 *s.* which is like the thing ſought, and the Queſtion ariſing upon 50 *lb*, therefore the Terms are duly ſtated Then, to make them all ſimple Numbers, and the Extremes like, I reduce both Extremes to ounces, and the middle Term to *ſh.* Then, I ſay, if 524 *oz.* coſt 90 *s.* 800 *oz.* muſt coſt more: Wherefore I multiply 90 *s.* by 800, and divide

divide the Product 72000 by 524, the Quote is 137 *s.* and the Remainder is 212; which reduced to *d.* and divided, gives 4 *d.* with 448 remaining; and this reduced and divided, gives 3 *f.* and 220 remaining: So the Anſwer is 137 *ſh.* 4 *d*: $3\frac{220}{524}$ *f.* or 6 *l.* 17 *s.* 4 *d.* 3 *f.* &c.

Qu. 4*th.* What are 5 yards of Ribbon worth, whereof 63 yards and 2 quarters coſt 5 *l.*? *Anſ.* 7 *ſh.* 10 *d.* 1 *f.* $\frac{242}{254}$

Work.

```
yd. qr.   l.   yd.
63 : 2 — 5 — 5
   4      1    4
254   —   5 — 20 qr.
          20
254)    2000 s. (7 s.
        1778
rem.     222
          12
254)    2664 d. (10 d.
        254
rem.     124
           4
254)     496 f (1 f
         254
rem.     242
```

Explan.

The Terms ſtated, and reduced according to the Rule, I find the Anſwer ought to be greater than the middle term, therefore I multiply 5 *l.* by 20, but the Product is leſs than the Diviſor; and ſo I reduce it to *ſh.* which makes 2000; and this divided by 254, quotes 7 *ſh.* The reſt of the Work is plain.

Qu. 5*th.* What Time will 7 Men be boarded for 25 *l.* when 3 Men paid 25 *l.* for 6 Months?

Anſ. 2 Months 16 Days (reckoning 28 Days to a Month.)

Work.

```
Men. Mo. Men.
 3 — 6 — 7
     3
7)  18   (2 Mo.
    14
rem. 4
    28
7) 112 da. (16 da.
```

Explanation.

The 25 *l.* is a ſuperfluous Number; then the Suppoſition is in the 3 Men and 6 Months, and the Demand regards the 7 Men: The Terms being all ſimple, I argue thus: If 3 Men are boarded 6 Months (for 25 *l.* or any Sum) 7 Men will be boarded for the ſame a ſhorter Time: Therefore I multiply 6 Months by 3, and divide the Product 18 by 7; whereby I find the Anſwer 2 Months 16 Days.

Qu. 6*th.* If the Carriage of 3 hundred weight coſt 10 *ſh.* for 40 Miles, how much ought to be carried for the ſame Price 25 Miles and 3 quarters?

Anſ. 4 Cw : 2 qr : 17 *lb.* $\frac{97}{103}$

Work.

```
M.   Cw.   M. qr.
40 — 3 — 25 : 3
 4           4
160 — 3 — 103
     160
103) 480  (4 Cw
     412
rem.  68
       4
103) 272   2 qr.
     206
rem.  66
      28
103) 1848 lb. (17 lb.
     103
     818
     721
rem.  97
```

Explanation.

The ſuperfluous Number is here 10 *s.* and the other 3 terms ſtated and reduced, I argue thus: If 3 Cw. is carried 160 qrs. of a Mile (for 10 *ſh.*) then a greater weight will be carried for the ſame Price 103 qrs. of a Mile: Therefore I multiply 3 by 160, and divide the Product 480 by 103, the Anſwer is, 4 Cw 2 qr : 17 *lb.*

Obſerve. The firſt Four Queſtions are what we call *The Rule of Three direct*; *i.e.* where the 3d Term, being greater or leſſer than the 1ſt, requires that the Anſwer be alſo greater or leſſer than the 2d Term. And the 2 laſt Queſtions are of the *Rule of Three Indirect* or *Reverſe*; where the 3d Term being greater or leſſer than the 1ſt, requires the 4th contrarily leſſer or greater than the 2d: But I have comprehended both in one General Rule. And from this Obſervation you learn how to know what Queſtions are of either kind.

In the following Collection of Queſtions I ſhall only ſtate the Proportion, ſet down the Anſwer, and leave the reſt of the Work to your

your own Exercise ; and, by comparing the Answer which you find, with what is here, you'll know whether it is right.

Qu. 7th. What is the Value of 8 Chalder : 3 quarters, and 5 Bushels of Corn (*English* Measure) at the rate of 1 *l.* 15 *sh.* the Chalder ?

Ans. 15. *l* : 11 *s* : 8 *d* 2 *f* $\frac{1}{2}$. State 1 Ch. — 1 *l* : 15 *s.* — 8 Ch. 3 qrs : 5 Bush.

Qu. 8th. When I bought 40 Gallons of Brandy for 6 *l* : 8 *sh* : 3 *d.* what is the Rate *per* Gallon? *G. l. s. d. G.*

Answer. 3 *s* : 2 *d* : 1 $\frac{9}{10}$ *f.* State 40 — 6 : 8 : 3 — 1.

Qu. 9th. In what Time will 13 Men finish a Work which 5 such Men could do in 3 Days : 8 Hours? *m. d. h. m.*

Answer. 1 da : 6 ho : 46 min : & $\frac{2}{13}$ State. 5 — 3 : 8 — 13.

Qu. 10th. If 3 Men are boarded 9 Months and 20 Days for 30 *l.* how many Men will the same Money pay for at that rate for 4 Months? *Mo. d. Men Mo.*

Answer. 7 $\frac{3}{12}$ or $\frac{1}{4}$ 9 : 20 — 3 — 4

Qu. 11th. 1 Man is boarded 3 Months for 5 *l.* what will he be owing if he remains at Board 8 Months and 7 Days? *mo. l. mo. da.*

Answer. 13 *l* : 15 *s.* State, 3 — 5 — 8 : 7

Qu. 12th. If I rode 60 Miles in 3 Days, when the Day was 14 Hours long (counting from Sun rising to setting) how long must the Day be, that I may ride 100 Miles in the same Time? *Mi. h. m.*

Answer. 23 ho : 20 min. State, 60 — 14 — 100

Qu. 13th. In what Time could I travel 50 Miles, the Day being 12 Hours long, at the rate of 50 Miles in 5 Days, when the Day is 16 Hours. *Ho. Da. Ho.*

Answer. 3 Days : 12 Hours. State, 12 — 5 — 16

Observe: If in this Question we put 5 Days : 7 Hours, instead of 5 Days, the State of the Question, according to the General Rule, is $\begin{matrix} ho. & da. & h. & ho. \\ 12 — & 5 : & 7 — & 16 \end{matrix}$ But then take care that the middle Term be reduced to Hours by 12, this being the suppos'd Number of Hours in these Days : And this being done, there's no other Multiplication ; and what remains to be done is, to divide the Hours produced by the Reduction, *viz.* 167 by 16, whereby they are reduced to Days of 16 Hours long, the Answer being 4 Days 3 Hours : So that the Question is in effect two Questions of Reduction, *viz.* 1st, of 5 d : 7 h. to Hours by 12, and these Hours again to Days by 16. Nor are the 4 Terms truly proportional ; for their State, according to the Operation, is this, $\begin{matrix} ho. & da. & ho. & ho. & da. & ho. \\ 12 — & 5 : & 7 — & 16 — & 4 : & 3 \end{matrix}$: But to compare them as to Proportion, we must reduce the mix'd Number ; *that is,* 5 Days 7 Hours, to Hours by 12 ; and 4 Days 3 Hours by 16 ; and so the 4 Terms are $\begin{matrix} ho. & ho. & ho. & ho. \\ 12 — & 67 — & 10 — & 67 \end{matrix}$; in which two Terms being equal, and the other two not so, the 4 cannot be proportional in any Order they can be taken.

The same will be also true tho' the middle Term be a simple Number, provided the 4th Term sought be not also a simple Number ; but if 'tis, then there is Proportion. *Thus,* if for 16 we put 15, the Answer of *Qu.* 13*th* is simply 4 Days. And these Numbers are proportional Indirectly, *viz.* $\begin{matrix} h. & da. & h. & d. \\ 12 : & 5 :: & 15 : & 4 \end{matrix}$; or these directly, $\begin{matrix} h. & h. & d. & d. \\ 15 : & 12 :: & 5 : & 4 \end{matrix}$

But as the given Numbers are $\begin{matrix} h. & d. & h. \\ 12 — & 5 — & 16 \end{matrix}$, the *Answer* is 3 *da* : 12 *ho.* and the 4 Terms are $\begin{matrix} h. & d. & h. & d. & h. \\ 12 — & 5 — & 16 — & 3 ; & 12 \end{matrix}$; and if we reduce the 2d and 4th Terms to one

Name (the one by 12, the other by 16) the 4 are $\overset{h.}{12} - \overset{h.}{60} - \overset{h.}{16} - \overset{h.}{60}$; which are proportional in no Order.

Qu. 14*th*. If a Piece of Cloth is 20 Yards in length, and $\frac{3}{4}$ in breadth, how broad is another Piece which is 12 Yards length, and contains as much Cloth as the other?

Answer. 1 Yard : 1 quarter. State, $\overset{yd.}{20} - \overset{qr.}{3} - \overset{yd.}{12}$

Qu. 15*th*. How much Shalloun of 1 Yard : 1 quarter breadth, will serve to line a Cloak of 5 Yards Cloth, 3 quarters broad?

Answer. 3 Yards State, $\overset{qr.}{3} - \overset{yd.}{5} - \overset{y.}{1} : \overset{q.}{1}$

Qu. 16*th*. If the Rate of Carriage is 1 Penny for 1 Pound weight carried 50 Miles, how far ought 1 Pound to be carried for 15 Shillings?

Answer. $37\frac{1}{2}$ Miles State, $\overset{d.}{1} - \overset{m.}{50} - \overset{ſh.}{15}$

Qu. 17*th*. When Wheat is at 12 *ſh. per* Buſhel, the 6 *d.* Loaf of Bread weighs 1 *lb.* 4 *oz.* (*Troy* weight) what ought it to weigh, the Wheat being 9 *ſh.* 6 *d.* the Buſhel?

Answer. 16 *oz* : $14\frac{2}{7}$ *dw*. State, $\overset{s.}{12} - \overset{lb.}{1} : \overset{oz.}{4} - \overset{s.}{9} : \overset{d.}{6}$

Note, There are here two ſuperfluous Numbers, *viz.* 1 Buſhel, and 6 *d.*

Qu. 18*th*. What was the Price of Wheat when the Penny-Loaf of Bread weigh'd 8 Ounces; the Statute being, that it muſt weigh 10 Ounces, the Wheat at 12 *ſh.* the Buſhel?

State, $\overset{oz.}{10} - \overset{s.}{12} - \overset{oz.}{8}$

Answer. 15 *s.*

Note, Here are two ſuperfluous Numbers, *viz.* 1 Buſhel and 1 Penny.

Qu. 19*th*. Three Pound weight of Bread coſts 2 *ſh.* 6 *d.* the Wheat at 14 *ſh.* the Buſhel; What is Wheat worth if I pay 2 *ſh.* for the ſame weight of Bread?

Answer. 11 *s* : 2 *d* : $1\frac{3}{5}$ *f.* State, $\overset{s.}{2} : \overset{d.}{6} - \overset{s.}{14} - \overset{s.}{2}$

Qu. 20*th*. What is the Intereſt of 64 *l.* for 1 Year, the Rate of Intereſt being 5 *l.* 10 *s.* to 100 *l*, for 1 Year?

State, $\overset{l.}{100} - \overset{l.}{5} : \overset{s.}{10} - \overset{l.}{64}$

Answer. 3 *l* : 10 *s* : 4 *d* : $3\frac{1}{5}$ *f.*

Qu. 21*ſt*. In what Time will 500 *l.* yield 40 *l.* Intereſt, when 86 *l.* does it in 4 Years 8 Months?

State, $\overset{l.}{86} - \overset{y.}{4} : \overset{m.}{8} - \overset{l.}{500}$

Answer. $20\frac{16}{85}$ Mon.

Qu. 22*d*. At 6 *l. per Cent. per Annum*, what Principal Sum muſt be employ'd to yield 6 *l.* in 2 Year : 6 Months?

State, $\overset{y.}{1} - \overset{l.}{100} - \overset{y.}{2} : \overset{m.}{6}$

Answer. 45 *l* : 3 *s* : 2 *d* : $2\frac{16}{51}$ *f.*

Qu. 23*d*. Of what Principal Sum did 20 *l.* Intereſt ariſe in 1 Year, at the rate of 5 *l. per Cent. per Annum*?

State, $\overset{l.}{5} - \overset{l.}{100} - \overset{l.}{20}$.

Answer, 400 *l.*

Obſerve: In this, and *Qu.* 20*th*, and all *Queſtions* where the Terms repreſent the ſame kind of Things, there is in the nature of the *Queſtion* ſome other Speciality, to diſtinguiſh them, which muſt be carefully obſerv'd, in order to make the 1ſt and 3d, 2d and 4th in all reſpects of one Application: So, in the *Queſtions* 20*th* and 23*d*, the Diſtinction is of Principal and Intereſt; in *Qu.* 14*th* and 15*th* it is Length and Breadth; and in *Qu.* 19*th* 'tis Price of Bread, and Price of Wheat.

Demonstration *of the preceding* Rule.

To this *Demonstration* I must premise these two things:

1°. When Numbers are consider'd complexly with the Things (or Quantities) they represent, or of which they are the Numbers (whence they are call'd *Applicate Numbers*) Then it's plain there can be no Comparison, or no Proportion, betwixt two such Numbers, unless they be of Things of the same kind, whose Quantities can in a proper sense be contain'd in one another. For example; There can be no Comparison or Proportion of 3 *sh.* to 4 yards, or 3 Men to 4 Days: But of any Quantity of Money to another; or of any kind of Weight or Measure to another; or of a Number of Men to Men, there is a proper Comparison: But *Observe*, that in this last Case the Number only is the Subject of Comparison.

2°. Of these Things wherein there is Subdivision with different Species and Denominations; Tho' two Numbers, either of the same or different Denominations, have a real Proportion, yet it is not the same as betwixt the two Numbers taken abstractly, or purely as Numbers, except when the Denominations are the same. Thus, 4 *lb.* and 7 *lb.* have a real Proportion, which is the same as that of the Numbers 4 and 7, taken abstractly; for, as 4 is $\frac{4}{7}$ of 7, so is 4 *lb* $\frac{4}{7}$ of 7 *lb.* But tho' 3 Ounces have also a real Proportion to 7 *lb*, 'tis not the Proportion of the Number 4 to 7, because 4 *oz.* are not $\frac{4}{7}$ of 7 *lb*; And the real Proportion of 4 *oz.* to 7 *lb*, reduced to that of abstract Numbers, is the Proportion of 4 to 112; *i. e.* 4 *oz.* to 112 *oz.* equal to 7 *lb.* *Universally*,

Of two Numbers, simple or mix'd, of the same kind of Things, (Money, Weight, Measure, &c.) the Proportion reduced to that of pure Numbers, is the Proportion of these two Numbers of the same Denomination to which the Given Numbers are respectively equal. *Exa.* The Proportion of 3 *sh.* to 8 *d.* is that of 36 to 8 (*viz.* 36 *d.* to 8 *d.*) Of 4 *l.* 6 *sh.* to 15 *sh.* 'tis 86 to 15, (*viz.* 86 *s.* to 15 *s.*) Of 9 *lb.* to 3 *lb.* 5 *oz.* 'tis 144 to 53 (*viz.* 144 *oz.* to 53 *oz.*) and so of others.

Follows the Demonstration,

1°. The three Given Numbers being stated according to the Rule, whereby the middle Term is like the Thing sought, and the Extremes like other, it follows from the first *Premise*, that the Extremes are the Terms to be compar'd, which contain the Proportion requir'd to be betwixt the middle Term and that sought: For, tho' the 1st and 2d Terms contain the Supposition upon which the Question arises, yet the given Proportion is originally betwixt the Extremes; and the Question express'd according to that Proportion is, to find a Number of Things like the middle Term, bearing the same Proportion to it as the 3d Term does to the 1st. Thus; In *Question 1st* the mix'd Proposition is, to find the Value of 15 yards of Cloth such, that 3 yards be worth 8 *sh.* But, the nature of the Question consider'd, it resolves plainly into this, *viz.* finding a quantity of Money so proportion'd to 8 *sh.* as 15 yards are to 3 yards; And therefore the Terms may be stated also thus: $\overset{y.}{3} - \overset{y.}{15} - \overset{s.}{8}$; whereby the 1st and 2d, 3d and 4th, are the compar'd Terms. But the other Way of stating them is more agreeable to the simple and obvious sense of the Question, and the Way of Reasoning with it upon which the Rule for multiplying and dividing is founded; which gives the true Answer according to the Proportion, as will be shewn

shewn in the next Article: And I conclude *this* with observing, that as the Extremes contain the Proportion which the 4th ought to have to the 2d, so, if they are not of one Name, (by being mix'd Numbers, or otherways) they ought, by *Premise* 2d, to be reduced to one Name, and then the Proportion is reduced to pure Numbers. And, for the reduction of the middle Term, 'tis chiefly done for Convenience in the following *Multiplication* and *Division*; and not to produce an abstract Number, for its Application must always go along with it in the Product and Quote made by the Extremes (now become abstract Numbers). Thus you see the *Reason* of the 1st and 2d Articles of the Rule.

2°. The *Question* being resolv'd into this, *viz.* finding a Number like the middle Term, and in such Proportion to it as the 3d to the 1st, in some Cases (which is Direct Proportion) or as the 1st to the 3d in others; (which is Indirect, with regard to the Order in which the Terms stand, for this is all the Meaning of the distinction). And, the Extremes being now Abstract Numbers, it's manifest from *Probl.* 1, *Ch.* 3, *B.* 4, that the 4th is truly found by multiplying the middle Term by one Extreme, and dividing the Product by the other, according to the Rule: For the 3d Term being greater or lesser than the 1st, and the Question requiring the 4th Term also greater or lesser than the 2d, that Proportion is Direct, and the 3d Term is the Multiplier by the Rule, as it ought to be by *Probl.* 1, *Ch.* 3, *B.* 4. But if the 4th ought to be contrarily lesser or greater than the 2d, then the 1st Term is the Multiplier, by the Rule, as it ought to be, since the real Proportion is in the reverse order of the Terms. And here also *observe*, that tho' the Terms were stated in the plain Order of the Proportion; *i. e.* the 1st and 2d Terms made like other (as above shewn) yet the Operation would prove the very same; therefore the other Way is chosen for another Reason, already mention'd.

3°. The Reason for Reduction of the Remainders to lower Denominations is obvious.

Wherefore the *Whole Rule* is Compleatly demonstrated.

But, for their sake who have pass'd over all the Theory of *Proportion* (which will certainly hinder their being in any tolerable degree Masters of the Application; for, at least, a few of the more fundamental Notions and Theorems ought to be well understood, even for the sake of common Affairs) I shall here add another easy and simple *Demonstration* of that part of this Rule which depends upon *Probl.* 1, *Ch.* 3, *B.* 4. *Thus*,

Take the First Question stated, *viz.* If 3 yards cost 8 *s.* what will 15 yards cost? I suppose (1°) that it is ask'd, If 1 yard cost 8 *s.* what will 15 yards? Here it's plain the Answer is 15 times 8 *s.* or 120 *s.* Again, (2°) Let it be ask'd, If 3 yards cost 8 *s.* what will 1 yard? Here it's as plain, that the Answer is the 3d part of 8 *s.* Now, since 120 *s.* (or 15 times 8 *s*) is the Value of 15 yards, only upon Supposition that 1 yard cost 8 *s.* If, instead of this, one yard cost but the 3d part of 8 *s.* (as 'tis when 3 yards cost 8 *s.*) then it follows that 15 yards cost but the 3d part of 120 *s.* And so you see the Reason both of the Multiplication and Division; and the same way of reasoning will answer in all Cases of the *Rule of Three Direct* (*i. e.* where the 3d Term is Multiplier).

Again; Take the 5th Question, If 3 Men take 6 Months to do any thing in, how long will 7 Men take? 1° say, If 3 Men take 6 Months, what will 1 Man take? It's plain he must take 3 times as much Time, or 3 times 6 Months (= 18 Months). 2° say, If 1 Man take 18 Months, how much will 7 Men take? Here it's as plain they take but the 7th part of 18 Months; whence the Reason of both parts is manifest: And the same Reasoning will hold in all Questions of the *Indirect Rule of Three*, (*i. e.* where the 1st Term is the Multiplier).

OBSERVA-

OBSERVATIONS *relating to the Application of the preceding* RULE, *and of Simple* Multiplication *and* Division.

1. It has been already observed, and I shall repeat it, that all Simple Questions in *Multiplication* and *Division* are really Questions of Proportion, and if the Numbers are applicate, they are Questions of the *Rule of Three*. Thus, to multiply 3 by 4, is to find a 4th Proportional to 1 : 3 : : 4 ; and to divide 12 by 3 is to find a 4th Proportional to 3 : 12 : : 1. To Apply this; suppose 1 Yard costs 3 *s.* to find the Value of 4 Yards is finding a Number of Money proportioned to 3 *s.* as 4 Yards to 1 Yard, that is, as the Number 4 to 1 : which is simply multiplying 3 *s.* by 4. Again, suppose 3 Yards cost 12 *s.* to find the Value of 1 Yard is finding a Number of Money proportioned to 12 *s.* as 1 Yard to 3 Yards, or the Number 1 to 3, which is simply dividing 12 *s.* by 3 to find a 3d part of it. And thus it is plainly in all Cases of the *Rule of Three* where the Extremes being of one Name, one of them is Unity. In all other Cases it is a mixt application of Multiplication and Division ; Where observe, that tho' the Multiplier and Divisor are, in the Question, applied to things of different kind from the number multiplied or divided, which in Simple Multiplication and Division I have fully shewn, in § 4, and 5. *Ch.* 7. *B.* 1. to be absurd, yet here that absurdity is removed by the extremes becoming, or being considered as abstract Numbers. Hence again,

2. We are further confirmed in the absurdity of all these Questions, proposed as Simple Questions of Multiplication, wherein both Terms are applicate, as the multiplying 3 *l.* 10 *s.* by 4 *l.* or by 4 *l* 6 *s.* and such like ; For if the Question belongs to Multiplication Simply, then it must resolve into a Question of the Rule of Three, wherein the Terms that contain the proportion (which in such Cases are the 1st and 2d) being of the same Name, the 1st Term or Divisor is Unity, which leaves the Solution upon the Multiplication. But the only Sense the Question can receive is, to find a Number like the 3d Term (or proposed Multiplicand) so proportioned to it as the 2d Term (or proposed Multiplier) reduced to its lowest Name, is to an Unit of that Name : It follows that the true *General Rule* for such Questions is, to reduce the proposed Multiplier to its lowest Name, and then multiply by it as an Abstract Number.

But if the Unit that regulates the Proportion is taken not of the lowest Name in the proposed Multiplier ; then, the Question is not in any sense Simple Multiplication ; for Division also is necessary to solve it. For *Exa.* If by multiplying 3 *l.* 10 *s.* by 4 *l.* 6 *s.* be meant finding a Number of Money so proportioned to 3 *l.* 10 *s.* as 4 *l.* 6 *s.* is to 1 *l.* then the Proportion is as 86 *s.* to 20 *s.* and the Solution is by multiplying 3 *l.* 10 *l.* by 86, and dividing the Product by 20, according to the *Rule of Three* ; and as many different Units as you can suppose for the 1st Term, so many different Questions, and consequently different Answers there must be. To sum up all ; if no qualification is expressed, then the solving of the Question by the *General Rule* last mentioned, is the only true Solution answering to the Notion of *Simple Multiplication :* And if any other Unit than of the lowest Name in the Multiplier is supposed, it is a Question of the Rule of Three, requiring both Multiplication and Division. And, at last I must observe, that the generality, who propose such Questions, expect the answer as if the Unit were of the highest Name in that kind to which the Multiplier belongs (tho' perhaps they are ignorant of the meaning of it) wherefore, to satisfy their Ignorance, state the proportion so, and work it accordingly. So the preceding Question stands thus, as 1 *l.* to 4 *l.* 6 *s.* so is 3 *l.* 10 *s.* to the Number sought ; And working by the Rule of Three, the prepared State and Answer of the Question is, as 20 *s.* to 86 *s.* so is (3 *l.* 10 *s.* or) 70 *s.* to 301 *s.* (15 *l.* 1 *s.*)

Several of our Writers on Arithmetick differ about the Solution of such Questions, without seeming to understand, at least not explaining the true meaning of them (tho' others

thers have declared the Absurdity) and their Ignorance appears further, by supposing that the Multiplier ought always to be of the same kind of thing as the Multiplicand; but if we consider the real meaning and import of such Questions, it appears plainly that the proposed Multiplier may be of any other kind of thing; so, if multiplying 3 *l.* 10 *s.* by 4 Yards 3 Qrs. can have no other Sense than finding a number of Money proportioned to 3 *l.* 10 *s.* as 4 Yards 3 Qrs. is to 1 Qr. (or 1 Yard or 1 Nail, for we may suppose any of them) the Question is as good Sense as if the Multiplier were 4 *l.* 3 *s.* And in fact, such Questions happen in all proportions of Money, and other things valued by Money.

Again, tho' they make no such absurd Questions in Division as dividing one kind of thing by a quite different kind (as 18 *l.* by 3 Yards) yet these are equally reasonable with those in Multiplication, when the necessary qualification is applied; That is, when the meaning is to find a number of one kind of thing so proportioned to another Number of the same kind, as an Unit of any other kind of thing is to any Number of this kind. For the Unit and that other Number being of one Name, they are as abstract Numbers; and the Question resolves into a Simple Division. So dividing 18 *l.* by 3 Yards in this Sense is only dividing 18 *l.* by 3, as happens in this Question. If 3 Yards cost 18 *l.* what will 1 Yard cost. But if the Unit is of another denomination, then the lowest in the divisor after reduction; both must be reduced, and the Solution takes in both Multiplication and Division; as in this Question, If 3 Qrs. cost 18 *l.* what will 1 Yard? that is, If 3 Qrs. cost 18 *l.* what will 4 Qrs?

3. *This Rule of Three* is the great Rule of Calculation in all kind of Affairs; but to give particular Directions for its Application in all the Variety of circumstances, where proportions arise is impossible; for Questions may be less or more complex; comprehending various Questions of proportion connected in their circumstances, either to bring out several Numbers required, or as so many necessary Steps towards the finding of one Number required; and besides the proportions contained in a Question, there may be other Operations of Addition and Substraction, Simple Multiplication or Division, necessary either to make out the Terms of a Proportion, or after the Proportions are solved to find some Number sought, or a Number to be further applied towards finding Numbers sought; in short to satisfy some condition of the Question in the progress of the Work.

The manageing of such Questions depends upon the Arithmetician's Judgment in distinguishing all the parts of the Question, and knowing what each requires according to the true Sense and Import of it, and of the several Operations of Arithmetick, and particularly of Proportion; of all which he must have a clear and ready Idea; and as there is no other general direction that can reach all Cases, the only thing more that can be done to help one to acquire the necessary Capacity for all useful Questions, is to make the application particularly to such Variety in all the common Subjects and Branches of Business, that who understands these may be supposed capable to do any other of the same, or any other useful kind. To this purpose are all the other common Rules that are brought in after the Rule of Three; which are applications of it chiefly; of which you have a large course in the following Chapters; and I shall add to this a few more complex Questions upon some of the former Heads, which come not so well under any of the following; but first I make this other general Observation.

4. If Fractions are among the given Numbers of a Question of the *Rule of Three*; The procedure is in all respects the same, having due regard to the nature and operations of Fractions; for if the extremes are (or be made by reduction) Simple Numbers, Integral or Fractional, of the same denomination, and the middle Term a Simple Integer or Fraction; the Multiplication and Division is to be performed by the Rules of Fractions, where Fractions are concerned: A few Examples are sufficient to explain this.

Exa.

Exa. 1. If $\frac{3}{5}$ of a Yard cost 4*s.* $6\frac{2}{3}$ *d.* What is the Value of $24\frac{5}{7}$ Yard? State $\frac{3}{5}$ Yard — 4*s.* $6\frac{2}{3}$ *d.* — $24\frac{5}{7}$ Yard. By reduct. 'tis $\frac{3}{5}$ Yard — $\frac{164}{3}$ Yard — $\frac{173}{7}$ Yd. the Product of $\frac{164}{3}$ and $\frac{173}{7}$ is $\frac{28372}{21}$ which divided by $\frac{3}{5}$ the Quote comes out $\frac{141860}{63}$ *d.* equal to $2251\frac{47}{63}$ *d.* or 9*l.* 7*s.* 7*d.* $2\frac{62}{63}$ *f.*

Exa. 2. If $4\frac{5}{9}$ Ounces cost 8*s.* what cost $30\frac{3}{7}$ Pounds? the Extremes being reduced first to Simple Fractions, it is $\frac{41}{9}$ Ounces — 8 *s.* — $\frac{213}{7}$ Pound, and the Extremes being again reduced to one Name, it is $\frac{41}{9}$ Ounces — 8 *s.* — $\frac{3408}{7}$ Ounces, then 8 Multiply, by $\frac{3408}{7}$ produces $\frac{27264}{7}$, which divided by $\frac{41}{9}$ quotes $\frac{245376}{287}$ *s.* equal to $854\frac{278}{287}$ *s.* or 42*l* : 14*s* : 11*d.* $1\frac{29}{287}$ *f.*

Exa. 3. If 3 hundred weight: 2 qrs. $14\frac{3}{4}$ *lb.* cost 68*l*: 10*s.* What cost $1\frac{3}{8}$ hund. weight? First it is, $406\frac{3}{4}$ *lb.* — 1370 *s.* — $\frac{11}{8}$ Cw, then $\frac{1627}{4}$ *lb.* —1370 *s.*—$\frac{11}{8}$ Cw. then $\frac{1627}{4}$ *lb.* — 1370 *s.* — $\frac{1232}{8}$ *lb.* and in this State the Question is compleatly reduced and prepared, and the Answer found by multiplying 1370 *s.* by $\frac{1232}{8}$ and dividing the Product by $\frac{1627}{4}$

Observe. That tho' we cannot easily, in every Case, know which of the Extremes is greatest, unless they are reduced to one Denominator; yet without this it is easy to know which Extreme is the Multiplier; because, suppose that upon the right hand to be either the greater or lesser, that will determine which of them is the Multiplier. Yet after all, if you reduce the extremes to one Denominator; you'll have no more trouble, because the common Denominator may be neglected, and the Operation performed with the Numerators; since that Denominator would be a Multiplier both in the Numerator and Denominator of the Quote, as it comes out first in Fractional form; and therefore both being divided by it (or which is the same, neglect it in the Operation) the Quote will still be the same. So to Multiply by $\frac{5}{8}$ and Divide the Product by $\frac{3}{8}$ is the same as Multiplying by 5, and Dividing by 3.

Mixt QUESTIONS *for the* Rule of Three.

1. If 1 Yard of Cloth cost 15 *s.* at first buying, and upon 540 Yards, there was of Charges (as Packing, Carriage, &c.) 5*l.* 10 *s.* What is the total cost? *Answer.* 410*l.* 10*s.* for if 1 cost 15 *s.* 540 cost 405*l.* to which add 5*l.* 10 *s.* the Sum is 410 *l.* 10 *s.*

2. A Gentleman has a Yearly Rent of 250*l.* If he lays up Yearly 80*l.* what has he to spend upon Living every Month? *Ans.* 14*l.* 3 *s.* 4 *d.* For 250 — 80 = 170. then if 12 Months have 170*l.* 1 Month has 14*l* : 3*s* : 4*d.*

3. Having bought 40 Yards of Cloth at 8 *s.* *per* Yard; and 70 Yards at 12 *s.* What is the Value of both Pieces? *Anf.* 58 *l.* for if 1 Yard cost 8 *s.* 40 cost 16 *l.* then if 1 Yard cost 12 *s.* 70 cost 42 *l.* and 16 *l.* + 42 *l.* = 58 *l.*

4. I bestowed 80 *l.* upon 2 Pieces of Cloth; one of them at 13 *s.* *per* Yard, the other at 16 *s.* whose total Value is 48 *l.* What quantity was in each?

Take 48 from 80 remains 32, the total Value of the first Piece; then find the Quantities by the total Value, and Value of 1 Yard.

5. Five Persons were boarded together at 4 *l.* *per* Quarter; they paid at entry 1 *l.* 10 *s.* a Piece; and having continued 2 Year 3 Quarters; How much do they owe? The Solution is thus made 5 × 4 = 20 *l*, due for every Quarter; then if 1 Quarter cost 20 *l.* what 2 Year 3 quarters? From the Answer of this, Substract 5 times 1 *l.* 10 *s.* paid at entry, the remainder is the Answer.

6. Two Posts travel the one directly *North* and the other *South*, from the same Place; the one Travels 3 Miles for every 2 the other Travels, who Travels at the rate of 36 Miles a Day: How far are they asunder at the 3d Days end? 1°. 3 × 36 = 108 Miles the last Post has Travelled; then say, as 2 to 3 so is 108 to the Miles the other Travels; the Sum of the Miles travelled by both is the Answer: But if they had Travelled the same way, then it is the difference of what each Travels. Or suppose the last named is 30 Miles before the other; to find in what time that other comes up with him, say, As 2 to 3, so is 36 to the Miles the other Travels in a Day; the difference is what of the distance is taken away every Day; whence find the time in which the whole will be taken away.

7. In what time will two Persons do a Work which one of them can do in 8 Days, and the other in 5 Days? Find by the given proportion how much of the whole Work each could do in 1 Day: Add these Answers together; and say, if both together can do so much as that Sum in 1 Day; in what time will they do the supposed Work. In all which Operations that supposed Work is represented by 1.

8. Four Men Drink at Table 16 Pennyworth of Wine: How many Men, each of whom Drinks but half of what each of the other does, will 22 Penny worth serve? Four of these last Men will Drink but 8 Pence worth; therefore say, if 8 *d.* serves 4, what will 22 *d.*?

2. Having bought 146 Yards of Cloth at a certain Price, the seller afterwards discompted 3 *l.* 10 *s.* *per. Cent*, and had in full Payment 248 *l.* What was the 1st Price of 1 Yard? Say, as 96 *l.* 10 *s.* is to 100 *l.* so is 248 *l.* to the first Price of the whole, By which find 1 Yard.

10. I laid out 10 *l.* upon a Parcel of Serges and Shallouns; the total Value of the Shallouns 60 *l.* and the total quantity of Serges 236 Yards; also for every 2 Yards of Serge, I had 3 of Shalloun: How much Shalloun was there? and, what was the Value of 1 Yard of each kind?

CHAP. II.

Contractions in the Rule of Three; *called Rules of* Practice.

CASE 1. WHen a Question in the Rule of Three being duly stated, and the Extremes simple Numbers of one Name; whether the middle Term be Simple or mixt; if the Extreme which by the general Rule is the Divisor, be 1, and the middle Term an Aliquot part of some superior Species; then divide the other extreme by the Denominator of that aliquot part, the Quote is the Answer in that superior Species; and if there is a remainder reduce, and find its Value.

Ta-

TABLE *of the Aliquot Parts of* MONEY.

s.	d.	is equal so, one	part of a Pound.
:	6		40th
:	10		24
1 :			20
1 :	8		12
2 :			10
2 :	6		8
3 :	4		6
4 :			5
5 :			4
6 :	8		3d
10 :			2

d :	f	is equal to	part of a Sh.
1 :			12th.
1 :	2		8
2 :			6
3 :			4
4 :			3d
6 :			2d

1 *f.* is $\frac{1}{4}$ part of 1 *d.*

Examples.

1. What is the Price of 67 yards of Cloth at 5 *s. per* yard? *Answ.* 16*l*: 15 *s.* found thus; The State of the Proportion is $\overset{yd.}{1} - \overset{s.}{5} - \overset{yd.}{67}$; And because the Divisor is 1, and the middle Term 5 *s.* which is a 4th part of 1 *l.* therefore I divide 67 by 4, the Quote is 16*l.* and 3 remains, which reduced to Shillings, aud divided by 4, quotes 15 *s.*

The *Reason* of this Practice is obvious; for if 1 yard cost $\frac{1}{4}$ of 1 *l.* 67 yards must cost 67 4th parts, or, which is the same thing, the 4th part of 67 *l.*

As the Manner of Application, and Reason, in all other Examples, may be easily understood from this one, I shall only state a few, with their Answers.

2. The Value of 54 Stone weight, at 10 *s.* ($\frac{1}{2}$ of 1 *l.*) *per* Stone, is 27 *l.* ($\frac{1}{2}$ of 54.)

3. The Value of 353 Yards, at 3 *s* : 4 *d.* ($\frac{1}{6}$ of 1 *l.*) *per* Yard, is 58 *l* : 16 *s* : 8 *d.* ($\frac{1}{6}$ of 353 *l.*)

4. The Value of 365 *lb* weight, at 3 *d* ($\frac{1}{4}$ of 1 *s.*) *per lb.* is 91 *s* : 3 *d*, or 4 *l* : 11 *s* : 3 *d.* ($\frac{1}{4}$ of 365 *s.*)

5. If 48 Men do a piece of Work in 8 Hours, in what Time will 1 Man do the same? *Answ.* 16 Days, which is the 3d part of 48, because 8 Hours is the 3d part of 24 Hours or 1 Day.

CASE 2d. If the Price of an Unit is an even number of Shillings, multiply the other Extreme (of the same Name with the Unit) by the half of that Number; double the first Figure of the Product for Shillings, and the remaining Figures to the left are Pounds in the Answer.

Exa. 1. What is the Value of 324 Yards at 6 *s. per* Yard? *Ans.* 97 *l* : 4 *s.* thus; multiplying 324 by 3 (the $\frac{1}{2}$ of 6) the Product is 972, which, according to the Rule, is 97 : 4 *s.* And *observe*, that it is very easy to set down the Shillings and Pounds separately, without writing first down the total Product and then separating them.

The *Reason* of this Practice is, that if we multiply the whole even Number of Shillings, the Product is the Answer in Shillings; which divided by 20, reduces it to Pounds, the remainder being Shillings: But if we multiply only the half of these Shillings, the Product is only the half of the Value in Shillings. Now suppose we multiply this Product by 2, to give the whole number of Shillings, and divide this last Product by 20 to reduce

them to Pounds; Then, because 20 is two times 10, it's plain that the Product made by the half of the given Price, being first multiply'd by 2, and this Product divided by 20, (or, which is the same thing, first by 2, and the Quote by 10) the last Quote will be the same as if that first Product were only divided by 10; because, to multiply by 2, and then divide the Product by 2, brings back the same Number that was multiply'd: Wherefore it's plain, that if the first Product is divided by 10, the Quote is the Answer in Pounds and 10th parts; and, because the Divisor is 10, therefore the Integral Quote, or Pounds, are express'd by the Dividend, excluding the first Figure on the right hand; and, because that Figure is the number of 10th parts, therefore double of it is the number of 20th parts, *i. e.* of Shillings; and thus every Part of the Rule is clear.

Observe; If the Price of an Unit consists of Pounds and Shillings, whose half reduced to Shillings is a Number by which we can easily multiply, so as to bring out the Product in one Line at the first step, as we may if that half doth not exceed 29, then we may also use this Method.

Exa. What is the Price of 467 Yards at 1 *l*: 14 *s.* *per* Yard? Here 1 *l*: 14 *s.* is 34 *s.* whose half is 17, by which multiplying 467, according to the Rule, the Answer is 793 *l*: 18 *s.*

CASE 3*d*. If the middle Term is not an aliquot part of some superior Integer, (the Divisor being always 1) yet it may be equal to the Sum of several aliquot parts; and then if you divide by the Denominators of each of these separately, and add all the Quotes, the Sum is the Thing sought. The *Reason* is plain; but I must observe, that in most Cases where the middle Term is not an aliquot part, the common Rule by Reduction is easier Work.

Exa. 1. If 1 yard cost 15 *s.* what cost 49 yards? *Ans.* 36 *l*: 15 *s.* found thus; 15 *s.* is 10 *s.* and 5 *s.* *viz.* the $\frac{1}{2}$ and $\frac{1}{4}$ of 1 *l.* So I take the $\frac{1}{2}$ of 49 *l.* which is 24 *l.* 10 *s.* and $\frac{1}{4}$, which is 12 *l.* 5 *s.* whose Sum is 36 *l.* 15 *s.*

Exa. 2. If 1 yard cost 12 *s.* 6 *d.* what cost 268 yards? *Ans.* 167 *l.* 10 *s.* thus; 12 *s.* 6 *d.* is 10 *s.* and 2 *s.* 6 *d.* *viz.* $\frac{1}{2}$ and $\frac{1}{8}$ of 1 *l.* Then $\frac{1}{2}$ and $\frac{1}{8}$ of 268 *l.* make together 167 *l.* 10 *s.*

Exa. 3. If 1 yard cost 4 *s.* 3 *d.* what is the Value of 140 yards? *Answ.* 29 *l.* 15 *s.* thus; for 4 *s.* it is 28 (*viz.* the 5th part of 140) and for 3 *d.* it is 35 *s.* (or 1 *l.* 15 *s.*) the 4th part of 140 *s.* because 3 *d.* is $\frac{1}{4}$th of 1 *s.* and the Sum is 29 *l.* 15 *s.*

Observe: If the middle Term is equal to the same aliquot part repeated, *i. e.* if it's any simple Fraction (not an aliquot Part) you may either divide by the Denominator, and then multiply the Quote by the Numerator; or rather first multiply by the Numerator, and then divide by the Denominator.

Exa. 4. If 1 yard cost 13 *s.* 4 *d.* the Value of 56 is 37 : 6 : 8 (*l. s. d.*); for 13 *s.* 4 *d.* make $\frac{2}{3}$ of 1 *l.* therefore I multiply 56 by 2, and divide the Product 112 by 3; the Quote is 37 *l.* 6 *s.* 8 *d.*

Again *Observe*, that if the middle Term is resolvable into Parts that are aliquot one of another, then it may often prove a convenient Practice, first to find the Answer for the greater Number, and then for the other which is its aliquot Part, by taking the like Part of the Answer for the former. *Exa.* If 1 yard cost 15 *s.* what 340 yards? Take the half of 340 for 10 *s.* and the half of this half for 5 *s.* your Sum is the Answer.

CASE 4. If the middle Term is so mix'd as to have in it any Number of the highest Species, first multiply that Number, and then the other Parts by some of the former Cases, if possible; and if this cannot be done, or not without much Operation, then we must take the common Method by Reduction:

Exa.

734			
4			
2936 *l*	: *s*	: *d.*	
244	: 13	: 4	
3180	: 13	: 4	

Exa. 1. If 1 yard cost 4 *l.* 6 *s.* 8 *d.* what cost 734 yards? *Answ.* 3180 *l.* 13 *s.* 4 *d.* For 4 *l.* multiply'd by 734, produces 2936 *l.* and for 6 *s.* 8 *d.* I take $\frac{1}{3}$ of 734, which is 244 *l.* 13 *s.* 4 *d.* and the Sum of both is 3180 *l.* 13 *s.* 4 *d.*

28 by 5 is =	140 *l.*	*s.*	*d.*
For 6 *s.*		8	: 8
For 10 *d.*	1	: 3	: 4
Sum,	149	: 11	: 4

Exa. 2. If 1 yard cost 5 *l.* 6 *s.* 10 *d.* what cost 28 yards? *Ans.* 149 *l.* 11 *s.* 4 *d.* thus; For 5 *l.* it is 140 *l.* for 6 *s.* I use *Case 2d*, and for 10 *d.* I take the 24th part of 28, by *Case 1st.*

For 1 *l.* =	38	: *s.*	: *d.*
For 6 *s* 8 *d*---	12	: 13	: 4
For 2 *s.* 6 *d*---	4	: 15	:
Sum,	55	: 08	: 4

Exa 3. If 1 yard cost 1 *l.* 9 *s.* 2 *d.* what cost 38 yards? *Answ.* 55 *l* 8 *s.* 4 *d.* thus; For the 1 *l.* 'tis 38, for 9 *s.* 2 *d.* which is equal to 6 *s.* 8 *d.* ($\frac{1}{3}$ of a Pound) and 2 *s.* 6 *d.* ($\frac{1}{8}$ of a Pound) I take $\frac{1}{3}$ and $\frac{1}{8}$ of 38 *l.* as in the Margin.

Suppose the Price of 1 yard 3 *l* : 7 *s* : 9 *d.* then no Method by aliquot parts is so easy as the common Method by Reduction: And I shall here *observe*, that as there may be different Ways of doing the same Question by aliquot parts, so the chusing of the best Way depends upon Experience. *Exa.* If 1 cost 15 *s* : 8 *d.* it may be done by considering 15 *s.* as $\frac{3}{4}$ of a *l.* or as $\frac{1}{2}$ & $\frac{1}{4}$; and 8 *d* as $\frac{2}{3}$ of a *s.* But more easily by resolving 15 *s* : 8 *d.* into 14 & 1 *s* : 8 *d.* Then, for the 14 *s.* work by *Case 2d*, and for 1 *s* : 8 *d.* by *Case 1st*: Also in *Exa.* 3 *d* we may work first by 1 *l* : 8 *s.* by *Case* 1; then for 1 *s.* and for 2 *d*; but the former Method is easier, after you perceive that 9 *s* : 2 *d.* is equal to 6 *s* : 8 *d.* and 2 *s* : 6 *d.*

CASE 5. If the Extreme which is the Multiplier is an aliquot part, or the Sum of certain aliquot parts, of the Unit which is Divisor, then take by Division such part or parts of the middle Term (whether this be a simple or mix'd Number): And if the Multiplier has also some Number of the same Species with the Unit, you must work for that Number separately by some of the former Cases, or the common Rule, then add all the Parts of the Answer.

Exa. 1. If 1 Pound weight cost 32 *l.* what cost 4 Ounces? *Ans.* 8 *l.* *viz.* $\frac{1}{4}$ of 32 *l.* because 4 *oz.* are $\frac{1}{4}$ of 1 *lb.*

Exa. 2. If 1 yard cost 3 *l* : 10 *s.* what cost 2 qr : 1 nail? *Ans.* 1 *l* : 19 *s* : 4 *d* : 2 *f.* thus; For 2 qr. I take $\frac{1}{2}$ of 3 *l* : 10 *s.* *viz.* 1 : 15, and for 1 nail I take $\frac{1}{16}$, which is 4 *s* : 4 *d* : 2 *f.* Total, 1 *l* : 19 *s* : 4 *d* : 2 *f.*

Exa. 3. If 1 *l.* buy 3 *Ct* weight : 1 *qr* : 7 *lb*, how much will 28 *l* : 5 *s* : 6 *d.* buy?

Ct.	*qr.*	*lb.*
3	: 1	: 7
28		
84		
7		
1	: 03	
92	: 03	

	Ct	*qr.*	*lb.*
4	3	: 1	: 7
10			
	3	: 8	: 12 *oz.*
		9	: 4$\frac{4}{10}$
	3	: 18	: 00$\frac{4}{10}$

Answ. 93 *Ct* : 2 *qr* : 18 *lb* : $\frac{4}{10}$ *oz.* which I find thus; 1°. For the 28 *l.* I multiply 3 *Ct* by it, which gives 84 *Ct*; then for 1 *qr.* I take $\frac{1}{4}$ of 28, is 7 *Ct*; and for 7 *lb.* I take $\frac{1}{16}$ of 28, is 1 *Ct* : 3 *qr*, or, which is the same, I take $\frac{1}{4}$ of 7 *Ct*, because 7 *lb* is $\frac{1}{4}$ of 1 *qr.* So the Total for 28 *l.* is 92 *Ct* : 3 *qr.* 2°. For 5 *s.* which is $\frac{1}{4}$ of 1 *l.* I take $\frac{1}{4}$ of 3 *Ct* : 1 *qr* : 7 *lb*, it is 3 *qr* : 8 *lb* : 12 *oz.* and for 6 *d*, which is $\frac{1}{10}$ of 5 *s*, I take $\frac{1}{10}$ of 3 *qr* : 8 *lb* : 12 *oz.* it is 9 *lb* : 4$\frac{4}{10}$ *oz.* So the total for 5 *s* : 6 *d.* is 3 *qr* : 18 *lb* : $\frac{4}{10}$ *oz.* And, to this adding 92 *Ct* : 3 *qr*, the Sum is 93 *Ct* : 2 *qr* : 18 *lb* : $\frac{4}{10}$ *oz.*

Observe:

Observe: If the Multiplier and the middle Term are both of the same kind of Things, then we may consider either as the Multiplier, as shall be most convenient for the Operation.

Exa. 4. If 1 *l.* gain 4 *s* : 6 *d.* how much is thereby gain'd upon 34 *l* : 10 *s*? *Answ.* 7 *l* : 15 *s* : 3 *d.* Which is found either of two Ways, *viz.* 1°. Multiplying 4 *s* : 6 *d.* by 34 *l* : 10 *s.* thus; 4 *s.* by 34, makes 6 *l* : 16 *s.* and 6 *d* by 34 makes 17 *s*; then 4 *s* : 6 *d.* by 10 *d.* or ½, makes 2 *s* : 3 *d*; and the Total is 7 *l* : 10 *s* : 3 *d.* Or, 2°, Multiplying 34 *l* : 10 *s.* by 4 *s* : 6 *d*, thus; 4 *s* : 6 *d.* is 2 *s* and 2 *s* : 6 *d*; Therefore I multiply 34 *l* : 10 *s.* by 2 *s.* or $\frac{1}{10}$, the Product is 3 *l* : 09 *s.* Then, by 2 *s* : 6 *d.* or $\frac{1}{8}$, it is 4 *l* : 6 *s* : 3 *d*; And the Total is, as before, 7 *l* : 15 *s* : 3 *d.*

	the Product of				the Product of 34 *l* : 10 *s.* by
4 *s* : 6 *d.*			34 *l* : 10 *s.*		
6 *l* : 16 *s.*	4 *s.* by 34		3 : 09		2 *s.* or $\frac{1}{10}$
17	6 *d.* by 34		3 : 06 : 3		2 *s* : 6 *d.* or $\frac{1}{8}$
2 : 3	4 *s* : 6 *d.* by ½				
7 : 15 : 3			7 : 15 : 3		

These are the *Chief* and *Fundamental* Practices by *Aliquot Parts*, which whoever understands will easily find many particular Abridgments depending upon the same Principles: But what I have done is sufficient here; Judgment and Experience will supply the rest better than a confus'd heap of *Particular Rules.*

CASE 6. When a Question of the *Rule of Three* is duly stated and reduced, according to the General Rule, 'twill often happen that you can easily discover a Number which will exactly divide the Extreme which is the Divisor, and some one of the other *two* Terms; substitute the Quotes of these Divisions in place of the Numbers divided, and work with them instead of these others; by this Means you'll have the Divisor and another Term reduced to smaller Numbers, and sometimes one of them will become 1, which leaves no more Operation but a simple Multiplication or Division.

Exa. 1. If 7 yards cost 56 *l.* what cost 35 yards? The Question stated is, 7 yards — — 56 *l.* — 35 yards; where it's easily perceiv'd that 7 divides both the Extremes, and the Quotes are 1 & 5: So that this Question 1 *yd* — 56 *l* — 5 *yd* will have the same Answer as the former, and is found simply by multiplying 56 *l.* by 5, which makes 280 *l.*

Exa. 2. If 250 *l.* buy 548 yards, what will 5 *l.* buy? The Extremes being both divided by 5, the Quotes are 50 and 1, and the Question will have the same Answer as this, 50 *l* — 548 *yds* — 1 *l.* which is solv'd by dividing 548 *yds* by 50; the Quote is 10 *yds* : 3 *qrs* : 3 *nai* : and $\frac{18}{50}$.

Exa. 3. If 27 yards cost 45 *l*, what cost 63 yards? Here the Extremes 27 & 63 being divided by 9, the Quotes are 3 & 7; and so the Question has the same Answer as this, 3 *yds* — 45 *l* — 7 *yds.* Again; 3 & 45 being both divided by 3, the Quotes are 1 & 15; and so the Question is reduced to this, 1 *yd* — 15 *l* — 7 *yds*, and the Answer 7 times 15, or 105 *l.*

You'll in many Questions discover at sight, or with a small attention, the Divisors that make this Abridgment, and also the Quotes; and in such Cases only is this Practice of any Value.

The *Reason* of this Practice is plain; for the two Numbers, equally divided, contain betwixt 'em the Proportion that ought to be betwixt the other given Number and that sought: But if two Numbers are equally divided, the Quotes (which are like aliquot Parts of them) are in the same Proportion.

From

From the ſame Principle it follows, that if the Diviſor, and any of the other two Terms, are Fractions; or one of them a Fraction, and the other a whole Number; if theſe two Terms are reduced to Fractions having the ſame Denominator, you may neglect the common Denominator, and work with the Numerators; becauſe Fractions, having a common Denominator, are in the ſame Proportion to one another as their Numerator.

Exa. If $\frac{3}{8}$ of a Yard coſt 16 *l.* what coſt $\frac{5}{8}$ of a Yard? This will have the ſame Anſwer as this, 3 *yds* — 16 *l* — 5 *yds.*

Exa. 2. If $4\frac{3}{5}$ Ounces coſt 28 *l.* what coſt 18 *lb.* or 4608 Ounces? The firſt Term reduced is $\frac{23}{5}$, and you may turn either 28 or 4608 into the form of a Fraction whoſe Denominator is 5, and then work with the Numerators; and ſo it will be either 23 *oz.*—140 *l* — 4608 *oz.* or 23 *oz* — 28 *l* — 23040 *oz.* The Anſwer is, $28048\frac{16}{23}$ *l.* In many Caſes this may be uſeful.

CHAP. III.

The Rule of Five.

THIS *Rule* is ſo call'd from having 5 Numbers given to find a 6th; of which 5 Given Numbers, 3 are conjoin'd in the form of a Suppoſition, and upon that a Queſtion is rais'd from the other 2, which with the Number ſought are reſpectively like in their Application to the former 3, and have the ſame Connection of Senſe; by which 'tis eaſy to know, at ſight, a Queſtion belonging to this Rule. Again; All Queſtions of this Rule are ſuch as include two Queſtions of the Rule of *Three*, ſo dependent upon one another, that the Anſwer of the firſt being made the middle Term of the ſecond, the Anſwers of both have the ſame Signification, and the laſt is the final Anſwer of the Queſtion. And *Obſerve*, that tho' many Queſtions include two Queſtions of the Rule of *Three*, yet they have not the Conditions here deſcrib'd (of which you'll meet with Examples afterwards); and, for the Solution of ſuch as have, take this:

RULE.

1°. Of the 3 Terms of Suppoſition ſet that one firſt down which is like the Thing ſought; towards the left hand of it ſet down the other two (it's no matter in what Order); then ſet the two Terms that raiſe the Queſtion towards the right hand of the former 3, in ſuch Order that of the 5 given Terms the 1ſt (counting from the left to the right) be like the 4th, and the 2d like the 5th: Then,

2°. Take the 1ſt, 3d, and 4th Terms in Order, of which make a Queſtion of the Rule of *Three* (aſſuming the 2d Term for a ſuperfluous one, to compleat the Senſe) and find the Anſwer: Then,

3°. Take the 2d (of the 5) the Number now found, and the 5th (of the 5) in Order, and of them make another Queſtion of the Rule of *Three*, (aſſuming the 4th Term of the 5 for a ſuperfluous Number) the Anſwer of this is the final Anſwer ſought.

QUESTIONS

QUESTIONS.

Qu. 1*st.* If the Carriage of 25 Stone weight, for 16 Miles, cost 15 *l*: 10 *s*. what will 40 Stone cost for 9 Miles?
Answ. 13 *l*: 19 *s*.

The State of the 5 Terms is,

St. *Miles* *l.* *s.* *St.* *Miles.*
25 — 16 — 15 : 10 — 40 — 9

The first Question of the *Rule of Three* is,

St. *l.* *s.* *St.*
25 — 15 : 10 — 40

by Reduction,

St. *s.* *St.*
25 — 310 — 40
40 *s.*
25 | 12400 (496.
100.
240
225
150
150.

2d Question.

m. *s.* *m.*
16 — 496 — 9
9
16 | 4464 (279 or 13 *l*:19 *s*
32
126
112
144
144

From the Sense and Connection of these two Questions it's plain, that we have the Answer of the Question propos'd; for, by the first we find what 40 Stone costs for 16 Miles, when 25 Stone cost 10 *s*. for the same Way. Then, by the second, we find what 40 Stone must cost for 9 Miles, when they cost 496 *s*. (the Answer of the preceding) for 16 Miles.

Qu. 2*d.* What Weight must be carried 12 Miles for 5 *l* : 4 *s*. when 18 Stone 10 *lb*. cost 15 *s* for 7 Miles?
Answ. 15 *st* : 5 *lb* : 3 *oz* : $14\frac{1}{2}$ *dr*.

State of the 5 Terms.

s. *m.* *St.* *lb.* *l.* *s.* *m.*
15 — 7 — 18 : 10 — 5 : 4 — 12

1st Operation.

s. *st.* *lb.* *l.* *s.*
15 — 18 : 10 — 5 : 4

by Reduction,

s. *lb.* *s.*
15 — 298 — 104
704 *lb.*
15 | 30992 (2066
20
099
90
92
90
2 remainder
16
15 | 32 (2 *oz*.
30
2 rem.
16
15 | 32 (2 *dr* $\frac{2}{15}$
30
2 remain.

2d Operation.

m. *lb.* *oz* *dr.* *m.*
7 — 2066 : 2 : 2 — 12

by Reduction,

m. *dr.* *m.*
7 — 528930 — 12
7
12 | 3702510
Quote, 308542 $\frac{6}{12}$ *dr*

which being reduced, is equal to

st. *lb.* *oz.* *dr.*
75 : 5 : 3 : $14\frac{6}{12}$

Here, by the 1st Operation, is found how much must be carried 7 Miles for 5 *l* : 4 *s*. Then, by the 2d Operation, how much must be carried 12 Miles for 5 *l* : 4 *s*.

As to the Work of these two Questions, I have done it at length, according to the most General Rules; but such as understand the Contractions, already explain'd in *Multiplication* and *Division*, and the *Rule of Three*, may do them easier thus:

In the 1st Question, the 1st Operation reduces at sight to this 5 *st* — 15 *l* : 10 *s* — 8 *st*. by dividing 25 and 40, both by 5; then we may easily multiply 15 *l* : 10 *s* by 8, without reducing it; the Product is 124 *l*. which divided by 5, quotes 24 *l* : 16 *s*. For the 2d Operation it stands thus, 16 *m* — 24 *l* : 16 *s* — 9 *m*. And here the middle Term is easily multiply'd by 9 without Reduction; the Product is 223 *l* : 4 *s*, which divided by 16, (or first by 2, and the Quote by 8) the Answer is 13 *l* : 19 *s*.

In the 2d Question, the 2d Operation may be done easier; for, without reducing the middle Term, it may be easily multiply'd by 7; the Product is, 14462 *lb* : 14 *oz* : 14 *dr*; which

which divided by 12 (or 1st by 2, and this Quote 7231*lb* : 7*oz* : 7*dr.* by 6, the Quote is 3*oz.* 14*dr.* which is 75*st.* 5*lb.* 3*oz.* 14*dr.* as before.

Observe. That as the first Operation ought to be carried down to the lowest Species, while there is any remainder; so the Fraction which that remainder makes ought to be taken into the second Operation, else the final Answer will be thereby deficient, either in some Integral part, or in the Fractional part of the lowest Species. So in the preceding Question 2d, the $\frac{1}{5}$ of a drop in the Answer of the first Operation, being neglected in the 2d, makes the Answer of this deficient, tho' it is only by a small Fraction of a drop; yet in other Cases the loss may be more considerable, and therefore it ought never to be neglected: But because the taking in this Fraction into the 2d Operation will often make the Work tedious and hard, for them that are not familiar with the practice of Fractions, I shall give you another Rule, whereby the Answer is found by one Division; and because this Rule depends upon the preceding, you must have a clear understanding of that, in order to be master of this; and therefore I would have you first apply the preceding Rule to all the following Questions, and then apply the other Rule which is this.

How to solve Questions of the Rule of Five *by one Division.*

RULE 1°. State the 5 given Terms as before directed, and then make the corresponding Terms (*viz.* 1st and 4th, 2d and 5th) simple Numbers of one Name, and the middle Term a simple Number if its mixt.

2°. Form a Question of the Rule of Three with the 1st, 2d and 4th (of the 5) as before; and mark which of the extremes (1st or 4th) would be the Divisor: Again, form a Question of the Rule of Three with the 2d, 3d and 5th (of the 5) and mark which would be the Divisor.

3°. Let these two Terms, which you find would be the Divisors in these simple Questions of the Rule of Three, be multiplied together, and of the Product make a Divisor; and for a Dividend, take the continual Product of the other three Terms (of the 5) and this Division being finished (in which the Quote is like the middle Term) gives the final Answer of the Question proposed.

Observe. If the Numbers by which the middle Term is multiplied, are such that it may be easily multiplied without reduction when it's a mixt Number, then it's better not to reduce.

Quest. 1. Done by one Operation. The Question being stated and reduced according as the 1st and 2d Articles require, I make a Question of the Rule of Three with the 1st, 3d and 4th Terms, and find that 25 is the Divisor: For if 25*st.* cost 310*s.* (for 16 Miles) 40*st.* will cost more (for the same 16 Miles) therefore 40 is the Multiplier, and 25 the Divisor. Again, I make a Question upon the 2d, pl3d, and 5th, and find 16 the Divisor; for if 16 Miles carriage (of 40*st.*) cost 310*s.* then 9 Miles must cost less; and so 9 is Multiier, and 16 Divisor.

St.	*M.*	*L.*	*Sh.*	*St.*	*M.*
25 —	16 —	15 :	10 —	40 —	9.

Or thus

St.	*M.*	*St.*	*St.*	*M.*
25 —	16 —	310	— 40 —	9

The rest of the Work is manifest, as in the Margin; and because the 1st single Operation had no remainder in the lowest Species, the final Answer is the same by both Rules.

St.	*M.*	*Sh.*	*St.*	*M.*
25 —	16 —	310 —	40 —	9

```
16        40
---      -----
400      12400
             9
         -----
   400)111600)
  Quote 279sh. or 13l. 19s.
```

or without reducing the middle Term; thus,

```
l.   s.
15 : 10
     40
-------
l 620 :
   9
-------
400)5580(13 l 19 s.
```

Observe that I assume the middle Term for the Answer of the 1st single Question, because its no matter what be supposed in order to discover the Divisor.

Question 2d. By one Operation.

```
Sh.   M.   St.  lb.   L.  Sh.    M.
15 — 7 — 18 : 10 — 5 : 4 — 12
             or
S.    M.     lb.        S.      M.
15 — 7. —   298    — 104 —    12
12            7
---         ----
180         2086
             104
180)216944(1205 lb.
    18
    --
     36  or 75 St. : 5 lb.
     36
     ---
      094
       90
      ---
       44 remain.
       16
180)704(3 z.
     54
     ---
     164 remains.
      16
180)2624)14 164/180 dr.
     180
     ---
     824
     720
     ---
     104 remain
```

Here I say, 1st, if 15 *Sh.* pay for (7 Miles carriage of) 298 *lb*, then 104 pays for more; and so 104 is the Multiplier, and 15 the Divisor. Again if 298 *lb.* was carried 7 Miles for 104*l.* a less weight must be taken 12 Miles (for the same Price) so 7 is the Multiplier, and 12 the Divisor: The rest of the Work is plain, as in the Margin. And here the final Answer differs from that found by the preceding Method (in which the Fraction of the 1st part was neglected) only in the Fraction of the lowest Species, which is here a little greater than the other.

The following Questions I leave wholly to your exercise, and only set down the Answers, as they are found by both the preceding Rules, that you may compare them with your own Answers. And observe, that in doing by two Operations I have always neglected the Fraction in the lowest Species of the 1st Operation.

Quest. 3. How far ought 3 *Ct.* and 2 qr. to be carried for 14 *s.* at the rate of 2 *l.* 10 *s.* for 14 *Ct.* carried 18 Miles? *Answer*, 20 Miles $\frac{4}{25}$ of a Mile, by both the Rules.

Quest. 4. If 246 *l.* board 9 Men 18 Months, how long will 48 *l.* board 5 Men? *Answer*, 6 Months 8 Days and $\frac{2}{5}$, by two Operytions; and 6 Month 9 Days $\frac{18}{1230}$ or $\frac{3}{205}$ by one Operation. Observe, I have taken 28 Days to 1 Month.

Quest. 5. How much will pay 8 Months board of 3 Men, when 24 *l.* 5 *s.* paid for 2 Year 4 Months of 7 Men? *Answer*, 2 *l.* 15 *s.* 5 *d.* $\frac{2}{30}$ or $\frac{1}{15}$ by two Operations: And 2 *l.* 15 *s.* 5 *d.* $\frac{3}{21}$ or $\frac{1}{7}$ by one Operation.

Quest. 7. How many Men will cut down 7 Acres of Wheat in 4 Days, when 6 Men cut down 12 Acres in 8 Days and 4 Hours? *Answer.* 4 Men with a Fraction equal to $\frac{9}{70}$ by two Operations; and by one Operation it is 5 Men with a Fraction equal to $\frac{43}{60}$.

Quest. 7. If I get 8 *oz.* weight of Bread for 6 *d.* the Wheat at 15 *s. per* Boll; what ought the Boll of Wheat to be, that I may get 12 *oz.* of Bread for 4 *d.* *Answer.* 6 *s.* 8 *d.* by both Methods.

Quest. 8.

Qu. 8. When Wheat is at 12 *s.* 10 *d. per* Boll, 7 Ounces of Bread cost 5 *d.* how much ought to be got for 8 *d*, the Wheat being 15 *s* ? *Answ.* 9 *oz* : 11 *dw* : 14 *gr.* $\frac{2}{5}$, by two Operations; and 9 *oz* : 11 *dw* : 14 *gr.* $\frac{16}{45}$ by one Operation.

Qu. 9. What ought to be the Price of 4 *lb* : 10 *oz.* of Bread, the Wheat being 16 *s* : 5 *d.* the Boll, supposing that when the Wheat is at 12 *s.* I get 8 *oz.* for 4 *d.*? *Ans.* 3 *s* : 2 $\frac{1}{2}$ *d.* by two Operations; but by one Operation 'tis 3 *s* : 3 *d* : 2$\frac{800}{1152}$ *f.*

Qu. 10. If 100 *l.* Principal Sum give 5 *l* : 10 *s.* in 1 Year, what is the Interest of 72 *l* for 5 Year 8 Months? *Answ.* By two Operations, 9 *l* : 4 *s* : 9 *d* : 1 *f* $\frac{60}{100}$, or $\frac{3}{5}$. And by one Operation, 9 *l* : 4 *s* : 9 *d* : 2 *f* $\frac{2}{5}$.

Observe, I have in this *Question* taken 12 Months to 1 Year.

Qu. 11. At the rate of 6 *l. per Cent. per Ann.* what Principal Sum will raise 48 *l.* in 2 Year 4 Months (supposing 12 Months to a Year)? *Answ.* By both Methods is 342 *l* : 17 *s* : 1 *d* : 2 *f* $\frac{6}{7}$.

Qu. 12. In what Time will 146 *l* : 10 *s.* Principal Sum raise 50 *l.* of Interest at the rate of 5 *per Cent. per Annum*? *Answ.* 6 Year : 10 Months : 16 Days, by two Operations; and by one it is 6 Year : 10 Months : 20 Days $\frac{188}{293}$.

Observe, I have here reckon'd 13 Months to a Year, and 28 Days to a Month; but, in Calculations of Interest, the most exact Way is, to take in no Denominations, but Years and Days (365 Days to 1 Year) and let the Time of a Question which is less than 1 Year, or that Part of it which exceeds a certain number of Years, be reckon'd in Days.

The *Reason* of this *Rule*, by one Division, will be easily understood by one Example. Thus,

Suppose 40 *l.* pay 7 Months board of 6 Men; to find how much 8 Men must pay for 5 Months. The State of the 5 given Terms is, 6 Men — 7 Mo. — 40 *l.* — 8 Men — 5 Mo. The 1st single Question of the *Rule of Three* is, If 6 Men pay 40 *l.* what must 8 Men in the same Time? The Answer is found by multiplying 40 *l.* by 8, and dividing the Product by 6; *that is*, we take the 6th part of 8 times 40, which may be express'd in a general fractional form thus, $\frac{40 \times 8}{6}$ Then the 2d single Question is, If 7 Months cost $\frac{40 \times 8}{6}$, what will 5 Months cost? And the Answer of this is found by multiplying the middle Term $\frac{40 \times 8}{6}$ by 5, and dividing the Product by 7: But a Fraction is multiply'd by multiplying its Numerator, and divided by dividing its Denominator; therefore the Answer is express'd thus, $\frac{40 \times 8 \times 5}{6 \times 7}$; which is the Quote of the continual Product of 40, 8, & 5, divided by the Product of 6 & 7, the Divisors of the two simple Questions: All which is according to the Rule. And, whatever the Question be, 'tis manifest that there will be always the same Reason; for, by expressing the Answer of the 1st simple Question in this general fractional Way, the Answer of the 2d will necessarily be express'd by a Quote made of a Divisor which is the Product of the Divisors of the two simple Questions, and a Dividend which is the Product of the other 3 given Terms.

OBSERVATION *relating to the preceding* RULE.

As I made no Distinction of a *Rule of Three Direct* and *Indirect*, so neither have I in the *Rule of Five*, as is commonly done, to no purpose but to make a needless Difficulty; since *Direct* and *Indirect* can be here understood no otherwise than as they relate to

 the

the two simple Questions of the *Rule of Three* contain'd in it. But I had this further Reason to make no such Distinction: That all the Questions that come under this Rule may be solv'd by two Applications of the *Rule of Three* that are either both *Direct*, or one *Direct* and the other *Indirect*. Thus, the 5 Terms being stated, and the two Questions of the *Rule of Three* consider'd according to the above Rule, if they are both *Direct*, then it may be other Ways solv'd, so as to make one *Direct*, and another *Indirect*: Or, if one is *Direct*, and the other *Indirect*, by the above Rule, it may be solv'd so, as to make both *Direct*; which is done, in both Cases, by making the 1st, 2d, and 4th Terms (of the 5) the 3 Terms of the 1st Operation; then making the Answer of this with the 3d and 5th (of the 5) the 3 Terms of the 2d Operation.

Exa. 1st. If 7 Men in 3 Months spend 100 *l.* how much will 12 Men spend in 5 Months? Here both the Operations, according to the preceding Rule, are *Direct*; But it may be done thus: (1°) If 7 Men take 3 Months (to spend 100 *l.*) how long will 12 Men take (to do the same)? They must take less Time; therefore this is *Indirect*: Then, whatever time 12 Men take to spend 100 *l.* they will spend more or less in 5 Months, according as 5 is more or less than the Answer of the 1st Question; therefore this is *Direct*.

Exa. 2d. If 7 Men spend 100 *l.* in 3 Months, in what Time will 12 Men spend 48 *l.*? By the preceding Rule the 1st Operation is *Indirect*, and the other *Direct*; but, do it the other Way, and they will be both *Direct*. Thus; if 7 Men spend 100 *l.* (in 3 Mo.) 12 Men will spend more (in the same Time); therefore this is *Direct*. Again; If 12 Men spend the Sum found by the 1st Question in 3 Months, they'll take more or less Time to spend 40 *l.* according as this is more or less than the Answer of the 1st Question; therefore this is also *Direct*.

That the final Answer, or Answer of the 2d Operation, will be the same in both these Methods, will appear from the Nature of the Thing; because both Ways there is a reasonable and natural Connection betwixt the two Operations, which take in all the Circumstances of the Question: Therefore I shall not trouble you with any farther Demonstration of it; and shall only add, That I chuse the first Method, because it leads, in a more easy and plain Way, to the Method of reducing both the Operations to one Division.

OBSERVATION *relating to other* Complex Questions.

All *Complex Questions* that are solvable by two Operations of the *Rule of Three*, so that the Answer of the 1st is a Term of the 2d, tho' they have not all Circumstances like those belonging to the *Rule of Five*, yet if we consider and perceive what are the Terms of these two Operations, 'twill be easy to reduce them to one Division, as we have done those of the *Rule of Five*; for, by expressing the Answer of the 1st Question Fraction-wise (as above) and placing that Expression where it should be in the 2d Question, we shall easily perceive which of the given Numbers are to be multiply'd for a Divisor, and which for a Dividend. I shall illustrate this by an Example: Suppose 14 yards of Cloth at the rate of 8 *s.* for 3 yards, are given for Sugar at the rate of 2 *s.* 8 *d.* (or 32 *d.*) for 5 *lb.* how much Sugar ought to be given? To do this at two Steps, I say, If 3 yards cost 8 *s.* what 14 yards? *Answ.* 37 *s.* 4 *d.* Then, if 32 *d.* buy 5 *lb.* what 37 *s.* 4 *d.*? And here, without finding the Answer of the first, I see that 3 and 32 are the Divisors; but then, because the 32 is *d*, make 8 *s.* = 96 *d.* and the 3 Numbers that produce the Dividend are *96*, 14, 5.

CHAP.

CHAP. IV.

Rule of Fellowship.

DEFINITION.

THIS Rule shews how, by two or more Independent Operations of the *Rule of Three*, to divide any given Number into unequal Parts, proportional to certain other given Numbers. 'Tis call'd the *Rule of Fellowship*, because the more common and useful Application is in the Division of Gains, Losses, or other things among Partners in Company: But, as there are also other Applications of it, I have made the Definition *Universal*; and, for the Solution, this is the

RULE. Add the given Numbers (to which these Sought are Proportional) into one Sum, which make the 1st Term; make the Number to be divided the 2d or middle Term; and the given Numbers (or parts of the 1st Term) make them severally the 3d Terms of so many distinct Questions of the *Rule of Three*; and the Numbers thus found are the Answers: The Reason of which is manifest.

To *prove* the Answer to be right, add them all into one Sum; and this ought to be equal to the middle Term, because the Numbers found are the several Parts of the middle Term, and the Parts must be all together equal to the Whole. But *Observe*, that if there are Remainders (tho' of the lowest Species) in this Division by which the Answers are found, these make Fractions, which are also to be added: In order to which, let the Remainders be all of the lowest Species, then add them, and divide their Sum by the common Divisor, (which is the common Denominator; and here there must be no Remainder) the Quote added to the Sum of the Integral parts of the Answer, will make it equal to the middle Term, if the Work is right.

Qu. 1st. Two Men (*A, B*) make a Common Stock, whereof *A*'s Part is 240 *l.* and *B*'s 360 *l.* After a certain time they make 80 *l.* Gain or Loss; What is each of their Shares? *Answ.* *A*'s 32 *l.* and *B*'s 48 *l.*

Stocks		*Operation*		*Gain or Loss.*
		l. *l.* *l.*	*l.* *l.* *l.*	
A	240 *l.*	If 600 — 80 — 240	600 — 80 — 360	
B	360	240	360	*A.* 32 *l.*
	600	600) 19200 (	600) 28800 (	*B.* 48
		Quote, 32 *l.*	48 *l.*	80

Qu. 2d. *A*, *B*, *C* make a Common Stock, whereof *A* has 246 *l.* *B* 392 *l*: 18 *s* *C* 278 *l.* They gain or lose 64 *l.* What is each Partner's Share of it? See the *Answers* following.

Stocks

	Stocks	
A.	246 *l.*	
B.	392 : 18 *s.*	
C.	278 :	
	916 : 18	

For *A.*

l. *s.* *l.* *l.*
916 : 18 — 64 — 246
Or,
s. *l.* *s.*
18338 — 64 — 4920
4920
18338) 314880 (17 *l.*
18338
131500
128366
3134 rem.
20
62680 (3 *s.*
55014
7666. rem.
12
91992 (5 *d*
91690
302 rem
4
1208 *f.*

For *B.*

s. *l.* *s.* *l.* *s.*
18338 — 64 — 7858 (*viz.* 392 : 18)
7858
18338) 502912 (27 *l.*
36676
136152
128366
7786 rem.
20
155720 (8 *s.*
146704
9016 rem.
12
108192 (5 *d.*
91690
16502 rem.
4
66008 (3 *f.*
55014
10992 rem.

For *C.*

s. *l.* *s.*
18338 — 64 — 5560
5560
18338) 355840 (19 *l.*
18338
172460
165042
7418 rem.
20
148360 (8 *s.*
146704
1656 rem.
12
19872
18338
1534 rem.
4
6136 *f.*

Shares of Gain or Loss.

	l. : *s.* : *d.* : *f.*		*Remainders.*
A.	17 : 3 : 5 : 0	—	1208
B.	27 : 8 : 5 : 3	—	10994
C.	19 : 8 : 1 : 0	—	6136
	64 : 0 : 0 : 0	=	18338 (1 *f.*

Observe: Since the 1st and 2d Terms are the same in all the Operations, if there are many Shares, 'twill be convenient to make a Table for the Divisor or first Term, especially if it's a great Number, and perform the Division as directed in § 2*d* of *Ch.* 6, *Book* 1*st.* And, for the same Reason, you may make a Table for the middle Term, as a common Multiplicand, and do the Multiplications as in § 2*d* of *Ch.* 5, *Book* 1; which will be convenient, especially if the middle Term consists of more, and a greater variety of significant Figures than the Extremes by which 'tis to be multiply'd. *Remember* also, that this Method can be follow'd only when you are to do the Work of the *Rule of Three* by the General Rule: But, in some particular Cases, you'll find other Contractions more convenient.

Again; where there are many Shares, 'twill be a useful Method to reduce the first two Terms to two others in the same proportion, but whereof the first is 1; which is done *thus:* The given 1st and 2d Terms being Simple Numbers (or made so) find a 4th Proportional to these two and 1; then say, As the 1st to the 2d, so 1 to a 4th; which will be like the middle Term. And in compleating this 4th, instead of the common Method of Reduction, carry it on decimally to more or fewer places, according as the given Parts of the 1st Term are greater or lesser Numbers: Then take this 4th Term for the 2d or

2d or middle Term, and 1 for the 1st, in all the Operations by which you find the Numbers sought. And so they are all found by *Multiplication*, because 1 is the Divisor. And, in these Multiplications, you may usefully apply the Tabular Method explain'd in *Ch.* 5, *B.* 1, observing always that whatever Denomination the 1 has, the 3d Term hath the same.

Thus, in *Exa. 2d*, say by the *Rule of Three*, as *916 l*: 18 *s*. is to *64 l*. so is 1 *l*. to a 4th Term, which you'll find to be this Decimal of a Pound, *viz*. .0698 &c. And in finding this, I either reduce *916 l*: 18 *s*. to 18338 *s*. and so the Proportion is as 18338 *s*. to *64 l*. So is 20 *s*. (equal to 1 *l*.) to .0698 &c. Or, more easily, by expressing the 18 *s*. Decimally it is, as 916 .9 *l*. to *64 l*. so is 1 *l*. to .0698 &c *l*. Then the 1st and 2d Terms, in all the Operations for the Answers are 1 *l*. and .0698 *l*. And then the 3d Terms being all in the Denomination of Pounds, like the 1st Term (by expressing the 18 *l*. decimally) the Answers are found by multiplying the several 3d Terms by .0698, as below, which produce the same Integral Answers as the preceding Method.

Observe: In the 1st and 2d Parts I have made but two Steps in the Multiplication, by multiplying with 24 and 27 at once.

.0698	.0698	392.9
246	278	.0698
4188	5584	31432
16752	18846	35361
		23574
17.1708	19.4044	27.42442
or	or	or
17 *l*: 3 *s*: 5 *d*.	19 *l*: 8 *s*: 1 *d*.	27 *l*: 8 *s*: 5 *d*: 3 *f*.

Again Observe, that we may also find the new middle Term by expressing the mix'd Numbers, not decimally, but by Reduction, saying, As 18 48 *s*. to *64 l*. so is 1 *s*. to a 4th, which will be .00349 &c *l*. which is different from the other: But, in using this, we must also express the 3d Term in Shillings, and then we shall have the same Answers; but the former Method is easiest.

As the Shares of *Gain* or *Loss* are, in these Questions, found by the total Gain or Loss and the particular Stocks; so, after the same manner, we may find the particular Stocks, from the total Stock and the Shares of Gain or Loss.

The following *Questions*, done after the same manner, shew the Application of this Rule to other Subjects.

Qu. 3d. *A*, *B*, *C* buy together 638 yards of Cloth, of the Value whereof *A* paid 20 *l*. *B* 260 *l*. and *C* 480 *l*. How much of the Cloth must each of them have? Add 200 *l*. 260 *l*. 480 *l*. into one Sum 940, and then divide 638 yards in proportion to the given Parts of 940.

Qu. 4th. There are 3 Horses *A*, *B*, *C*; in the same Time that *A* can eat 5 Bolls of Oats, *B* can eat 7, and *C* 9. How must 25 Bolls be parted among them, that they may begin and end at the same time? Add 5, 7, and 9, the Sum is 21; then divide 25 in proportion to the given Parts of 21.

Qu. 5th. There was a Mixture made of 3 different kinds of Wine, in which for every 3 Gallons of one kind there were 4 of another, and 7 of the third; How much of each kind is in a Mixture of 146 Gallons? Add 3, 4, 7, *&c.*

Qu. 6th. Three Butchers pay among them 40 *l*. for a Grass-Inclosure, into which they put 200 Cows, whereof *A* had 80, *B* 100, and *C* 120; How much ought each to pay? Or, what they pay being given with the total Number of Cattle, we may find how many belongs to each.

Qu. 7th. A Father left his Estate of 1000 *l*. among 3 Sons, in such manner that for every 2 *l*. that *A* gets, *B* shall have 3, and *C* 5; How is the Estate to be divided?

Of

Of Fellowſhip *with* TIME.

When Stocks continue unequal Time in Company, ſo that a Conſideration muſt be made of the Time, as well as of the Stock, this is call'd *Fellowſhip with Time*; for which this is the

RULE. Let all the Stocks be of one Denomination, and alſo the Times; then multiply each Partner's Stock by his Time, and divide the Gain or Loſs in proportion to theſe Products.

Qu. 8th. *A* had in Company 45 *l.* for 3 Months; *B* 58 *l.* for 5 Months; and *C* 92 *l.* for 7 Months; at the end of which they find 48 *l.* gain'd; What is each Partner's Share?

The Products are, for *A* $45 \times 3 = 135$; for *B* $58 \times 5 = 290$; and for *C* $92 \times 7 = 644$; whoſe Sum is $135 + 290 + 644 = 1069$. Then the Proportions are, as 1069 *l.* to 48 *l.* ſo is 135, 290, 644 ſeverally to the proportional Shares of 48 *l.*

The *Reaſon* of this Rule can be no other than an Agreement of Parties, that their Shares of Gain or Loſs ſhall be ſo proportion'd to one another, as thoſe Sums of Intereſt which, at any rate *per Cent. per Annum,* might be gain'd by the particular Stocks, in the time of their continuance in the Common Stock. *Now,* that the Rule is agreeable to this Suppoſition, I thus ſhew: By multiplying the Particular Stocks and Times, we reduce the Queſtion to another State, *viz.* wherein the Particular Stocks are equal to thoſe Products, and in which therefore the Shares of Gain muſt be proportion'd to thoſe Products; and the Times all equal to an Unit of the Denomination of Time multiply'd: So 45 *l.* bearing Intereſt for 3 Months is equivalent to 3 times 45, or 135, for 1 Month, at any Rate of Intereſt: And ſo of the reſt. Conſequently the 48 *l.* gain'd in 7 Months is truly proportion'd to thoſe Products.

Qu. 9th. Suppoſe *A* put in 40 *l.* and at 4 Months end took out 10 *l.* and at 2 Months thereafter put in 30 *l.* *B* put in 50 *l.* and at 3 Months put in 20 *l.* At 8 Months end they balance their Accounts, and find 18 *l.* gain'd; What is the Share of each?

In ſuch Queſtions, where each Partner's Stock varies by Addition and Subtraction, we muſt conſider how long each Part of the varying Stocks continued in Company, and multiplying them by their Times, the Sums of theſe Products are the Numbers by which the Diviſion is to be made; as here.

A had 40 *l.*	then 30 *l.*	then 60 *l.*	*B* had 50 *l.*	then 70 *l.*
for 4 *Mo.*	2 *Mo.*	2 *Mo.*	for 3 *Mo.*	5 *Mo.*
160	60	120	150	350

The Sum of *A*'s ſeveral Products is $160 + 60 + 120 = 340$. Of *B*'s is $150 + 350 = 500$. Then $340 + 50 = 840$. And as 840 to 18 *l.* ſo 340, & 500 ſeverally to the Shares of 18 ſought.

There are other *Queſtions* of a kind with theſe, and wrought the ſame Way; as, the following.

Qu. 10th. Three Perſons, *A*, *B*, *C*, hire together certain Paſture-Ground for 24 *l.* in which *A* keeps 40 Cows for 4 Months: *B* keeps 30 Cows for 2 Months; and *C* keeps 36 Cows for 5 Months: How much of the Rent ought each of them to pay?

Multiply each Perſon's Number of Cows by the Time they were kept, and by theſe Products proportion the Rent.

And

And if the Partners take out and put in Cattle at different Times, then work as in *Quest. 9th.*

To the preceding Questions I shall add the following Collection, in which the Student will find an Useful Exercise.

Qu. 11th. *A*, *B*, and *C* make a Stock, whereof *A* has 20 *l.* *B* 30 *l.* They gain 36 *l.* whereof *C* got 16 *l.* What was *C*'s Stock, and the Gain of *A* and *B* ?

Take 16 from 36, and the Remainder 20 is the Sum of the Gain of *A* and *B*; which being divided in proportion to their Stocks, gives their Shares: Then find *C*'s Stock in such proportion to his Gain, as *A* or *B*'s Stock to his Gain.

Qu. 12th. *A* put into a Common Stock 20 *l.* and B 144 Ducats; they gain'd 60 *l.* of which *A* got 38 *l.* What was the Ducat valued at ?

Take 38 from 60, the Remainder 22 is *B*'s Gain: Then say, As 38 *l.* (*A*'s Gain) to 20 *l.* (his Stock) so is 22 *l.* to a 4th Term, which is *B*'s Stock: Then, if 144 Ducats give that Stock, what's 1 Ducat worth ?

Qu. 13th. *A*, *B*, and C make a Common Stock of 468 *l.* with which they trade, and gain a certain Sum, whereof the Shares of *A* and B together make 64 *l.* of *B* and C 58 *l.* of *A* and *C* 70 *l.* What is the particular Stock and Gain of each Partner ?

Add 64, 58, and 70, the Sum 192 *l.* is double the total Gain, because each Partner's Share is twice contain'd in it; therefore the half of it 96 *l.* is the total Gain: From which take 64 *l.* (*A* and *B*'s Share) the Remainder 32 is *C*'s Share; which taken from 58 *l.* (B and *C*'s Shares) leaves 26 *l.* for *B*'s Share; which taken from 64 *l.* (*A* and *B*'s Share) leaves 38 *l.* for *A*'s Share: Then having the particular Gains, divide the total Stock proportionally.

Qu. 14th. *A* has in Stock 35 *l.* and *B* 20 *l.* They agreed, that the Gain be divided so as *A* have 10 *per Cent.* and *B* only 8; How is 40 *l.* to be divided betwixt them ?

Find what's due to 35 *l.* at the rate of 10 *per Cent.* and to 20 *l.* at the rate of 8 *per Cent:* then divide the total Gain 40 *l.* in proportion to those Sums; for, the only Meaning such a Question can have is, that the Gain be proportion'd to what 35 would draw of 10 *per Cent.* and 20 of 8 *per Cent.* and not, that *A* has really 10 *per Cent.* and *B* 8, for their Stocks; for they will have more or less, according as the total Gain happens to be.

Observe. Mr. *Hill*, without expressing any particular Stocks, supposes 120 *l.* Gain, and *A* to gain 10 *per Cent.* *B* 8; and, to solve the Question, he bids us suppose their Rates of Gain *per Cent.* to be their Stocks, and in that Proportion to divide 120 *l.* but he has neglected to explain something necessarily suppos'd in this Solution, *viz.* That their real Stocks were equal: In which Case, be these Stocks what you will, the Gains are proportional to the several Rates *per Cent.* But, if 10 *l.* and 8 *l.* are their real Stocks, then the Solution is wrong, and we ought to find what's due to 10 *l.* at 10 *per Cent.* and to 8 *l.* at 8 *per Cent.* and by these Sums proportion the Gain.

Qu. 15th. *A* and B were in Company thus: *A* had 50 *l.* in Stock for 10 Months, and *B* had his Stock in for 8 Months, and receiv'd equal Share of the Gain; What was *B*'s Stock ?

Since their Gain was equal, so must the Products of their Stocks and Times; wherefore multiply *A*'s Stock and Time, *viz.* 50 *l.* by 10, the Product is 500; which divide by *B*'s Time 8, the Quote 62 *l* : 10 *s.* is *B*'s Stock. Or, which is the same, make this Proportion; as *B*'s Time 8 Months to *A*'s Time 10 Months, so reciprocally *A*'s Stock 50 *l.* to *B*'s 62 *l* : 10 *s.*

Observe: If we suppose *A*'s Gain is to *B*'s in any other Proportion, as 2 to 3, then, because the Gains are proportional to the Products of Stock and Time, say, As 2 to 3, so is 500 *l.* (the Product of *A*'s Stock and Time) to a 4th, *viz.* 750 *l.* (the Product of *B*'s Stock

and Time); which therefore divided by 8 (*B*'s Time) the Quote is 93*l*: 15*s*. for *B*'s Stock.

Qu. 16*th*. *A* receives of Gain 20*l*. for 8 Months, *B* 25*l*. for 7 Months, and *C* 36*l*. for 5 Months; the ſum of the Products of their Stocks and Times is 520*l*. What were their Stocks?

Divide 520*l*. in 3 parts proportion'd to 20*l*. 25*l*. and 36*l*. then divide each of theſe parts by the reſpective Times, 8 Mo. 7 & 5, the Quotes are the Stocks ſought.

Obſerve: If inſtead of the particular Times the Stocks were given, and the Times requir'd, the Operation is the ſame; for 520 being reſolv'd into 3 parts proportion'd to the Gains, divide theſe parts by the Stocks, and the Quotes are the Times.

Qu. 17*th*. *A* gains 20*l*. and his Stock is 15*l*. more than *B*'s, whoſe Gain is 12*l*. What are the particular Stocks?

Say, As the difference of the Gains is to the difference of the Stocks, ſo is each of the particular Gains to the correſpondent Stocks.

For, ſince the ſum of the Gains is to the ſum of the Stocks as each Gain to its Stock, then, from the nature of Proportion, the difference of Gain is to the difference of Stock as each Gain to its Stock.

Qu. 18*th*. *A* gains 20*l*. in 6 Months, *B* 18*l*. in 5 Months, and C 28*l*. in 9 Months, whoſe Stock is 72*l*. What are the Stocks of *A* and *B*?

Multiply *C*'s Stock and Time, the Product is 648*l*. Then, as 28*l*. (*C*'s Gain) to 648*l*. ſo are 20*l*. and 18*l*. to the Products of *A* and *B*'s Stock and Time; which being found, divide them by their Times, and the Quotes are the Stocks.

If, inſtead of the real Sums of Gains, there were given 3 Numbers in the ſame Proportion as the real Gains, the Work is the ſame. Or ſuppoſe, inſtead of the Particular Gains, that *A* has $\frac{1}{3}$ of the whole Gain, and *B* $\frac{2}{5}$, then we muſt add theſe Fractions, and take the Sum from 1, the Remainder is the Fraction of the total Gain which C has; and then uſe theſe Fractions as the Particular Gains.

Again; If their particular Gains and Stocks are given, with the Time of one Partner, to find the Times of the reſt, the Work is alſo the ſame.

Qu. 19*th*. *A*, *B*, *C* have a Common Stock of 1000*l*. *A* gains 100*l*. for 9 Months, *B* 80*l*. for 12 Months, and *C* 120*l*. for 8 Months; What were the Particular Stocks?

Divide each Partner's Gain by his Time, and then divide 1000*l*. into 3 parts proportion'd to thoſe Quotes. The *Reaſon* of this is, that if the Times are equal, the Stocks are in proportion to the Gains; and if the Gains are equal, the Stocks muſt be reciprocally as the Times; and conſequently neither being Equal, the Stocks are as the Gains directly, and as the Times reciprocally; *that is*, as the Quotes of the Gains divided by the Times. Or, it may be ſhewn this Way: Let g, s, t repreſent the *Gain*, *Stock*, and *Time* of one Partner, and G, S, T thoſe of another; then, becauſe the Gains are in proportion to the Products of Stock and Time (as already demonſtrated) and theſe Products being repreſented by st. ST, it is $g : st :: G : ST$; but by equally dividing the relative Terms, *viz.* g and st by t, and G and ST by T, the Quotes are ſtill proportional; that is, $\frac{g}{t} : s :: \frac{G}{T} : S$.

Obſerve: If inſtead of the total Stock and particular Times (as above) were given the particular Stocks and total Time to find the particular Times, the Solution is after the ſame Way, and for the ſame Reaſon, *viz.* dividing the particular Gains by their Stocks, and proportioning the Times to thoſe Quotes.

Qu. 20*th*. *A* hath 200*l*. more Stock than *B*, but *A* continued his only 5 Months, and *B* 9, and drew equal Gains; What are the Stocks?

Say, As the Difference of Times to the Difference of Stocks, ſo is *A*'s Time to *B*'s Stock, and *B*'s Time to *A*'s Stock; or, having one Stock, by that and the Difference find the

the other. The *Reason* of this is, that when the Gains are equal, the Stocks are reciprocally as the Times; and therefore, from the Nature of Proportion, the Difference of the Times is to the Difference of the Stocks, reciprocally as the particular Times to the Stocks; *i. e.* as *A*'s Time to *B*'s Stock, or as *B*'s Time to *A*'s Stock.

Qu. 21*st*. *A*, *B*, and *C* have 100*l.* to be divided among them, in such manner that 2 times *A*'s Share be equal to 3 times *B*'s, and 4 times *B*'s be equal to 5 times *C*'s; What are their Shares?

'Tis plain by the Conditions, that as oft as *A* gets 3, *B* must have 2; also as oft as *B* gets 5, so oft must *C* get 4: Then I say, As 5 to 4, so is 2 to $1\frac{3}{5}$, so that as oft as *B* gets 2, so oft *C* gets $1\frac{3}{5}$; but so oft also *A* gets 3; therefore the Proportions of the Shares sought are $3 . 2 . 1\frac{3}{5}$, or 15 . 10 . 8, according to which 100*l.* is to be divided.

Suppose the Conditions thus; $\frac{1}{2}$ of *A*'s Share is equal to $\frac{2}{3}$ of *B*'s, and $\frac{2}{7}$ of *B*'s equal to $\frac{5}{7}$ of *C*'s; we may find the Proportions of their Shares the same Way as before.

Qu. 22*d*. A Father, ignorant of Arithmetick, orders his Estate of 500*l.* to be divided among three Sons, so as the eldest get $\frac{1}{2}$, the second $\frac{3}{5}$, and the third $\frac{4}{7}$; What is each Son's Part?

Here 'tis impossible to give them these Shares, because $\frac{1}{2}$, $\frac{3}{5}$, and $\frac{4}{7}$ exceed the whole; and therefore the Meaning of the Question must be understood to be, the dividing 500*l.* into 3 Parts that bear such Proportion to one another as these Fractions: And the like is to be understood of all Divisions propos'd in this manner, whether the Parts propos'd exceed, or come short of, the Thing to be divided.

Qu. 23*d*. 'Tis propos'd to divide 300*l.* among 3 Persons, so that *A* get 6*l.* more than $\frac{1}{2}$, *B* 12*l.* more than $\frac{1}{3}$, *C* 8*l.* less than $\frac{2}{3}$; What gets each?

According to the most obvious sense of this Question, the Meaning of it is, that the Shares be in proportion to the Sum of 6*l.* and $\frac{1}{2}$ of 300*l.* for *A*; 12*l.* and $\frac{1}{3}$ of 300*l.* for *B*; and $\frac{2}{3}$ of 300*l.* wanting 8*l.* for *C*: But *Jeake* (from whom I take it) understands it in another Sense, which indeed I think no Body could ever find in it, as 'tis propos'd; *viz.* that the Shares be such, as if 6*l.* be taken from *A*'s, 12*l.* from *B*'s, and 8*l.* added to *C*'s, the Remainders in the former, and the Sums in this, be to one another as $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{2}{3}$, and so the Solution is made thus: Take 6 and 12 from 300, and to the Remainder add 8, then divide this Sum in 3 Parts proportional to $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{2}{3}$, and to these Shares add and subtract the Sums propos'd. The *Reason* of the Work is plain, according to the Sense he puts upon it.

Qu. 24*th*. Three Persons, *A*, *B*, and *C*, buy a Ship, of the Price whereof *A* paid $\frac{5}{8}$, *B* $\frac{3}{7}$, and *C* 140*l.* How much Money paid *A* and *B*? and, What Part of the Ship had *C*?

Add the Fractions $\frac{5}{8}$ and $\frac{3}{7}$, and take the Sum from 1, the remainder is the Part of the Ship belonging to *C*; then say, If *C*'s Part cost 140*l.* what cost the Sum of *A* and *B*'s Parts? And having found that, divide it into 2 Parts, proportion'd to one another as $\frac{5}{8}$ to $\frac{3}{7}$.

Qu. 25*th*. There were at a Feast 20 Men, 30 Women, and 15 Servants; for every 10*s.* that a Man paid, a Woman paid 6, and a Servant 2; How much did every Man, Woman, and Servant pay of 24*l*?

Multiply 20 by 10, 30 by 6, and 15 by 2; then divide 24*l.* in 3 parts proportion'd to these Products (*viz.* 200, 180, and 30) and you have the Total paid by the 20 Men, 30 Women, and 15 Servants: Each of which Sums being divided by their respective numbers of Persons, gives the Payment made by each Individual.

Suppose the Conditions such, that a Man pays 3 times as much as a Woman, and 2*s.* more; that a Woman pays double of a Servant, and 1*s.* more; To find their Shares, multiply 2 by 20, and 1 by 30, the Products 40 and 30 equal to 70*s.* take from 24*l*, the

Remainder is $20l:10s$. Then, becauſe a Man pays triple of a Woman, ſuppoſe a Man pays 3, a Woman pays 1; and becauſe a Woman pays double of a Servant, if a Woman pays 1, a Servant pays $\frac{1}{2}$; ſo their Proportions are $3:1:\frac{1}{2}$, or, in whole Numbers, 6 . 2 . 1; *that is*, 6 for a Man, 2 for a Woman, and 1 for a Servant. Multiply theſe by their reſpective Numbers of Perſons, the Products are 120 for 20 Men, 60 for 30 Women, and 15 for 15 Servants: Then divide $20l:10s$. in 3 parts, in proportion to one another as are 120, 60, and 15, and divide theſe parts by their reſpective numbers of Men, Women, and Servants, the Quotes are what each Man, Woman, and Servant pays of the $20l:10s$. Laſtly, to a Man's Share of this add 2 *s*. to a Woman's 1, and you have their compleat Payments of the whole 24 *l*.

Obſerve: If, inſtead of adding, it had been propos'd to ſubtract, as if a Woman pays 1 *s*. leſs than the double of a Servant, then add 30 *s*. to 24 *l*. (ſubtracting what a Man pays more than triple of a Woman); and, in the laſt Part, inſtead of adding, ſubtract 1 from the Woman's Part of the Sum divided.

Qu. 26th. A Father dying, left his Wife with Child, to whom he bequeath'd, if ſhe had a Son, $\frac{1}{2}$ of his Eſtate, and $\frac{2}{3}$ to the Son: But, if ſhe had a Daughter, $\frac{1}{3}$ to her, and $\frac{2}{3}$ to her Mother. It happen'd that ſhe had both a Son and a Daughter; How ſhall the Eſtate be divided to anſwer the Father's Intention?

As the Father plainly deſign'd the Son to have double of the Mother's Part, and the Mother double of the Daughter's Part, therefore for every 1 the Daughter got the Mother muſt have 2, and the Son 4; and in proportion to theſe Numbers 1 . 2 . 4 muſt the Eſtate be divided.

[This is a Queſtion propos'd by a *Roman* Lawyer, in the 28th Book of the *Digeſts*, which he thinks is juſtly ſolv'd after this manner.] Again,

Suppoſe that the Mother had a Son and Daughter who liv'd, but her ſelf dy'd in the Birth; How is the Eſtate to be divided betwixt the Son and Daughter? *Nicholas Tarteglia* makes this Suppoſition in his *Arithmetick*, and ſolves the Queſtion thus; ſays he, "Had the Mother liv'd, the Proportions are 1 . 2 . 4, as above, therefore the Eſtate "muſt be divided in proportion of 1 for the Daughter to 4 for the Son. But I doubt the Juſtice of this Solution; for tho' this Proportion betwixt the Son and Daughter's Parts, in caſe of the Mother's Life, is a Conſequence of the Father's plain Intentions, with reſpect to the Mother and Son or Daughter, yet never having a Son and Daughter both together in his view, this Solution ſeems to have no Foundation. And I rather think the Solution ought to be thus: Find the Parts belonging to Mother, Son, and Daughter, then divide the Mother's Part betwixt the Children, according to the Rule of Heirſhip in the Country where the Queſtion ariſes.

Under this Head of *Fellowſhip* are alſo comprehended the Calculation of Gains or Stocks betwixt a Merchant and Factor.

Queſtions *of* FACTORSHIP.

Queſtion 1. A Merchant delivers to his Factor 100 *l*. allowing him to join to it 30 *l*. and values his Service worth 40 *l*. what ſhare of the gain ought the Factor to have?

There are two ways of ſolving this Queſtion: The generality of Authors do it thus, Add 30 *l*. to 40 *l*. the Sum is 70 *l*. then divide the Gain in two Parts, in proportion as 100 *l*. to 70 *l*.

Another Method is this; Subtract 40 from 100 which leaves 60, and proportion the Shares of Gain to 60 (for the Merchant) and 70 (*i. e.* 40 and 30) for the Factor.

If the Merchant and Factor determine the meaning of their Agreement to either of these ways of stating the Proportion, there is no more Question; but without this the last Method seems the more reasonable, because the Gain is made upon the real Stock 130 *l.* and not upon the imaginary one 170 *l.* And the more obvious sense of valuing the Factors Service at 40 *l.* seems to be, the allowing him the Gain of 40 *l.* of the real Stock more than what he actually puts in, which must consequently be deducted from the Merchants Stock, and added to his.

Quest. 2. A Merchant's real Stock being 100 *l.* and the Factors 30 *l.* who received $\frac{1}{3}$ of the Gain: What was his Service valued at?

To proceed upon an imaginary Stock, say as $\frac{2}{3}$ to $\frac{1}{3}$, or as 2 to 1, so is 100 to 50, from which take 30, the remainder 20 is the Answer.

But upon the real Stock, find the 3d part of 130, from which take 30, the remainder is the Answer.

Quest. 3. A Merchant's real Stock being 100 *l.* and the Factors Service valued at 20 *l.* who received $\frac{1}{2}$ of the Gain; What was the Factor's real Stock?

To proceed upon an imaginary Stock, it is 80 *l.* because 20 and 80 makes 100 equal to the Merchant's. By the other Method it is only 60, because 20 and 60 make 80, the half of 160, the total real Stock.

Quest. 4. The Merchant's real Stock being 100 *l.* and the Factor being allowed $\frac{1}{4}$ of the Gain for his Service, what real Stock must he join to have $\frac{1}{3}$ of the Gain?

When the Factor gets $\frac{1}{4}$ (without any real Stock) his Service is there valued at 25 *l.* the 4th of the real Stock 100 *l.* or 33 *l.* 6 *s.* 8 *d.* the $\frac{1}{4}$ of the imaginary Stock 133 *l.* 6 *s.* 8 *d.* found by adding the $\frac{1}{3}$ of 100 *l.* to 100 *l.* Then with this Value of his Service, proceed to find the real Stock that he must have to get $\frac{1}{3}$ Gains by the Methods of Question 3d.

Observe, that in all the preceding Questions we may suppose 2 or more Merchants with the Factor; it will be easy to apply the same Rules, by adding the Stocks of all the Merchants into one Sum, and considering that as one Stock; and then, besides whats already demanded, it may also be demanded to find the Gain of each Merchant; thus, by the first Method of an imaginary Stock, what remains to the Merchants after the Factor's part is deducted, must be divided in Proportion to their real Stocks.

By the real Stocks we must divide the Factor's estimation into parts proportioned to the Merchants real Stocks, and take the parts answering to each from itself, the remainders are the Numbers by which the Merchants shares are to be proportioned.

Quest. 5. A Merchants real Stock being 120 *l.* and the Factors 60, they agreed, that at a Years end the Factor should have $\frac{1}{2}$ of both the Stocks and Gain, but they broke up at 8 Months end, having gained 150 *l.* How much ought the Factor to have?

Here 'tis plain the Factor for 12 Months Service was to have not only the Gain of 30 *l.* of the Merchants Stock, but also 30 of the Stock itself; so that his Service was valued at 30 *l.* real Stock; but the Society lasting only 8 Months, 'tis plain he ought only to have 20 *l.* Pound (which is to 30 *l.* as 8 Months to 12) and this added to his own 60, makes 80 *l.* which he receives of the real Stock; and the Merchants part being 100 *l.* then it is as plain the 150 *l.* Gain must be divided into 2 parts, proportion'd to these Stocks 80 and 100.

Observe, *Buteo*, from whom I take this Question, finds fault with this Method of solving it (which he says both *Lucas* and *Stephanus* have followed) and does it himself thus.

He says, that since upon supposition of the Society continuing 12 Months, the Factor was to have the half, therefore his Service was valued at 60, to make his real Stock of 60 equal to the Merchants 120; so that the Society continuing only 8 Months, his estimation is only 40, which added to his real Stock 60, makes 100; and the Merchants being 120: Therefore the Sum of the real Stocks (*viz.* 120 and 60) and the Gain (150, which makes 330) ought to be divided into 2 Parts, proportional to 100 for the Factor, and 120 for the Merchant.

Now

Now all the fault that *Buteo* finds with the other Method, is, that he sees no Reason why the Merchants Stock of 120 should be diminished; but to me the Reason is obvious, because, tho' he puts in 120, yet part of it belongs to the Factor for his Service; and if the Society had continued 12 Months according to the Conditions, 30 *l.* of the Merchants 120 would have been given to the Factor; and for the same Reason the Society continuing 8 Months, 20 of the Merchants must be given to the Factor, and Gain proportionally.

But, in his way of solving the Question, he estimates the Factor's service otherwise than the plain Conditions of the Question; and so brings in an imaginary Stock, and by Consequence a false Proportion.

CHAP. V.

QUESTIONS *concerning* Loss *and* Gain.

QUEST. 1. A Parcel of Goods being bought for 60 *l.* and sold for 75 *l.* what was the rate of Gain *per Cent?* I say if 60 *l.* gain 15 *l.* What will 100 *l.* Gain. the answer is 5 *l.*

Quest. 2. Having bought 18 Gallons of Brandy for 12 *l.* how may I sell 1 Gallon, and Gain at the rate of 8 *per Cent?* I find what 1 Gallon cost, it is 13 *s.* 4 *d.* Then I say if 100 give 108, what will 13 *s.* 4 *d.* give? it is 14 *s.* 4 *d.* 3 *f.* $\frac{21}{21}$.

Quest. 3. Having sold 11 Yards of Cloth for 4 *l.* 16 *s.* and thereby Gain'd at the rate of 10 *per Cent.* What was the prime Cost of 1 Yard? First I find 1 Yard is sold for 9 *s.* 6 *d.* then if 110 comes of 100 (prime cost) To what prime cost, at that rate, does 9 *s.* 6 *d.* belong? *Answer.* 8 *s.* 7 *d.* 2 $\frac{6}{11}$ *f.*

Q. 4. Having sold 2 Yards of Cloth for 11 *s.* 6 *d.* I gained at the rate of 15 *per Cent;* but had I sold it for 12 *s.* what is the rate of Gain *per Cent?* I say as 11 *s.* 6 *d.* is in proportion to 115 *l.* so is 12 *s.* to a 4th Term, which I find to be 120 *l.* and so 20 *l.* is the Answer of the Question.

Observe. This Question is in Substance, and Numbers, the same with Mr. *Hill*'s 8th Question of Loss and Gain; but neither his Operation nor Answer are the same. He states it thus; As 11 *s,* 6 *d.* is to 15 *l.* so is 12 *s.* to 15 *l.* 13 *s.* with a very small Fraction. But this state of the Proportion is quite wrong; because 11 *s.* 6 *d.* and 15 *l.* are not similar Terms; the 1st being a Sum of prime Cost and Gain put together, and the other only an Article of Gain; which shews that the 2d Term ought to be 115 *l.* which is the sum of the prime Cost 100 *l.* and is the Gain made upon it at the same rate with the Gain included in 11 *s.* 6 *d.* Whoever understands the nature of Proportion, will find no difficulty to perceive the Reason of this; yet for the sake of others, I'll shew, by another way of Solving the Question, that the first is the true Answer. Thus, from the 1st Sale and rate of Gain find the prime Cost of 1 Yard, it is 10 *s.* (for as 15 to 100, so is 11 *s.* 6 *d.* to 10 *s.*) then the 2d Sale being 12 *s.* the Gain is 2 *s.* therefore say, If 10 Gain 2, what will 100 Gain? It is 20.

In the following Questions I consider the Forbearance of *Money;* or *Time* allowed for Payment.

Quest 5.

Qu. 5. Having bought a parcel of Goods for 18 *l.* and sold the same immediately for 25 *l.* with 4 Months Credit, What is gain'd *per Cent. per Annum?* Say by the Rule of *Five*, If 18 *l.* in 4 Months gain 7 *l.* what will 100 *l.* gain in 1 Year?

Observe 1°. If the Gain *per Cent. per Ann.* is given (suppose 12 *per Cent.*) to find the Time that ought to be allow'd, then say, If 100 *l.* gain 12 *l.* in 1 Year, in what Time must 18 *l.* gain 7 *l*?

(2°) Or, if the Rate of Gain is given, with the prime Cost and Time, to find the selling Price, say, If 100 *l.* in 1 Year gain 12 *l.* what must 18 *l.* gain in 4 Months? Which is to be added to the prime Cost.

(3°) If the Rate of Gain, Time, and selling Price are given to find the prime Cost, as, suppose 4 Months allow'd for payment of 25 *l.* by what was gain'd at the rate of 12 *l.* *per Cent. per Annum*, to solve this, you must first find 4 Months Interest of 100 *l.* at the propos'd Rate, which add to 100, then say, As that Sum is to 100, so is 25 *l.* to a 4th Term; which is the Sum sought.

Qu. 6. Having bought 40 Gallons of Brandy at 3 *s. per* Gallon, by an Accident there was lost of it 6 Gallons, at what rate *per* Gallon may I sell the rest, with 8 Months Credit, and gain upon the whole prime Cost at the rate of 10 *per Cent. per Annum?* Find the Value of 40 Gallons at 3 *s.* 'tis 6 *l*; then say, If 100 *l.* in 1 Year gain 10 *l.* what will 6 *l.* gain in 8 Months? Add that Gain to 6 *l*, the Sum is the Value at which the whole remaining 34 Gallons are to be sold with 8 Months Credit: From which find the Price of 1 Gallon.

Qu. 7. Having paid 14 *s.* for each of 100 yards of Cloth, I propose to gain 25 *per Cent.* ready Money; and if I sell it upon Time, to have moreover 10 *per Cent. per Annum* for the Forbearance: What must be the Price of 1 yard with 6 Months Credit, to make both these Gains? First find the ready Money Price of the whole at 25 *per Cent.* Gain; then find what Gain or Interest that will give in 6 Months at the rate of 10 *per Cent. per Ann.* which being added to it, the Sum is the Price of the whole forborn 6 Months; by which find the Price of 1 yard.

CHAP. VI.

Questions of Bartering.

Qu. 1*st.* A Gives 4 Hogsheads of Wine, at 9 *l. per* Hogshead, to *B*, for Raisins at 7 *d* *per* Pound; How much weight of Raisins ought *A* to receive?

Say, If 1 Hogshead give 9 *l.* what will 4? The Answer is, 36 *l.* Then, if 7 *d.* buy 1 *lb.* what will 36 *l.* buy? *Answ.* $102\frac{6}{7}$ *lb.*

Observe this Fraction $\frac{6}{7}$, if it cannot be given in real Quantity by reducing to lower species, then *A* must give 102 *lb.* Raisins and 6 *d.* in Money, because $\frac{6}{7}$ of 1 *lb* weight is 6 *d.* And, in all Cases, when any Fraction of a Quantity cannot be given, its Value in Money must be given.

Qu. 2*d.* If I get 120 Gallons of Brandy, at 4 *s.* the Gallon, for 28 Bolls of Wheat, how is 1 Boll sold?

Say, If 1 Gallon cost 4 *s.* what 128 Gallons? The *Answer* is, 480 *s.* Then, if 280 Bolls cost 480 *s.* what 1 Boll?

Qu. 3*d.* *A* and *B* barter thus: *A* gives 120 yards of Cloth, such that 3 yards $\frac{1}{4}$ cost 15 *s* : 9 *d.* *B* gives part Stockings at 7 *s.* the pair, part Hats at 0 *s.* 6 *d.* a piece, and gives an equal number of Hats and pairs of Stockings; How many were of each?

Find

Find the total Value of *A*'s 120 yards of Cloth, then add the Value of 1 Hat and 1 pair of Stockings; and by the Sum divide the Value of the Cloth, the Quote is the Answer: And if there is a Remainder, it shews that the Value of the 120 yards cannot be exactly given, according to the propos'd Conditions; and therefore, besides the Number express'd by the Quote, *B* must give *A* so much Money as that Remainder express'd in the denomination of the Dividend; *i. e.* so many *l.* or *s.* or *d.* as the Dividend is, because that is the Value of such a Fraction of one Hat and one Pair taken together.

Observe; If the proportion of the Number of Hats and Pairs of Stockings is suppos'd to be any other than that of Equality; *Exa.* as, 2 Hats for 5 pair of Stockings, then we must add the Value of 2 Hats to the Value of 5 pair of Stackings, and make that Sum the Divisor: Then the Quote being multiply'd by 2, gives the number of Hats, and by 5, gives the number of pairs of Stockings; the Remainder is to be taken the same way as before.

Qu. 4th. *A* and *B* barter thus: *A* has 27 yards Silk-stuff worth 2 *s.* ready Money, but in Barter he will have 2 *s.* 3 *d.* *B* has Hats worth 7 *s.* a piece ready Money; How many of them must he give to *A* for his 27 yards of Stuff? and, What is the Price of a Hat in Barter to equal the advancement of *A*'s Price?

Find the Number of Hats by the ready Money Prices, and for the rais'd Price of the Hats say, As 2 *s.* to 2 *s* : 3 *d.* so is 7 *s.* to the Answer.

Observe: When the Prices of each Party are rais'd proportionally, then it's manifestly indifferent to find the Quantity sought, either by the ready Money Prices or the advanc'd Prices; and therefore the finding *B*'s advanc'd Price in order to find the Quantity, is no ways necessary; as most of our Authors seem to think, by their taking this Method; which has led them into Mistakes, as you'll find below.

Qu. 5th. *A* hath 100 yards Camblet at 16 *d. per* yard ready Money, which he puts away in Barter at 18 *d.* to *B*, taking of him Stockings at 5 *s.* the pair, which are worth but 4 *s* : 6 *d.* ready Money: How many pairs must he give? and, Which of them gains, also how much, by the Bargain?

If the Ready-money and Barter Prices were proportional, neither Party could gain; but the asking *who gains* supposes it's otherwise, or at least uncertain: And therefore we must find the number of Stockings by the advanc'd Prices at which the Barter was actually made. Then find the total Value of the 100 yards at 16 *d.* and of the number of Pairs (found) at 4 *s* : 6 *d*, the Comparison of these Values shews who gains, and how much. And *observe* also, that if there is a Remainder in finding the number of Pairs, as that is so many Pence to be given by *B* to *A*, so, in comparing the real Values, it must be added to the Value of *B*'s Stockings. But if there is no Remainder, you may find the Gains thus; *A* gets 2 *d.* a yard advance, which is 200 *d.* upon the whole: *B* gets 6 *d.* a pair advance, which multiply by his number of Pairs, and compare that Product with 200 *d.* Or you may take another Method, *thus*; say, If 16 *d.* is advanc'd to 18 *d.* what ought 4 *s.* 6 *d.* to be advanc'd to? *Answ.* 5 *s* : 3 *f.* whereby it's plain B is the Loser, who puts his Stockings away at 5 *s.* whereas he ought to have 5 *s* : 3 *f.* so he loses 3 *f.* on every Pair; which multiply'd by the number of Pairs, the Product is what he loses on the whole.

Qu. 6th. Two Merchants have various kinds of Goods to barter; *A* has 735 yards *Indian Silk* at 8 *s* : 6 *d per* yard ready Money, and in Barter 10 *s.* also 532 Canes at 3 *s.* a piece ready Money, and in Barter 3 *s* : 4 *d.* also 16 Pieces of Muslin at 4 *l.* the Piece ready Money, and in Barter 4 *l* : 10 *s.* *B* has scarlet Cloth at 1 *l. per* yard ready Money, Glass Manufacture at 1 *s* : 8 *d. per* Pound-weight ready Money, and a finer kind at 2 *s* : 4 *d.*

How

How many yards of Cloth, and pounds of each kind of Glass, of each a like number, (*i. e.* of yards, and pounds of each Glass) must *B* give to *A*, advancing his Goods proportionally also in Barter?

This Question I take from Mr. *Hatton*, who solves it in this manner, *viz.* He finds *A*'s whole Goods to be worth, at the Barter Price, 528 *l* : 3 *s* : 4 *d.* then he takes the sum of the Prices of 1 of each of *A*'s Things (*viz.* 1 yard Silk, 1 Cane, and 1 Piece of Muslin) at the Ready-Money Price, which make 4 *l* : 11 *s* : 6 *d*; and also at the Barter Price, which make 5 *l* : 3 *s* : 4 *d.* the Difference of those Sums is 11 *s* : 10 *d.* He takes also the Value of an Unit of each of *B*'s Things, which make 1 *l* : 4 *s.* Then he says, If 4 *l* : 11 *s* : 6 *d.* advance 11 *s* : 10 *d*, what ought 1 *l* : 4 *s.* to advance? The Answer is, 3 *s* : 1 *d.* and $\frac{270}{1098}$, which makes the Sum of the Barter Price of a Unit of each of the Things to be 1 *l* : 7 *s* : 1 $\frac{270}{1098}$; And, lastly, by this dividing *A*'s total Value at the Barter Price, the Quote is $389\frac{26280}{35712}$; and so many Units of each of his Things ought *B* to give.

But this Solution is false; which probably I had taken little Notice of, had not the Author made it very remarkable, by complaining how little such Questions are understood; and, that as he seems to think this a Curious and Useful one, so he tells us plainly he expects the Thanks and Applause of the Publick for it; and, that we may see he deserves it, makes what he calls a *Demonstration* of the Truth of his Solution: But I shall easily shew the Error of it; in order to which, I shall first propose

Another Solution. Find the total Value of *A*'s Things at the Ready-money Price, it is $9123\frac{1}{2}$. *s.* which divide by 24 *s.* the Sum of 1 of each of *B*'s Things at the Ready-money Price, the Quote is 380.1458 &c, and so many of each of his Things ought *B* to give to *A*, which is considerably less than the other Answer.

Now, since it be an undoubted Truth, that if the Price of each Party's Goods are advanc'd proportionally, the same quantity must come out, whether we calculate by the Ready-money or the Barter Prices; and the Ready-money Prices being given, there can be no Error in this Method: Therefore the last Answer must be the true one.

This is sufficient to shew, that Mr. *Hatton*'s Answer is wrong: And it shews us also, that the Error must lie in his Way of calculating the Advancement upon *B*'s Things (for, if that were right, his Solution and mine would bring out the same Quantity). But I shall more particularly explain the Error of his Way of finding *B*'s Advancement, and then shew the True Way.

In the first place, it can't be deny'd that since the Barter ought to be equal, therefore the total Value of all that *A* and *B* deliver ought to be equal, both in the Ready-money and the Barter Prices, and consequently the Advance on the wholes (or what the Value of the Barter Prices exceeds that of the Ready-money Prices) ought also to be equal.

Again; If the number of *A*'s Things were equal, then every 4 *l* : 11 *s* : 6 *d.* (the Value of one of each) which is contain'd in the total Value, would advance 11 *s* : 10 *d.* and then of consequence every 1 *l* : 4 *s.* (the Value of one of each of *B*'s Things; that are equal in Number) would advance in the same proportion: But *A*'s Things not being equal in Number, every 4 *l* : 11 *s* : 6 *d.* does not advance 11 *s* : 10 *d*; for, by taking 1 of each, we can only make 16 setts in each of which there are 3 Things whose total Value is 4 *l* : 11 *s* : 6 *d.* because there are but 16 Pieces of Muslin; then there remains 516 Canes, and 719 yards Silk, of which we can make 516 setts of 2 Things, each of which setts advance 1 *s* : 10 *d.* (the Difference of the Ready-money and Barter Prices of 1 yard Silk and 1 Cane); and, lastly, there remain 203 yards Silk, each of which advances 1 *s* : 6 *d.* Therefore the Advance of 1 of each of B's Things can't be truly found by any of these Advances.

But this will be farther clear by another Unquestionable Method of finding the Advance on a sett of 1 of each of *B*'s Things, which is this; Since the total Value of *A* and *B*'s Things must be equal, both at the Ready-money and Barter Prices, and consequently the total Advance equal, therefore I find the total Value of *A*'s Things at the Ready-money Price, which is $9123\frac{1}{2}$ *s.* and also at the Barter Price, which is $10563\frac{1}{3}$ *s.* the Difference is $1439\frac{5}{6}$ *s.* the total Advance. Again; Since *B*'s Things are equal in Number, therefore 'tis plain that the Value of every sett of 1 of each makes the same Advance, and which must be a proportional part of the total Advance to the total Value, because 1 of each is an Aliquot of the whole; and therefore, I say, if the total Value at the Ready-money Price $9123\frac{1}{2}$, Advance $1439\frac{5}{6}$, what will the Value of 1 of each of *B*'s Things (*viz* 24 *s.*) advance? And I find 3 *s* : 9 *d.* $\frac{24687}{54741}$, which makes the advanc'd Value of a sett of 1 of each of *B*'s Things to be 27 *s* : 9 *d* $\frac{24687}{54741}$ instead of 27 *s* : $1\frac{270}{1098}$ *d.* as Mr *Hatton* makes it. And, to conclude, if we seek *B*'s Quantity by the Barter Price now found for 1 of each, 'tis the same as already found by the Ready-money Prices; for the total Value at the Barter Price (*viz.* $10563\frac{1}{3}$) being divided by 27 *s* : 9 *d* $\frac{24687}{54741}$, the Quote is 383.1458 &c. as before. And thus the two Methods, by the Ready-money and Barter Prices, confirm one another, when the Barter Price is truly found; and finish the Demonstration of Mr. *Hatton*'s Error.

Observe again, That if it be demanded what is the Advance upon a single one of each of *B*'s Things, this has no determinate *Answer*; because it may be any Thing we please, so that the Sum of the Advances upon a single one of each kind be equal to the Advance found for one of each taken together.

Observe also, that if instead of an equal number of each of *B*'s Things we suppose their Numbers to be in any other proportion, as 2 . 3 . and 4; then take the Value of 2 of the one kind, 3 of the other, and 4 of the other; add all these into one Sum, and find the Advance upon that Sum by the Advance of the whole; *thus* you get the Barter Price for 2 of one kind, 3 of another, and 4 of another, taken all together: And by this dividing the total of *A*'s Goods, the Quote shews how many times we are to take 2 of one, 3 of another, and 4 of another, and consequently how many of each.

Qu. 7th. *A* and *B* barter thus; *A* hath 100 yards Cloth at 12 *s.* a yard Ready-money, but in Barter he will have 13 *s* : 6 *d*, and will have also $\frac{1}{4}$ of the Barter-value in Ready-money. *B* hath Sugar at 8 *d.* a pound; How much Sugar ought *B* to deliver? and, How is it to be rais'd to equal the Barter?

Find the total Value of 100 yards Cloth at 13 *s* : 6 *d.* then take $\frac{1}{4}$ of it; which being the Money *B* is to pay, find how much Cloth at 12 *s.* that will buy, and subtract it from 100 yards. Then find how much Sugar at 8 *d.* ought to be given for that Remainder of Cloth at 12 *s.* For raising the Price of the Sugar, 'tis plainly in proportion as the Cloth is rais'd, *i. e.* As 12 *s.* to 13 *s* : 6 *d.* so is 8 *d.* to the advanc'd Price of the Sugar.

Observe: If in this Question the quantity of *A*'s Goods is not given, there can nothing be requir'd but to find how *B* is to advance his Price to equal the Barter.

Such a Question I find in several Authors, but they all solve it after another manner, which is this; They take the propos'd part as $\frac{1}{4}$ (which is to be paid in Money) of *A*'s Barter Price, and subtract it both from that Price and the Ready-money Price; then say, As the Remainder of the Ready-money Price to that of the Barter Price, so is *B*'s Ready-money Price to his Barter Price sought. But this Solution is without Foundation; for there appears no Reason for advancing *B*'s Price in any other proportion than *A*'s; as by this Method it is rais'd in a greater proportion. But if this Method is reasonable at all, it will be so also when the quantity of *A*'s Goods is given to find what *B* must deliver. In which Questions these Authors always use the Barter Price to find the Quantity sought. Let us then apply this *Supposition*, with their Method, to the preceding Question. It's plain, in the first place, that as much Ready-money as *B* pays to *A*, he ought

to

to have Cloth for it at the Ready-money Price, because so much is not barter'd, but bought for Ready-money: And then, for the remainder of the Cloth we are to find how much Sugar must be given for it; and if we do this by a Barter Price greater in proportion to the Ready-money Price than *A*'s is, there's a manifest Injustice done to *A*. I shall *observe* in the last place, that if for the Ready-money which *B* pays he gets Cloth from *A* at the Barter Price, then indeed the Barter Price of *B*'s Goods must be found, according to this other Method, in a greater proportion than *A*'s; which will make the Barter equal, by correcting the Injustice done to *B*, in giving him Goods at a greater than the Ready-money Price, even when he pays Money for 'em: But this Method is a ridiculous going round about to no purpose, and committing two Errors, or doing two pieces of Injustice, that one may correct the other, when there is a more simple and natural Way of doing.

Qu. 8th. *A* has 40 pair of Stockings at 3 *s.* Ready-money, or 3 *s.* : 8 *d.* in Barter; but he is willing to discompt 3 *per Cent.* of his Barter Price, to have $\frac{1}{4}$ of it paid in Ready-money. *B* has Cloth at 10 *s. per* yard Ready-money; How many yards must he deliver, with the Money that *A* requires? and, What is the Rate of his Cloth to equal the Barter?

Take 3 from 100, and say, As 100 to 97, so is 3 *s* : 8 *d.* to a 4th Number which is to be taken for the Barter Price; then do the rest of the Work as in the preceding Question. But if *A* will, besides his Ready-money, gain 3 *per Cent.* say, As 100 is 103, so is 3 *s* : 8. to a 4th Number which is to be taken for his Barter Price.

Qu. 9th. *A* barters with *B* 40 *lb.* of Cloves at 6 *s.* the pound Ready-money, and 7 *s* : 6 *d.* in Barter, but is willing to lose 10 *per Cent.* to have $\frac{1}{3}$ Ready-money: What is the Ready-money Price of 1 yard of Velvet deliver'd by *B* at 21 *s.* to equal the Barter? and, How much was deliver'd?

Say, As 100 to 90, so is 7 *s* : 6 *d.* to 6 *s.* 9 *d.* which is the true Barter Price after the Discompt of 10 *per Cent.* Then find how much Velvet (at 21 *s.*) is equal in Value to 40 *lb.* of Cloves at 6 *s.* 9 *d.* And for the Ready-money Price of the Velvet say, As 6 *s* : 9 *d.* to 6 *s.* so is 21 *s.* to the Thing sought.

I take this Question from Mr. *Hill*, with this difference; that I suppose *A*'s Quantity to be given, and *B*'s sought, which he does not; his Demand being only to know the Ready-Money Price of the Velvet; which he finds thus: Having found the 6 *s* : 9 *d.* as before, he takes $\frac{1}{3}$ of 7 *s* : 6 *d.* (*viz.* 2 *s* : 6 *d.*) from it self, and also from 6 *s* : 9 *d.* the Remainders are 5 *s.* and 4 *s* : 3 *d.* then says, As 5 *s.* to 4 *s* : 3 *d.* so is 21 *s.* to the Thing sought. This Method is in general like that censur'd above, in Question 7*th*; but it's yet farther wrong, and can be brought to no sense; for the $\frac{1}{3}$ Ready-money which *A* demands can be understood no other way than as $\frac{1}{3}$ of that Price at which he is willing actually to put away his Cloves, which is 6 *s* : 9 *d.* (*viz.* 7 *s* : 6 *d.* deducing 10 *per Cent.*) and therefore we are to take $\frac{1}{3}$ of 6 *s* : 9 *d.* and not of 7 *s* : 6 *d.* Again; by taking that $\frac{1}{3}$, (whether it be of 7 *s* : 6 *d.* or 6 *s* : 9 *d.*) from each of them, he does not bring in *A*'s Ready-money Price at all into the Calculation, and so it may be suppos'd to be any thing we please; whereby the same *Answer* will be found in all Suppositions, which is absurd. The Method upon his General Principle ought to have been this, *viz.* Take $\frac{1}{3}$ of 6 *s* : 9 *d.* from it self, and from 6 *s.* then, as 4 *s* : 6 *d.* to 3 *s* : 9 *d.* so 21 *s.* to the Thing sought.

I was the more surpriz'd at his Method of solving this Question, that in another Question, wherein *A* proposes to gain 10 *per Cent.* and have $\frac{1}{3}$ Ready-money, he proceeds in the Way last directed, *thus*; Having found the Price at which *A*'s Goods are put away (with the 10 *per Cent.* included) he takes $\frac{1}{3}$ of that from it self and the Ready-money Price, and by these Remainders finds *B*'s Barter Price, which is the Thing sought in that Question.

Qu. 10*th.* *A* has 100 yards Cloth at 8 *s.* Ready-money, and in Barter 10 *s.* *B* has Raiſins at 6 *d. per* pound Ready-money, and he will have ¼ of what he puts away paid in Money; How much Raiſins muſt he deliver? and, What Rate do they bear in the Barter? alſo, How much Money muſt *A* give *B*?

1°. 100 yards at 8 *s.* is worth 40 *l.* Then find how much Raiſins at 6 *d.* muſt be given for 40 *l.* Add the 3d part of that quantity to it, the Sum is the total weight that *B* muſt deliver; and the Value of that 3d part added (which is plainly the 4th part of the whole Sum) ſhews the Money that *A* has to pay. For *B*'s Barter Price, find it in proportion to the Ready-money Price as *A*'s Price to his Ready-money Price.

CHAP. VII.

Of Tare *and* Tret.

BY *Tare* is commonly meant the *Weight* of the Caſk, Cheſt, or Bag in which Goods are put up, and whoſe Weight can be known ſeparately from that of the Goods; and which being ſubtracted from the *groſs* Weight (or that of the Caſk, &c. and Goods together) the remainder is the Weight of the Goods alone, and is call'd *The Nett Weight.*

But if the *Tare* is not known ſeparately, and an Allowance made for it at ſo much *per* Hundred weight, or Hundred yards, &c then the Deduction of the *Tare* is by the *Rule of Three*; which the following Examples will ſhew.

There is another Allowance made for Duſt, Waſte, Refuſe, or in lack of Goods, call'd *Tret*, which is allow'd and calculated after the ſame Way.

Exa. 1*ſt.* At 7 *lb. Tare* or *Tret* to 112 *lb.* groſs, what is the Tare, and alſo the Nett Weight, when 746 *lb.* groſs was receiv'd? Say, As 112 *lb.* to 7 *lb.* ſo is 746 *lb.* to the Tare ſought; which ſubtracted from 746 *lb.* the Remainder is the Nett weight.

Exa. 2*d.* At 5 *lb.* tret to 112 *lb.* groſs, what groſs Weight muſt be receiv'd, when 84 *lb.* Nett was paid for? and, How much is allow'd? Subtract 5 from 112, the Remainder is 107: Then ſay, As 107 to 112, ſo 84 to the Groſs Weight ſought; the Difference of which and 84 is the Allowance. Or thus; As 107 to 5, ſo is 84 to the Allowance ſought; which add to 84, the Sum is the Groſs Weight ſought.

Thus from the Groſs weight, Nett weight, and Allowance, or any two of theſe in one Caſe given, with any one of them in another Caſe, we may find the other two in that other Caſe.

Obſerve. There are ſometimes two Allowances deducted out of the ſame Quantity, firſt *Tare*, and then *Tret*: After the *Tare* is deducted, the Remainder is call'd particularly *Subtle Weight*, out of which the *Tret* is deducted, and the laſt Remainder is call'd *Nett Weight.*

Exa. 3*d.* *Tare* being allow'd at 4 to 112, and *Tret* at 5 to 112, what is the *Nett Weight* in 87 *lb. Groſs*?

Say, As 112 to 108 (*viz.* 112 leſs 4) ſo is 87 *lb.* to the *Subtle*: Then, as 112 to 107 (*viz.* 112 leſs 5) ſo is the *Subtle* to the *Nett.*

And here *Obſerve*, that if you multiply 108, 107, and 87 continually, alſo 112 by 112, and divide that Product by this, the Quote is the Nett Weight ſought.

CHAP.

CHAP. VIII.

Alligation.

DEFINITION.

ALLIGATION is the *Rule* of mixing ſeveral Simples of the ſame kind, but of different Prices or Qualities, ſo as the Compound may be of a middle Price or Quality: In which there are two principal Caſes, call'd *Alligation Medial* and *Alternate.*

CASE I. *Medial.*

Having the *Rates* (*i. e.* the Price of any quantity of each of ſeveral Simples, or any other Quality by which they are diſtinguiſh'd) and *Quantities* to be mix'd, To find the Rate of the Mixture,

RULE. Find, according to the given Rates, the Value of each given Quantity; then taking the ſum of theſe Quantities, and the ſum of their Values, ſay, If that ſum of Quantities give that ſum of Values, what will any other Quantity give? And you'll find the Rate of the Mixture.

Examples *in which Regard is had to the different Prices of Things.*

1*ſt.* A Merchant has 13 Gallons of Wine at 17 *s.* *per* Gal. 11 Gallons at 15 *s.* and 19 Gallons at 14 *s.* If theſe are mix'd, what's the Price of 1 Gallon of the Mixture?

If 1 Gal. gives	17 *s.*	13 Gal. give	221 *s.*			
1 —	15 —	11 —	165			
1 —	14 —	19 —	266			
				s.	*d.*	*f*
	If	43 —	652 — 1 Gal. gives	15 :	01 :	$3\frac{35}{43}$

If we ſuppoſe 6 Gallons of Water (whoſe Value is nothing) mix'd with theſe, the Proportion is this; If 49 Gallons coſt 652 *s.* what 1 Gallon?

2*d.* A Farmer mixes 7 Bolls of Wheat at 15 *l.* *per* Boll with 9 Bolls : 3 Buſhels at 10 *l.* *per* Boll, and 6 Bolls at 12 *l* : 14 *s.* *per* Boll; What is a Peck of the Mixture worth?

If 1 Boll coſt	15 *l.*	7 Bolls coſt	105 *l.*			
1 —	10 —	9 : 3 Buſh. —	97 : 10 *s.*			
1 —	12 : 14 *s.*	6 : —	76 : 04			
				s.	*d*	*f.*
	If	22 : 3 —	278 : 14 — 1 Peck gives	15	3	$3\frac{3}{91}$

Obſerve: I ſuppoſe here that 4 Pecks make 1 Buſhel, and 4 Buſhel 1 Boll.

Examples *in which Regard is had not only to the* Price, *but to the* Quality.

1. *Of mixing* METALS.

Observe: An Ounce of *Pure Gold* being reduced into 24 equal Parts, these Parts are call'd *Caracts*; but Gold is often mix'd with some baser Metal, which in the Mixture is call'd the *Alloy*; and according to the proportion of pure Gold which is in every Ounce, so the mixture is said to be so many Caracts fine: Thus; if only 22 Caracts of pure Gold, and 2 of Alloy, it's 22 Caracts fine: If 20 Caracts of pure Gold, and 4 of Alloy, it is 20 Caracts fine: If there is no Alloy, it's 24 Caracts fine, or pure Gold.

Exa. 3*d.* A Goldsmith mixes 7 Ounces of Gold 23 Caracts fine, with 13 Ounces 19 Caracts fine; What's the Quality of the Mixture? *Answ.* $20\frac{2}{5}$ Caracts.

If 1 *Oz.* has 23 *Car.* of pure Gold, 7 *Oz.* have 161 *Car.*
1 — 19 — 13 — 247
If 20 have 408 — 1 *Oz.* has $20\frac{2}{5}$ *Caracts.*

Suppose there is to be mix'd with these 4 *oz.* of Brass or other Alloy, then add 4 to 20, and the Proportion is, If 24 *Oz.* have 408 *Car.* what 1 *Oz*?

Observe: Silver is valued by the Ounces of pure Silver in a Pound, and 12 Ounces (*Troy* weight) being a Pound, therefore it's call'd 11 or 10, &c. Ounces fine, which has 11 or 10, &c. Ounces pure Silver in the Pound.

2. *Of mixing* MEDICINES *according to their different Degrees of* Heat *and* Cold, *or* Dryness *and* Moisture.

Heat and *Cold*, also *Dryness* and *Moisture*, in Medicines, are distinguish'd by different Degrees, thus; There is suppos'd a certain Quality call'd *Temperate*, differing from which there are suppos'd to be 4 degrees of *Heat*, and 4 of *Cold*; also 4 degrees of *Dryness*, and 4 of *Moisture*; all which make 9 degrees of that Quality which regards *Heat* and *Cold*, and as many regarding *Dryness* and *Moisture*, which we may call the Common Quality; so that the 4th degree of *Cold* or *Moist* is call'd the 1st degree of the Common Quality; the 3d degree of *Cold* or *Moist* is the 2d degree of the Common Quality; and so on, as in this

TABLE.

Degrees of the Com. Qual.		
9	4	Degrees of Hot: Dry.
8	3	
7	2	
6	1	
5	0 temperate	
4	1	Degrees of Cold: Moi.
3	2	
2	3	
1	4	

Now, if Medicines are to be mix'd with Regard to these Qualities, then the Degrees of the particular Qualities being given, they must be reduced to the Common Quality, and the Operation made with the Numbers of that; as in this *Example*.

Exa. 4*th.* An Apothecary mixes several Simples, *thus*; 4 ounces cold in the 3d degree, 7 ounces cold in the 1st degree: 5 ounces temperate, and 8 ounces hot in the 4th degree; What is the Quality of the Mixture?

The Qualities given reduced to the common one, are, 2d, 4th, 5th, 9th; then multiplying each Quantity by its degree of the common Quality, the Products are $4 \times 2 = 8 . 7 \times 4 = 28 . 5 \times 5 = 25 . 8 \times 9 = 72$. The Sum of these Products is $8 + 28 + 25 + 72 = 133$; which being divided by the Sum of the Quantities,

ties, (*viz.* $4 + 7 + 5 + 8 = 24$) the Quote is $5\frac{1}{24}$: Which shews the Quality of the Mixture to be betwixt the 5th and 6th degree; *i. e.* betwixt Temperate and the first degree of Heat.

Case 2. *Alternate.*

Having the *Rates* of several Simples to be mix'd, and the *Rate* of the *Mixture*; To find such quantities of the Simples as, being mix'd together, shall bear that common Rate,

Observe: The Mixture Rate must be taken betwixt the highest and lowest Rate of the Simples; else, 'tis plain, the Mixture will not bear that Rate, but will be either of a greater or lesser Rate, as the Simples are either all of a greater or lesser Rate.

Rule. 1°, The Rates being all of (or reduced to) one denomination, and refer'd to Quantities of one denomination, 2°, set the Rates of the Simples in a Column under one another, and the Mixture Rate upon the left hand of these. Then, 3°, connect or link together the several Simple Rates, so that every one less than the Mixture be link'd with some one greater, or with as many as you please that are greater; and every one greater with one less, or with as many lesser as you please. 4°. Take the Difference betwixt the Mixture Rate and that of the several Simples, and write it against all the Simples with which that one (whose Difference it is) is link'd; then, the Sums of the Numbers (of Differences) standing against every Simple Rate, are such quantities of the several Simples, against which they stand, as answer the Question.

Qu. 1. A Merchant would mix Wines at 14 *s.* 19 *s*, 15 *s*, and 22 *s.* *per* Gallon, so as the Mixture may be worth 18 *s.* What quantity of each may be taken?

s. 18	
14	1
15	4
19	4
22	3

or

s. 18	
14	4
15	1
19	3
22	4

or

18		
14	1 : 4	5
15	1	1
19	4 : 3	7
22	4	4

or

18		
14	4	4
15	1 : 4	5
19	3	3
22	4 : 3	7

or

18		
14	1 : 4	5
15	4	4
19	4	4
22	4 : 3	7

or

18		
14	1	1
15	1 : 4	5
19	4 : 3	7
22	3	3

or

18		
14	1 : 4	5
15	1 : 4	5
19	4 : 3	7
22	4 : 3	7

Here the Simples are link'd all the Ways possible, by each of which there is a different Solution, *thus*; In the first Method 14 and 19 are link'd, and 15 with 22: Then the difference betwixt 18 and 14 is 4, which I set against 19; and the difference of 18 and 15 is 3 set against 22: The difference of 18 and 19 is 1 set against 14; and the difference of 18 and 22 is 4 set against 15: And these differences are the Answers, *viz.* 1 Gal. of 14 *s.* Wine, 4 Gal. of 15 *s.* Wine, 4 Gal. of 19 *s.* and 3 Gal. of 22 *s*; which being mix'd together, each Gallon is worth 18 *s.* The same Way understand the 2d Method of linking the Simples. For the 3d Method, 14 *s.* is link'd both with 19 and 22, and 19 both with 14 and 15: Therefore the difference of 18 and 14, which is 4, is set against both 19 and 22; and the difference of 18 and 19 (*viz.* 1) set against both 14 and 15; and thus there come two differences against 14 and 19, which being summ'd, the Answers in this Method are, 5, 1, 7, 4 Gal. of the Simples against which these Numbers stand. From this 'tis easy to understand the other Methods.

The same *Rule* is applicable to mixing *Metals* or *Medicines*, as in the following Question.

Qu. 2.

Qu. 2. To mix Gold 18 Caracts fine, with Gold of 13 Caracts, of 14 Caracts, of 21, and 23, so as the Mixture may be 15 Caracts fine; What Quantities may be taken of each?

		Oz.			Oz.			Oz.			Oz.
	13	3:8		13	6		13	3:6		13	3:6:8
	14	6		14	3:8		14	6:8		14	6
15 .	18	2	or.15	18	1	or.15	18	2	or.15	18	2
	21	1		21	2		21	2:1		21	2:1
	23	2		23	1		23	1		23	2

These are a Part only of the various Ways of linking the Simples in this Question.

The Reason *of the preceding* Rule.

To *demonstrate* that the preceding *Rule* produces true Answers, I shall,

1°, Suppose only two Simples, as, *Wine* of 15 *s.* a Gallon, and of 22 *s.* to be sold at 18 *s.* the Given Rates stand, according to the Rule, as here; and the Quantities sought are 4 Gallons of the 15 *s.* Wine, and 3 Gallons of the 22 *s.* Wine, which being sold in a mixture at 18 *s.* I say, there is precisely as much gain'd by the one Quantity as is lost by the other; for, each Gallon at 15 *s.* gains 3 *s.* the difference of 15 and 18, and therefore 4 Gallons gain 4 times 3 *s.* Again; each Gallon at 22 *s.* loses 4 *s.* the difference of 18 and 22, therefore 3 Gallons lose 3 times 4 *s*; but 4 times 3 *s.* is equal to 3 times 4 *s.* therefore the Gain and Loss are equal, and consequently the Quantities mix'd do justly bear the propos'd Rate.

	15	4
18		
	22	3

The same Reason is manifestly good in all Cases of two Simples mix'd according to this Rule, from the Way of placing the Differences alternately against the Simple Rates. Again,

2°. However many Simples there are, and with however many others every one is link'd, since 'tis always a lesser with a greater than the Mixture price, therefore there is a balance of Gain or Loss upon the Quantities taken from every linking of two Simples; and consequently there must be a balance on the whole: So that the Rule is good in all Cases.

Practical Observations *relating to the preceding Case.*

1*st Obs.* These Questions of *Alligation Alternate* are of the kind which the Algebraists call *Indeterminate Problems*; *i. e.* which have an infinite number of different Answers; for finding which, their Art gives an universal Rule: But, for the Rule here given, it is limited in its immediate Effect, to the different Answers found by the various Methods of linking the Simples; which can be done only a certain limited number of Ways: *Yet* from this Rule we can find an infinite number of other Solutions. *Thus*;

(1°.) Take any Method of linking the Simples; then take the Quantities arising from that Method; and if you encrease or diminish each of them in the same proportion (*i. e.* by equal *Multiplication* or *Division*); these new Quantities are also true Answers, for that very Reason that they are proportional to those arising immediately from the Linking and Differences; because, if two Quantities of two Simples make a balance of Gain or Loss, with respect to the Mixture price, so must double or triple, or the half or third part, or any other proportion of these Quantities. And, because these Quantities may

may be encreas'd or diminiſh'd in an infinite variety of Proportions, therefore 'tis plain that we may proceed to an infinite variety of Solutions.

2°. Or, if we only encreaſe or diminiſh the alternate or correſpondent Differences of any pair of Simples that are link'd together, or of any two or more pairs, leaving the [illegible] as they are, we may thus alſo proceed to an infinite number of Solutions.

II *Obſ.* Beſides the Rates of the Simples and Mixture given, the Q[illegible] limited either to a certain total quantity of the Mixture, or to a certa[illegible] one or more of the Simples.

(1°.) If the Limitation is to a certain total Quantity, then, if the [illegible] found by any one Way of linking the Simples is the given Total, the Q[illegible] Otherwiſe (or without trying all the Ways of linking) take any one Meth[illegible] or diminiſh the quantity of each Simple found by that linking, in proportion as the [illegible] Total is greater or leſs than that Total found by the linking.

Qu. 3*d*. A Merchant mixes Wines at 14 *ſh*, 12 *ſh*, 15 *ſh*, 18 *ſh*, and 22 *ſh*. the Gallon, to be ſold at 17 *ſh*. and would make in the whole 100 Gallons; What Quantity may he take of each?

	12	1	1	$5\frac{15}{17}$
	14	1	1	$5\frac{15}{17}$
17	15	5	5	$29\frac{7}{17}$
	18	5 : 3	8	$47\frac{1}{17}$
	22	2	2	$11\frac{13}{17}$
		Sum	17	100

	12	1 : 5	6
	14	5	5
17	15	5	5
	18	5	5
	22	5 : 3 : 2	10
		Sum	31

If the Given Total were 17 or 31, the Queſtion is ſolv'd: But, to make 100 gallons, I take either of theſe Ways of linking, as the firſt, and ſay, As 17 Gal. to 100, ſo is 1 to $5\frac{15}{17}$; ſo is 5 to $29\frac{7}{17}$, ſo is 8 to $47\frac{1}{17}$, ſo is 2 to $11\frac{13}{17}$; which being ſet againſt the Correſpondent Simples, the Sum makes up 100 Gallons.

After this manner, if you know the Total, and alſo the Particulars, of any Mixture, you may find how much of each ſpecies is in any quantity of the Mixture.

(2°.) If the Limitation is to a certain Quantity of one of the Simples, then, if the ſame Quantity happen upon that Simple in any one Way of linking, the Queſtion is ſolv'd; otherwiſe you muſt raiſe or diminiſh the reſt, in proportion as the limited Quantity of that Simple is greater or leſſer than the Quantity of it found by the linking.

Thus, in *Qu.* 3*d*, ſuppoſe the Mixture ought to have 2 Gallons of the 22 *s.* Wine; then the firſt Way of linking ſolves the Queſtion. And if it ought to be 4 Gallons of 22 *s.* ſay, As 2 Gallons (ſtanding againſt 22 *s.*) is to 4, ſo is 1 to 2, ſo is 5 to 10, and ſo is 8 to 16: And ſo the Quantities ſought are 2 Gallons of 12 *s.* and alſo of 14 *s.* 10 Gallons of 15 *s.* and 16 Gallons of 18 *s.*

If the Simple whoſe Quantity is limited is only once link'd, we need do no more than raiſe or diminiſh the Quantity of that one Simple with which it is link'd, and leave the reſt as they are. So, in the preceding Suppoſition, 22 *s.* is only link'd with 15 *s.* and therefore raiſing 5 Gallons againſt 15 *s.* to 10 Gallons (which is as 4 to 2) we may take the reſt as they ſtand: For thus there is ſtill a Balance preſerv'd in the Gain and Loſs.

But if the Simple whoſe Quantity is limited is link'd with more than one, we may take this Method: Take that part of the Quantity ſtanding againſt each of theſe Simples (with which the limited one is link'd) which is the difference of the Mixture Rate and the limited Simple, and raiſe or diminiſh it proportionally: The Quantities thus found muſt be added to the other parts of the Quantities againſt theſe other Simples. *Thus*, in *Qu.* 3*d*,

 ſuppoſe

suppose 'tis requir'd to have 8 Gallons of the 12 *s.* Wine; if the 2d way of linking is chosen, then 12 being join'd with 18, which has but one Difference against it, *viz.* 5, (the Difference of 17 and 12) I say, As 6 to 8, so is 5 to $6\frac{2}{3}$, the Quantity to be taken of the 18 *s.* Wine. Again; 12 being join'd with 22, and the Difference of 17 and 12 being 5, (one of the Differences against 22) I say as before, As 6 to 8, so is 5 to $6\frac{2}{3}$; which being taken instead of 5 against 22 *s.* makes the Total of that Simple $11\frac{2}{3}$ Gall.

(3°.) If the Limitation is to a certain Quantity of more than one Simple, *work thus*; Take these simple Rates, with their given Quantities, and find by *Case* 1. what Rate the Mixture of these by themselves would bear; then take the sum of their Quantities given, with their Mixture-Rate now found, and place that Rate in the Question, instead of the Rates of these Simples; and then the Question is the same as a Limitation to one Simple which is the Total of the Given Quantities now reduced to one Mix'd Rate; by which therefore find the Quantities of the rest, as in the preceding Article, *Thus*; Suppose in *Qu.* 3*d*, that there ought to be 3 Gallons of 12 *s.* Wine, and 7 of 14 *s.* the Mixture Price at which these may be sold is 13 *s* : 4 *d* : and $\frac{4}{5}$; and this being reduced to 5th parts of a Penny, is 804: Therefore the Rates of the other Simples, and also the Mixture, must be reduced to 5th parts of a Penny, and the Question will stand as below; in which, according to the way of Linking chosen, the Quantities are 60 . 300 . 216 and 120; but of the Wines whose Quantities are limited, and whose Mixture Price is 804, (5ths of a Penny) the Quantity ought only to be 10 (*viz.* 3 of the one, and 7 of the other); and therefore the rest are diminish'd accordingly to 50, 36 . and 20; which are true Answers to the Question.

	804	60	10
1020	900	300	50
	1080	216	36
	1320	120	20

Take Notice, That if the Mixture Rate of the Simples limited is such that the Given Mixture Rate is not a Medium, when that other Mixture Rate is placed as a Simple, then the Limitation makes the Question impossible.

III *Obs.* If a Mixture is made of several Simples whose Rates are known; with the Rate of the Mixture, and total Quantity mix'd, we may find how such a total Quantity might be mix'd of these Simples to bear the Given Rate, by the 1st Article of the 2d *Observation*. But it is to be observ'd, that the Mixture has perhaps been made after another manner. So, in *Qu.* 1, the 1st and 2d Ways of Linking make the same total Quantity.

IV *Obs.* When the Quantity of one Simple is limited, if that Simple is an exact *Medium* betwixt some other two (exceeding the one as much as it wants of the other) then, having link'd the Simples any one way, if the limited Quantity is less than what is found by the linking, take the half of what it is less, and add so much to each of these two Simples betwixt which it is an exact Medium; for thus the total Quantity found by the linking is preserv'd, since what is taken less in one, is made up out of others; and what is so taken less and more than the Quantities found by the linking, are of equal Value, because the middle Price is an exact Medium betwixt the other two; Therefore the Rate of the Mixture is never alter'd.

Again; If the Limitation is greater than what is found by the linking, take the half of what it exceeds from the Quantities of each of those betwixt which the Price whose Quantity is given is an exact Medium; but if this half is greater than these Quantities, you must take another Method.

Thus, in *Qu.* 3d, let the Limitation be to 2 Gallons of the 15 *s.* Wine, then take the first way of linking, in which the Quantity of that Wine is 5 Gallons; then, because 15 *s.* is an exact Medium to 12 and 18, I take the Difference of 2 and 5, which is 3, and its half

half $1\frac{1}{2}$. I add to 1 and 8 the Quantities againſt 12 and 18, the Sums are $2\frac{1}{2}$ and $9\frac{1}{2}$: The reſt of the Quantities ſtand as they are

But if the Limitation is to 8 Gallons of the 15 *s.* Wine, which exceeds 5 by 3, then, becauſe $1\frac{1}{2}$ is greater than 1 againſt 12, therefore I cannot take the Method preſcrib'd, with this Way of Linking, but with the ſecond Way it can be done. And if it could not be done with either of theſe Linkings, we muſt either find one in which it can be done, or ſolve the Queſtion by the General Rule in Article (2°.) *Obſ.* II.

V *Obſ.* From the Method explain'd in the preceding *Obſervation*, it's plain how we may, in ſome Circumſtances, limit both the total Quantity and ſome one of the Simples, *thus*; if the Simple which is limited is an exact Medium betwixt other two, then take any one Linking, and proportion the Quantities to the Total limited; then apply the Method of the laſt *Obſervation*, if poſſible.

VI *Obſ.* In Mixtures one Ingredient may be ſuch, as to bear no Value in the Mixture, but only to encreaſe the Quantity, and diminiſh the Value; Therefore let its Rate be repreſented by 0, as Water mix'd with Wine; Braſs, or other Alloy, mix'd with Gold and Silver.

Exa. 1. If 8 Gal. of Wine at 9 *s.* *per* Gallon, 12 Gal. at 8 *s.* are mix'd together; How much Water muſt be added to make the Mixture worth only 6 *s.* *per* Gallon? I find the Mixture Rate of the 8 Gal. and 12 Gal. then I take 20 Gal. at that Rate to mix with Water whoſe Rate is 0: Which is done by the Method of *Article* 2d, *Obſ.* II.

Exa. 2. A Goldſmith would mix Gold, 18 Caracts fine, 20 Caracts fine, 24 Caracts, and a quantity of Alloy to make the Mixture 19 Caracts fine; How much may be taken of each? Repreſent the Rate of the Alloy by 0, and proceed as in *Caſe 2d.*

VII *Obſ.* Beſides the mixture of Liquors, or any other kind of Things, the ſame Rules are applicable where *Perſons* are the Subjects, *thus*:

Exa. 1. 8 Men being boarded at the rate of 6 *l.* a Quarter for every Man, 6 Women at 5 *l.* for each Woman, and 4 Children at 2 *l.* for each Child; How much does each Perſon pay a Quarter, taking them at an equal rate, one with another? This is plainly a Queſtion of *Caſe* 1, and to be ſolv'd after that manner.

Exa. 2. If the Quarter's Board for a Man is 5 *l*, for a Woman 4 *l*, for a Child 3 *l*, and for a Servant 1 *l.* how many Men, Women, Children, and Servants may be taken to board, ſo as their Board, at an equal Rate, may come to 3 *l*: 5 *s.* for each Perſon? This is a plain Queſtion of *Caſe* 2d.

I add the following Queſtions *for a further Exerciſe.*

Qu. 4*th.* There is a Mixture of 40 Gallons of Wine worth 10 *s.* a Gallon, part of which is at 8 *s*, part at 9 *s*, at 12 *s*, and 14 *s*. What ſhall be added to it, to make the Mixture worth only 11 *s*?

To ſolve this Queſtion, The Simple to be added muſt be of ſuch a Price that the new Mixture Price lie betwixt it and the former; therefore, if there is not ſuch a Simple among theſe which are already in the Mixture, another to anſwer the Queſtion muſt be brought in, and the Solution is thus made: Take the 40 Gallons at 10 *s*, and find how much of any Simple of greater value than 11 *s.* (becauſe this is greater than 10 *s*) muſt be mix'd with theſe 40 Gallons to bear 11 *s.* in Mixture: Which is done by the Method of *Article* 2d, *Obſ.* II. preceding.

Qu. 5*th* A Mixture being made thus; 14 *lb.* weight of Sugar at 7 *d. per lb*, 16 *lb.* at 9 *d*, and 30 *lb.* at 10 *d.* How much, and of what kind, may be added, that in every *lb.* of the Mixture there be 6 *oz.* of the Sugar at 10 *d*, without changing the quantity of the other kinds in every *lb*? and, What Rate will the Mixture bear?

First, find how much of the 10 *d.* Sugar is in 1 *lb.* of the Mixture, as it ſtands already; then (1) if that is more than 6 *oz.* (as in this caſe 'tis 8 *oz.*) ſubtract 6 *oz.* from it, and multiply the difference (2) by the total number of *lb*'s (60), the Product is 120 *oz.* Now, if we could take out 120 *oz.* of the 10 *d.* Sugar, and in its place put in 120 *oz.* of any other kind than what is already in the Mixture, it's plain the Conditions of the Queſtion would be anſwer'd: Therefore, in the firſt place, we ſhall add 120 *oz.* of ſome other kind, as if as much of the 10 *d.* Sugar were actually taken out; but becauſe it is not ſo, we muſt add to it as much of all the other kinds (including that new kind ſuppos'd to be already added) as ſhall make each *lb* (of this ſum) have as much of each kind as are in each *lb* of the other 60 *lb*; which is done *thus:* Find how many times 6 *oz.* are contain'd in 120, [and if there were a Remainder, as here there is none, I add that Remainder of the 10 *d.* Sugar to the Mixture, (*i. e.* to the 120) whereby there will be no Remainder, and the Quote will be 1 more than it was with the Remainder] this Quote ſhews how many *lb.* are to be added to the given total Mixture (60 *lb.*) that each may have 6 *oz.* of 10 *d.* Sugar; which are to be made up *thus:* Firſt, there is the 120 *oz.* of the 10 *d.* Sugar ſuppos'd at firſt to be taken out (for as much of a new kind put in) and now as 'twere put back again, together with the Remainder laſt mention'd (where there is any) then you muſt find how much of each of the other Simples [including that new kind whereof there is ſuppos'd to be already added 120 *oz.* for as much of 10 *d.* Sugar ſuppos'd to be taken out] there is in each of the preceding 60 *lb.* Theſe Quantities muſt be multiply'd by the Difference of 60 *lb.* and the new total Quantity or Mixture; the Products are what muſt be taken of each of theſe other Simples (beſides the 120 *oz.* already ſuppos'd to be added of the new kind) and thus the Conditions of the Queſtion are anſwer'd.

For the *Rate* of this New Mixture, it's found by *Caſe* 1.

But, Secondly, if there is in the Mixture leſs than 6 *oz.* of 10 *d.* Sugar to every *lb.* (as ſuppoſe there be only 4 *oz.*) then the difference is 2 *oz.* which multiply'd by 60, makes 120 *oz.* or 7 *lb* : 8 *oz.* Take double of this, with as much as is already in the Mixture of 10 *d.* Sugar (*viz.* 30 *lb.*) the Sum is 45 *lb.* to be added of 10 *d.* Sugar. Then I conſider how much of each of the other kinds is in the Mixture, and from the total of theſe I take 120 *oz.* (or 7 *lb* : 8 *oz.*) in any manner, *i. e.* all out of one kind, if poſſible, or part of one and part of another, till the whole is ſubtracted, marking what hereby remains in each kind; then as much of each kind as theſe Remainders being alſo added to the Mixture, will bring the Total (which is now 120 *oz.* more than double of what it was before) to ſuch a Mixture as ſhall have 6 *oz.* of 10 *d.* Sugar to every *lb*, and the ſame quantities of each of the reſt as were at firſt to every *lb.*

Qu. 6*th*. There is a Mixture made of Wheat 6 *s*, 7 *s*, and 8 *s.* the Buſhel; the total Quantity is worth 100 *l.* and the proportion of the particular Quantities are thus; For every 2 Buſh. of 6 *s.* there are 3 of 7 *s*; and for every 3 of 6 *s.* there are 4 of 8 *s.* How much of each kind is in the Mixture? and, What is the Rate of the Mixture?

First, I reduce the Proportions of the Quantities to a Series of three Numbers, *thus:*

	6	2 :	3	6
Sh.	7	3		9
	8		4	8

As 2 buſh. (of 6 *s.*) to 3 buſh. (of 6 *s.*) ſo is 3 buſh. (of 7 *s.*) to a 4th, which is $4\frac{1}{2}$ buſh. (of 7 *s.*) ſo that for every 3 buſh. of 6 *s.* there are $4\frac{1}{2}$ of 7 *s.* and 4 buſh. of 8 *s.* And multiplying all by 2, to make 'em Integers, the proportional Quantities of each are 6 . 9 . 8. Next I find the Mixture Price from theſe Quantities; and then, by that Rate, I find

I find how much the 100 *l.* will buy; and then I divide that Quantity into three parts proportional to 6 . 9 . 8.

Qu. 7th. A Mixture is to be made of Wine at 18 *s.* *per* Gallon, at 16 *s.* and 9 *s.* How much may be taken of each to make 48 Gallons that shall be worth in all 28 *l* : 16 *s* ?

Find what 1 Gallon of the Mixture is worth, *viz.* 12 *s.* then find what Quantities may be taken of the several Simples, to make a Mixture at 12 *s.* and the total 48 Gallons.

Observe : By this Method you can always have at least one Solution to all Questions of this kind (*viz.* wherein the Rate of each Simple, the total Mixture, and total Value are given) provided the Question is possible; as 'tis not if the Mixture Rate found be not betwixt the highest and lowest of the Simples; and provided also that the Nature of the Subject does not limit the Answer to Integral Numbers, as when the Numbers sought are *Animals*, (see the following *Example*) for then, tho' the Question be possible, and hath several Solutions, yet none of them may be found by this Method, because of Fractions brought by it into the Answers: And this Method will have the same Defect in any Subject, if we limit the Answer to Integers.

For *Example*; Apply the Numbers of the preceding Question to another Subject, *thus*: 48 Persons, Men, Women, and Children, pay all together at a Feast 28 *l* : 16 *s.* whereof every Man pays 18 *s.* every Woman 16 *s* and every Child 9 *s.* How many were of each? By the preceding Method we shall find 9 Men, 9 Women, and 30 Children; which satisfie the Question: Tho' it has also another Solution (which cannot be found by this Method) *viz.* 2 Men, 18 Women, and 28 Children.

Exa. 2d. 15 Birds cost in all 5 *s.* whereof there were Partridges at 7 *d.* Quails at 5 *d.* and Larks at 2 *d.* How many were of each? If we try this by the preceding Rule, we can find no Answer, because of Fractions; and indeed it has but one Solution in Integers, which is found to be 3 Partridges, 5 Quails, and 7 Larks.

Qu. 8th. A Mixture was made of 10 Gal. Wine, 4 Gal. Brandy, and 12 Gal. Water. Out of the whole was drawn 8 Gal. and as much Water put in to fill it up: Then was drawn out 10 Gal. after which was put in 6 Gal. of Wine: Again there was drawn out 5 Gal. and 3 Gal. of Brandy put in. How much Wine, Brandy, and Water is at last in the Mixture?

Find how much of each Species is in the 8 Gallons first drawn out, by which you'll know what remains of each: To which is added a Gallon of Water; then find how much of each is contain'd in the 10 Gall. next drawn out, whereby you'll know what remains of each: To which is added 6 Gall. Wine; then find how much of each is in the 5 Gall. drawn out, and you'll know how much remains of each; to which 3 Gall. of Brandy is to be added.

CHAP.

CHAP. IX.

Of Exchange.

THE *Reduction* of different Coins, or any Denominations of Money (whether they have Real Coins anſwering to them, or not) from one to another, *i. e.* finding how many of one Species or Denomination are equal in Value to a given Number of another; with other Queſtions relating to the ſame Subject, is what I here call *Exchange.*

Obſerve I. If the Queſtion is about the Reduction of ſuch Species as are aliquot Parts one of another, as Pounds, Shillings, and Pence, the Work is only ſimple *Multiplication* or *Diviſion*; as has been ſhewn in *Book* 1, *Ch.* 7, § 4 & 5; but, in all other Caſes, there is a variety of Applications of the *Rule of Three.*

II. To reduce any number of one Species to another, there muſt always be known the Proportion betwixt the two Species, either immediately, or by the immediate Proportions betwixt each of them and one or more other Species.

III. When we know any two Numbers of different Species that are of equal Value, theſe two Numbers expreſs the Proportions, or Rate of *Exchange*, of theſe Species, whether they belong both to one Country, or to different Countries, [as, if we ſuppoſe 3 Guineas of *Britain* equal to 3 *l*: 3 *s.* or 5 *s.* Sterling equal to 3 Guilders *Amſterdam.*] But, in Common Acceptation, when we ſpeak of *Exchange*, it's underſtood of the Species of different Countries; and the *Rate* of *Exchange* is expreſs'd by ſo much of the one Place equal to an *Unit* of a certain Species of another: So, if 5 *s.* Sterling are equal to 3 Guild. *Amſterdam*, tho' this does really expreſs the Proportion, yet more commonly we ſay, The *Rate of Exchange* is at 1 *l.* Sterl. to 12 Guild. (found by the given Proportion; thus, if 5 *s.* give 3 Guild. in what 20 *s.*) or as 1 Gu. to 1 *s*: 8 *d.* Sterling.

IV. The Values of the Coins and Monies of different Countries; in what Specie's Denominations they exchange with one another, with the Limits of the Common *Rotes* of *Exchange*, (for they vary often) are Matters of Fact which belong to another province: My Buſineſs here is, to teach the Calculation, or ſhew how the Rules of *Aaithmetick* are apply'd to this Uſeful Subject: And, to do this to the beſt advantage, I ſhall give more ſuitable Numbers of Queſtions, and diſtinguiſh them under ſuch Heads as comprehend Things of moſt ordinary Occurrence; and then add a few other Queſtions, that are leſs common, that from the Whole you may be Maſter of what can happen upon this Subject, that depends upon *Common Arithmetick.*

§ I.

§ I. QUESTIONS *relating simply to the* Par *of* Exchange, *or finding the Number of one Species equal to a Given Number of another; the Proportion being given directly, or by the mediation of other Species.*

(1°) If the Proportion is given betwixt two Species directly, the Solution is by one Operation of the *Rule of Three.*

Quest. 1. A Merchant at *Amsterdam* paid 150 Guilders for 13 *l* : 15 *s.* Sterling, receiv'd by his Correspondent at *London* ; What is the *Rate of Exchange ?* or, What is 1 Guilder valued at in English Money ? Say, If 150 Guild. give 13 *l* : 15 *s.* what will 1 Guilder ? *Answ.* 22 *d.* Or, if it's ask'd what 1 *l.* Sterling is valued at in Dutch Money, say, If 13 *l* : 15 *s.* give 150 Guild. what 1 *l ?*

Qu 2*d.* A Merchant at *London* receiv'd 100 *l.* Sterling, for the value paid by his Correspondent at *Paris* at the rate of 3 *s* : 6 *d.* Sterl. *per* Crown ; How many Crowns were paid at *Paris ?* Say, If 3 *s* : 6 *d.* give 1 Cr. what 100 *l* ?

(2°) When the Proportion is given between each of the two Species in question, and a third Species, the Solution is by two Operations of the *Rule of Three.*

Qu. 3*d.* If I would exchange 200 Ducats, worth 7 *s.* a piece, for Dollars at 4 *s* : 8 *d.* a piece, how many Dollars ought I to have ? Say, If 1 Ducat give 7 *s.* what 200 ? *Answ.* 1400 *s.* Then, if 4 *s* : 8 *d.* give 1 Dollar, how many will 1400 *s* ?

Or thus : If 4 *s* : 8 *d.* give 1 Dol. how many will 7 *s* ? *Answ.* $1\frac{1}{2}$; so that 1 Ducat is worth $1\frac{1}{2}$ Doll. Then say, If 1 Duc. give $1\frac{1}{2}$ Dol. how many will 200 give ?

Observe : Had it only been requir'd to find the direct Proportion betwixt Crowns and Dollars, then say, If 7 *s.* give 1 Ducat, how many will 4 *s* : 8 *d* ? *Answ.* $\frac{2}{3}$ of a Ducat. Or say, If 4 *s* : 8 *d.* give 1 Dol. how many will 7 *s* ? *Answ.* $1\frac{1}{2}$ Doll.

Qu. 4*th.* Exchange from *London* to *Amsterdam* being at 1 *l.* Sterling for 34 *s.* Flem. and from *Amsterdam* to *Paris* at 5 *s.* Flemish for 1 Crown, what is the *Exchange* betwixt *London* and *Paris*, according to that Course ? *Answ.* $6\frac{4}{5}$ Crowns for 1 *l.* Sterling, or 2 *s* : $11\frac{5}{17}$ *d.* for 1 Crown ; found thus : If 5 *s.* give 1 Cr. what 34 *s* ? It is $6\frac{4}{5}$ Cro. which is the Value of 1 *l.* because 34 *s.* is equal to 1 *l.* Or say, If 34 *s.* give 1 *l.* what 5 *s* ? It is 2 *s* : $11\frac{5}{17}$ *d.*

Lond.	*Amst.*	*Par.*
1 *l.* =	34 *s.*	
	5 *s.* =	1 Cr.

Observe : If 'tis demanded to find the Value of 500 Cr. in English Money, according to that Course of Exchange ; then, having found the Rate of Exchange, say, If 1 Cro. give 2 *s.* $11\frac{5}{17}$ *d.* what 500 Crowns ? or, If $6\frac{4}{5}$ Cr. give 1 *l.* what 500 Crowns ?

If a Sum English (as 100 *l.*) is given to find its Value in Crowns, 'tis only stating the Terms otherwise, according to the *Rule of Three*, thus ; If 1 *l.* give $6\frac{4}{5}$ Cr. how many will 100 *l.* give ?

(3°) When the Proportion is given betwixt one Specie and another, betwixt this other and a third, betwixt this third and a fourth, and so on as far as you please, to find the Exchange betwixt the first and last Species ; the Solution is by one fewer Operations of the *Rule of Three* than there are different Species, in the manner of the following Question.

Qu. 5*th.*

Qu. 5*th.* Exchange betwixt *London* and *Amsterdam* being at 1 *l.* Sterling for 38 *s.* Flemish: betwixt *Amsterdam* and *Frankfort* at 6 *s.* Flemish for 65 Cruitzers; betwixt *Frankfort* and *Paris* at 56 Cruitzers for 1 Crown; what is the Exchange, according to that Course, betwixt *London* and *Paris*? *Answ.* 1 *l.* for $7\frac{20}{27}$ Crowns; found thus: Set down the Given Terms as in the Margin, and work from the left Hand to the right, *thus*; say, If 6 *s.* give 66 Cru. what will 38 *s*? The Answer is 418 Cruit. which is set under *Frankf.* Then say, If 54 Cru. give 1 Crown, what 418 Cru.? The Answer is $7\frac{20}{27}$ Cr. which is set under *Paris.* And thus the Exchange betwixt *London* and *Paris* is found to be at 1 *l.* for $7\frac{20}{27}$ Crowns.

Lond.		*Amst.*		*Frankf.*		*Paris*
1 *l.*	=	38 *s.*				
		6 *s.*	=	66 Cru.		
				54 Cru.	=	1 Cr.
1	=	38 *s.*	=	418 Cru.	=	$7\frac{20}{27}$ Cr.

Observe, (1°) However many different Places are propos'd, you go thro' them all in the same manner.

(2°) The undermost Line shews not only the Exchange betwixt the first and last Places, but also betwixt any two of them; the Quantities in that Line being evidently all equal in Value, from the Nature of the Operation.

(3°) If there are two different given Species for one Place, they must be reduced to one Species: So, if the Exchange betwixt *Amsterdam* and *Frankfort* were express'd by 36 Stivers for 66 Cruitzers, then we must first reduce the 38 *s.* to Stivers, or the 36 Stivers to Shillings, by the known relation of *Stivers* and *Shillings*; which is 6 Stivers to 1 Shilling. Or, if it were 36 Stivers for 1 Florin, then you must also reduce the 1 Florin to Cruitzers, or the 54 Cruitzers to Florins. And if this Reduction cannot be done, *i. e.* if the relation of these Species is not known, the Question cannot be solv'd.

(4°) If there's another Given Quantity of the first and last Place, to find a Quantity of equal Value in the other, it's a plain Application of the *Rule of Three*, from the *Rate* of Exchange found (as was observ'd upon the preceding Question).

Some Contractions of the preceding RULE *explain'd.*

(5°) All these Operations of the *Rule of Theee* may be reduced to one Division, *thus*; Multiply the *Consequents* of all the *Proportions* [*i. e.* the Numbers upon the right hand, or the first Number under every Place] continually into one another; also all the *Antecedents*, except the first, [*i. e.* the Numbers upon the left hand, or second Number under every Place] make the first Product *Dividend*, and the second *Divisor*, and the *Quote* is the Number sought of the Species of the last Place, equal to the Number under the first Place; *thus*, in the preceding Example, 38 multiply'd by 66, produces 2508, (the number of Crowns being 1, does not multiply) then 54 by 6, produces 324; and 2508 divided by 324, produces $7\frac{20}{27}$.

The *Reason* of this will be manifest, by considering how the several Operations of the *Rule of Three* are made; for the Answer of the first Operation is the Quote of 38 multiply'd by 66, and the Product divided by 6; which we may express thus, $\frac{38 \times 66}{6}$; then for the second Operation it is, the last Answer multiply'd by 1, (which is still only $\frac{38 \times 66}{6}$) and the Product divided by 54, which is $\frac{38 \times 66}{6 \times 54}$; according to the Direction now given: And how many Places soever there be, the Reason is manifestly the same.

Again;

Again; If any *Divisor* and *Dividend* can be divided by any Number without a Remainder, then if we take the two Quotes, and divide the one of them by the other, there will arise the same Quote as from the given *Dividend* and *Divisor*; therefore this Work may be much contracted, *thus*; If among the Numbers that produce the Divisor there are any the same with what are among those that produce the Dividend, cast them out of both (*i. e.* do not use them in the Multiplication): And if any two Numbers, one belonging to the Divisor, and one to the Dividend, can each be divided without a Remainder by any the same Number; Take these Quotes in place of those given Numbers, in forming your *Divisor* and *Dividend*. *So*, suppose in the preceding Example it were 1 *l.* English for 54 *s.* Flemish, then this 54 (which belongs to the Numbers that form the Dividend) and the 54 Cruitzers (which belong to the Numbers that form the Divisor) may both be cast out. And, as the Question is already put, tho' there is no Number the same, yet for 6 and 66 we may take 1 and 11, the Quotes of 6 and 66 divided by 6; also for 38 and 54 we may take 19, 27, the halfs of the former, and so the Dividend is $19 \times 11 = 209$, the Divisor is 27, and the Quote $7\frac{20}{27}$.

§ II. QUESTIONS *wherein* Gain *and* Loss, *and the Allowances to* Factors, *are consider'd.*

Qu. 6th. *A* of *London* draws upon *B* of *Amsterdam* 500 Guilders at 22 *d. per* Guild. for which *B* redraws upon *A* at 21 *d. per* Guilder, with Provision at $\frac{1}{2}$ *per Cent.* and 10 Guilders of Brokerage; How much will *A* pay? and, Whether has he gain'd or lost? *Ans.* He pays $11018\frac{3}{4}$ *d.* and loses $18\frac{3}{4}$: found thus; As 1 Guil. to 22 *d.* so is 500 Guil. to 11000 *d.* receiv'd by *A* for the Draught; then, as 100 to $\frac{1}{2}$, so 500 to $2\frac{1}{2}$, the Provision due to *B*; which, with 10 Guild. Brokerage, added to 500, makes $512\frac{1}{2}$, for which he draws on *A*; therefore say, As 1 Guild. to $21\frac{1}{2}$ *d.* so $512\frac{1}{2}$ to $11018\frac{3}{4}$ *d.* which *A* pays for the redraught; so that he loses $18\frac{3}{4}$ *d.*

Observe, (1°) If *A* had remitted to *B*, with Orders to remit the Value back again; then having found what *B* receives by *A*'s Remittance, from that subtract *B*'s Provision and Brokerage, the Remainder is what he remits to *A*; whose Value being found, the comparison of that and what *A* paid for the Remittance shews what he gains or loses.

(2°) If *A* draws upon *B*, and afterwards remits the Value to him, he must add the Provision due to *B*, to the Sum which *B* paid, and remit the Total.

(3°) If *B* by *A*'s Order draw upon him, and afterwards remits him the Value, then *B* deducts his Provision and double Brokerage from what he receiv'd by the Draught, and remits the Remainder.

(4°) In all the Cases where *A* draws or remits, you must also consider what Brokerage it may have cost him, in order to know his Gain or Loss.

Qu. 7th. If Exchange from *London* to *Amsterdam* is at 1 *s*: 10 *d.* for 1 Guilder, and to *Paris* at 3 *s*: 8 *d.* for 1 Crown; also from *Amsterdam* to *Paris* at 40 Stivers for 1 Crown; whether is it most profitable that *London* remit directly to *Paris*, or by way of *Amsterdam* [*i. e.* remit to *Amsterdam*, to be remitted from that to *Paris*]? Find what 1 *l.* Sterling is worth at *Paris*, according to the Course with *Amsterdam* (by Question 4th) and also according to the *Exchange* directly with *Paris*; the Comparison of these Values of 1 *l.* gives the Answer.

Observe, 1, what Allowance is due to the Factor at *Amsterdam* is to be deduced from the Money he receives (*i. e.* the value of 1 *l.* Sterling in Guilders) and the Remainder is what he remits to *Paris*.

2. If there are more than 3 Places in the Question, and the Exchange is given betwixt one and another in a series, as in *Quest.* 5*th*, and also betwixt the first and last; to find which is most profitable, to Remit from the first to the last Place directly, or thro' all these Places; then you must find what is the Exchange betwixt the first and last Place according to the Courses thro' all the other Places, and compare that with the given Exchange betwixt the first and last Place.

Qu. 8*th*. A Merchant at *London* remits to *Amsterdam*, at the rate of 18 *d.* Sterling *per* Guilder: His Correspondent at *Amsterdam* remits the same by Order to *Bourdeaux*, at 3 Guilders *per* Crown, rebating $\frac{1}{3}$ *per Cent.* for his Provision: How much will be receiv'd at *Bourdeaux* for 10 *l.* Sterling paid at *London*? *Answ.* $44\frac{8}{27}$ Crowns, *thus*, As 18 *d.* to 1 Guilder, so is 10 *l.* to $133\frac{1}{3}$ Guilders: Then, as 100 to $\frac{1}{3}$, so is $133\frac{1}{3}$ to $\frac{4}{9}$, the Provision; which taken from $133\frac{1}{3}$, there remain $132\frac{8}{9}$. Then, as 3 Guild. to 1 Crown so is $132\frac{8}{9}$ to $44\frac{8}{27}$.

Again; Suppose 'twere ask'd what was paid at *London* when 500 Crowns were receiv'd at *Paris*? To find this say, As 1 Cro. to 3 Guild. so 500 Cro. to 1500 Guild. which were paid at *Amsterdam* for the Remittance to *Paris*: Then, $\frac{1}{3}$ from 100, there remain $99\frac{2}{3}$; and as $99\frac{2}{3}$ to 100, so is 1500 to $1505\frac{5}{299}$, the Guilders receiv'd at *Amsterdam* by the Remittance from *London*. Lastly, As 1 Guild. to 18 *d.* so is $1505\frac{5}{299}$ Guild. to $27090\frac{90}{299}$ *d.* the Engl. Money sought. Or thus, As $99\frac{2}{3}$ to 100, so is 500 Cr. to $501\frac{201}{299}$, the Crowns that would have been got at *Paris* had no Allowance been deduced at *Amsterdam*: Therefore say, As 1 Cro. to 3 Guild. so $501\frac{201}{299}$ to $1505\frac{5}{299}$, the Guild. receiv'd at *Amsterdam*. By which find the English Money; as before.

Again, let us suppose the French and English Money are both known, as, 430 Cro. and 100 *l*, Sterling; To find the Allowance *per Cent.* which the Factor at *Amsterdam* has; say, As 18 *d.* to 1 Guild. so 100 *l.* to $1333\frac{1}{3}$ Guild. which was receiv'd at *Amsterdam*. Then find how many Crowns this is worth at 3 Guild. *per* Crown; if the number were 430, then there was no Allowance deduced at *Amsterdam*, but it is $444\frac{4}{9}$; from which take 430, the Remainder is $14\frac{4}{9}$: Then say, As $444\frac{4}{9}$ to $14\frac{4}{9}$, so is 100 to the Allowance upon 100.

Qu. 9*th*. Suppose *all* as in the preceding Question, with this further, That the Merchant at *London* draws upon *Bourdeaux* for the Crowns receiv'd there at 50 *d. per* Crown, paying 5 *s.* Brokerage; allowing also $\frac{1}{2}$ *per Cent.* to his Correspondent at *Paris*; What does he gain or lose by this Negotiation? *Answ.* He loses 1 *l* : 1 *s* : $4\frac{7}{27}$ *d.* which is discover'd thus; As 100 to $\frac{1}{2}$, so is $44\frac{8}{27}$ Crowns (receiv'd at *Paris*) to $\frac{299}{1350}$, the Allowance; which taken from $44\frac{8}{27}$, leaves $44\frac{101}{1350}$; which being drawn upon *Paris* at 50 *d. per* Crown, there is receiv'd at *London* for it $2203\frac{20}{27}$ *d.* or 9 *l* : 3 *s* : $7\frac{20}{27}$ *d*; from which subtract the Brokerage 5 *s.* the Remainder is 8 *l* : 18 *s* : $7\frac{20}{27}$ *d.* which the Merchant at *London* had clear for the Draught upon *Paris*: But he paid 10 *l.* for the Remittance; therefore he has lost 1 *l* : 01 *s* : $4\frac{7}{27}$ *d.*

Observe: If the Correspondent at *Bourdeaux* remits the Value in his hands to *London*, then, from the Sum receiv'd by the Remittance from *Amsterdam*, he deduces his Provision and Brokerage, and remits the Remainder to *London*.

Qu. 10*th.* A Rix-dollar is worth at *Amsterdam* 50 Sols, and at *Coningsberg* 90 Gros; the Exchange betwixt *Coningsberg* and *Amsterdam* is at 230 Gros for 6 Florins (equal to 120 Sols); whether is it most profitable that *Amsterdam* remit to *Coningsberg* in Specie, or by Exchange? *Answer.* 'Tis best to remit by Exchange: Which is discover'd thus; As 50 *s.* to 90 Gros, so 120 *s.* to 216 Gros, which must be remitted in Specie, (*i. e.* its Value in Rix-dollars) but by the Exchange there will be 230 Gros for 120 *s.* therefore 'tis best to remit by Exchange; the Difference is, 14 Gros will be had more for every 6 Florins.

Observe: If *Amsterdam* is to remit to *Coningsberg* 1000 Gros, and you would know which to chuse, and what is sav'd upon the whole; find what 1000 Gros will cost, both in Specie and by Exchange, and then you'll have the Difference, *thus*; As 90 Gros to 50 *s.* so 1000 to 555 *s.* which must be paid in Specie. *Again*; As 230 Gros to 120 *s.* so 1000 Gros to $521\frac{17}{23}$ *s.* to be paid by Exchange.

Qu. 11*th.* *Amsterdam* draws upon *Rouan* 400 Crowns at 87 *d.* Flemish *per* Crown, for which *Rouan* redraws upon *Amsterdam* at 90 *d.* with $\frac{1}{2}$ *per Cent.* for Provision; what has *Amsterdam* gain'd or lost? Say, As 1 Crown to 87 *d.* so 400 Crowns to 34800 *d.* receiv'd by Draught. *Again*, adding 2 for Provision to 400 Crowns, say, As 1 Crown to 90 *d.* so 402 to 36180 paid for the Redraught; so that *Amsterdam* has lost 1380 *d.*

Qu. 12*th.* *Amsterdam* remits to *Paris* 1000 Crowns at 78 *d. per* Crown, paying for Brokerage 540 *d.* which *Paris* remits to *Amsterdam* at 80 *d. per* Crown, rebating $\frac{1}{2}$ *per Cent.* for Provision; What is gain'd or lost? Say, As 1 Crown to 78 *d.* so 1000 Crowns to 78000; to which add 540, the Sum is 78540 *d.* given out. Again, subtract 5, the Provision, from 1000 Crowns, remains 995; then, As 1 Crown to 80 *d.* so 995 to 79600 *d.* receiv'd: So there is gain'd 1060 *d.*

Qu. 13*th.* If Exchange betwixt *Midleburg* and *London* is at 35 Sols for 1 *l.* Sterling, also betwixt *Midleburg* and *Amsterdam* at $1\frac{1}{2}$ *per Cent.* of Advance for *Midleburg*, (*i. e.* $101\frac{1}{2}$ *s.* at *Amsterdam*, worth 100 *s.* at *Midleburg*) at what Rate ought *Amsterdam* to remit to *London*, to receive the Return by *Midleburg* at the 'foresaid Course, and gain 5 *per Cent.*? Find the Course betwixt *London* and *Amsterdam*, after the manner of Question 4*th*, it is 1 *l.* for $35\frac{21}{40}$ *s.* Then, to remit to *London* with 5 *per Cent.* Gain, he must pay less than the Course, in proportion as 100 is less than 105; therefore say, As 105 to 100, so $35\frac{21}{40}$ to $33\frac{35}{42}$, which is the Rate at which he ought to remit to *London*. Or you may find this Answer at once, *thus*; As 105 to $101\frac{1}{2}$, so 35 to $33\frac{35}{42}$; the Reason of which you'll find easily, by comparing the two former Proportions, *viz.* $100 : 101\frac{1}{2} :: 35 : 35\frac{21}{40}$ and $105 : 100 :: 35\frac{21}{40} : 33\frac{35}{42}$; where the two middle Terms of the one being the two Extremes of the other, and the Products of Extremes and Means being equal from the nature of Proportion, it follows that $101\frac{1}{2} \times 35 = 105 \times 33\frac{35}{42}$; and hence $105 : 101\frac{1}{2} :: 35 : 33\frac{35}{42}$.

§ III. QUESTIONS *relating to what is call'd* The Arbitration *of* Exchange.

Qu. 14*th*. *A* of *Rochel* orders *B* of *Amsterdam* to draw upon him at $97\frac{3}{4}$ Sols for 1 Crown, and to remit the same to *Hamburg* at 34 Sols for 1 Dollar. *B* cannot draw, but at 97 Sols for 1 Crown; How ought he to remit to follow his Order? *Answ.* At $33\frac{17}{23}$ Sols for 1 Dollar: Found *thus*; As $97\frac{3}{4}$ to 97, so 34 to $33\frac{17}{23}$; for the Course being below the Order in the Draught to *Rochel*, it ought to be so proportionally in the Remittance to *Hamburg*. Or, the Reason of the Work may be conceiv'd *thus*; If for 1 Cro. which *A* pays at *Rochel* he gets only 97 Sols at *Amsterdam* by the Course, instead of $97\frac{3}{4}$ which was his Order, then for 1 Dollar he receives at *Hamburg* he ought to pay proportionally fewer Sols than 34, which was his Order.

Roch.		*Amst.*		*Hamb.*
Cr.		Sols.		
1	=	$97\frac{3}{4}$		Doll.
		34	=	1
1	=	97		

Qu. 15*th*. *A* of *Amsterdam* orders *B* of *Paris* to draw upon him at $99\frac{1}{2}$ Den. for 1 Crown, and to remit the same to *London* at $49\frac{3}{4}$ *d.* Sterling for 1 Crown. *B* can remit at $50\frac{1}{2}$ *d.* for 1 Crown; How ought he to draw to follow his Order? *Answ.* At 101 Den. for 1 Cro. Found *thus*; As $49\frac{3}{4}$ to $50\frac{1}{2}$, so is $99\frac{1}{2}$ to 101; for *A* receiving more than his Order at *London*, for 1 Crown paid at *Paris*, he ought to pay proportionally more than his Order at *Amsterdam*, for 1 Crown receiv'd at *Paris*.

Amst.		*Par.*		*Lond.*
Den.		Cr.		d.
$99\frac{1}{2}$	=	1	=	$49\frac{3}{4}$
		1	=	$50\frac{1}{2}$

Qu. 16*th*. *A* of *Coningsberg* orders B of *Amsterdam* to remit to *Rouan* at $103\frac{1}{2}$ *d.* Flemish for 1 Crown, and to draw the same on him at 225 Gross for 1 *l.* Flemish, *B* cannot draw under 230 Gross; How ought he to remit to follow his Order? *Answ.* At $101\frac{1}{4}$ *d.* Flem. for 1 Cro. Found thus: As 230 to 225, so is $103\frac{1}{2}$ to $101\frac{1}{4}$. For if *A* pays at *Coningsberg* more Gros than Order (for 1 *l.* receiv'd at *Amsterdam*) he ought to receive proportionally more Crowns than Order at *Rouan* (for $103\frac{1}{2}$ *d.* paid at *Amsterdam*) or, which is the same thing, he ought to pay proportionally less than Order at *Amsterdam*, for 1 Crown receiv'd at *Rouan*.

Con.		*Amst.*		*Rou.*
		d.		Cro.
Gro.		$103\frac{1}{2}$	=	1
225	=	1 *l.*		
230	=	1 *l.*		

Qu. 17*th*. *A* of *Cologn* orders *B* of *Amsterdam* to remit to *Dantzick* at 1 *l.* Flemish for 230 Gros, and to draw upon him at 100 for $102\frac{1}{2}$ to be paid at *Cologn*; but *B* remits at 1 *l.* for 228 Gros, and draws at 100 for 102; Has he follow'd his Order? and, if not, is *A* Gainer or Loser? *Answ.* He has not follow'd his Order, and *A* is Loser: Which is discover'd *thus*; As 230 Gross to 228, so is $102\frac{1}{2}$ to $101\frac{14}{23}$, which is the Number to be paid at *Cologn* for 100 receiv'd at *Amsterdam*, to make the Course and Order proportional in Drawing and Remitting; for if *A* gets less than Order at *Dantzick*, he ought to pay less than Order at *Cologn*, but by the Course he pays 102, therefore is a Loser.

Col.		*Amst.*		*Dant.*
		1 *l.*	=	230 gr.
$102\frac{1}{2}$	=	100		
		1 *l.*	=	228 gr.
102	=	100		

Qu. 18*th.* *A* of *Amsterdam* orders *B* of *London* to draw upon *Rouan* at 35 *d.* Sterling *per* Crown, and to remit the same to him at 1 *l.* Sterling for 35 Sols. But *B* draws at $34\frac{1}{2}$ *d.* *per* Crown, and remits the same at 1 *l.* for $36\frac{1}{2}$ Sols: Has he follow'd the Order? and, if not, is *A* Gainer or Loser? *Answ.* He has not follow'd his Order, and *A* is Gainer: Which is discover'd *thus*; As $36\frac{1}{2}$ Sols to 35, so is 35 *d.* to $33\frac{41}{73}$, the Number to be receiv'd at *London* for 1 Crown paid at *Rouan*, to make the Course and Order proportional in Drawing and Remitting; for it's plain, that if *A* receives at *Amsterdam* more Sols than Order, for 1 *l.* paid at *London*, he ought to pay at *Rouan* proportionally more Crowns than Order (for 35 *d.* receiv'd at *London*); or, which is the same thing, to receive proportionally less at *London*, for 1 Crown paid at *Rouan*; but he receives $34\frac{1}{2}$ *d.* and therefore is Gainer.

Amst.	*Lond.*	*Rou.*
	35 *d.* =	1 Cr.
35 Sols =	1 *l.*	
	$34\frac{1}{2}$ *d.* =	1 Cr.
$36\frac{1}{2}$ =	1 *l.*	

Qu. 19*th.* *A* of *Rouan* orders *B* of *Amsterdam* to draw upon him at 97 Den. for 1 Crown, or upon *London* at 35 Skilings for 1 *l.* Sterling. According to the Course, *B* can draw upon *Rouan* at 98 Den. for 1 Crown; and upon *London* at $35\frac{1}{2}$ Sk. for 1 *l.* Sterling: Which of them ought B to chuse to serve his Employer best? *Answ.* He ought to draw upon *London*: Which is thus discover'd; If *A* is willing to pay 1 Crown at *Rouan* for 97 *d.* receiv'd at *Amsterdam*, or to pay 1 *l.* at *London* for 35 Sk. receiv'd at *Amsterdam*; then 'tis plain, that to follow his Order, if *B* receives 98 *d.* instead of 97 (for 1 Crown) he ought to receive proportionally more than 35 Sk. (for 1 *l.*) And that proportional Number is $35\frac{35}{97}$; (for, As 97 to 98, so is 35 to $35\frac{35}{97}$) which is less than $35\frac{1}{2}$ receiv'd by the Course, so that the Course from *Amsterdam* to *London* exceeds the Order more in proportion than from *Amsterdam* to *Rouan*; and *A* will have more Money in proportion lying at *Amsterdam*, for the same Sum paid at *London*, than if the Draught were upon *Rouan*.

Rou.	*Amst.*	*Lond.*
1 Cro. =	97 *d.*	
	35 Sk. =	1 *l.*
1 Cro. =	98 *d.*	
	$35\frac{1}{2}$ Sk. =	1 *l.*

Qu. 20*th.* *A* of *Dantzick* Orders *B* of *Amsterdam* to remit to him, at 1 *l.* Flemish for 220 Gros; or to *Hamburg* at $33\frac{1}{2}$ Skilings for 1 Dollar; but the Course is at 1 *l.* for 218 Gros at *Dantzick*, and 34 Skilings for 1 Dollar at *Dantzick*; Which ought *B* to chuse to serve his Employer best? *Answ.* He ought to remit to *Dantzick*: Which is discover'd *thus*; Since *A* is willing to pay 1 *l.* at *Amsterdam* for 220 Gros at *Dantzick*, or to pay $33\frac{1}{2}$ Sk. at *Amsterdam* for 1 Dollar at *Hamburgh*; therefore if he can get only 218 Gros for 1 *l.* he ought to have proportionally more than 1 Dollar at *Hamburgh* for 34 Skil. at *Amsterdam*; or, which is the same thing, he ought to pay less than 34 Sk. for 1 Dollar: Therefore say, As 220 Gros to 218, so is $33\frac{1}{2}$ Sh. to $33\frac{43}{220}$, which would be paid at *Amsterdam* for 1 Dollar at *Hamburgh*, if the Course and Order were proportional to both Places; but by the Course *A* must pay 34 Sk. which is more than he would pay in proportion of the Order and Course to *Dantzick*; Therefore it's best that *B* remit to *Dantzick*.

Dantz.	*Amst.*	*Ham.*
220 gr. =	1 *l.*	
	$33\frac{1}{2}$ Sk. =	1 dol.
218 gr. =	1 *l.*	
	34 Sk. =	1 dol.

Observa-

Observation *relating to the last* 7 Questions.

There's one General Method may be taken with all these Questions, which is this: Reduce all the Given Proportions to such Numbers, as that those under the middle Place be the same in all (as 'tis in *Quest.* 15*th*); and you may also chuse that Number what you please; and then, from the Numbers under the first and last Place, you will easily find the Answer. *Thus*, in *Quest.* 14*th* say, As $97\frac{3}{4}$ is to 1, so is 34 to $\frac{136}{391}$. So that the Exchange betwixt *Rochel* and *Amsterdam* is reduced to 34 Sols for $\frac{136}{391}$ Crowns. Again say, As 97 to 1, so is 34 to $\frac{34}{97}$; the Question will stand as in the Margin. Then say, As $\frac{136}{391}$ to $\frac{34}{97}$, so 1 Dol. to $2\frac{3}{194}$ Dollars: So the Remittance to *Hamburgh* ought to be at 34 Sols for $1\frac{3}{388}$ Dollars; which is the same Proportion as found the other Way, only in different Numbers. The Reason of the Work will be in some Cases clearer by this Method; but the Work often more tedious.

Roch.	*Amst.*	*Hamb.*
$\frac{136}{391}$ Cr. = 34 *s.*		
	34 =	1 Dol.
$\frac{34}{97}$ = 34		

§ IV. *Containing a few* Questions *of another kind than any of the preceding, for a farther Exercise upon this Subject.*

Qu. 21*st*. A Merchant would exchange 200 *l.* Sterling for Dollars or Crowns: He is offer'd Dollars at 4 *s*: 6 *d.* which are worth but 4 *s*: 3 *d.* or Crowns at 5 *s.* worth but 4 *s*: 8 *d.* Which of them shall he take to lose least? and, How many will he receive? Find how many Dollars at 4 *s*: 6 *d.* and Crowns at 5 *s.* he would get for 200 *l.* then find the Value of that number of Dollars at 4 *s*: 3 *d.* and that number of Crowns at 4 *s*: 8 *d.* the Comparison of these Values will shew which is of greatest Value; and the Value of that which is the greatest, compar'd with 200 *l*, shews what he loses.

Qu. 22*d*. A Merchant at *Amsterdam* drew Bill upon *London* for 300 *l.* Sterling, receiving the Value in Crowns at 4 *s*: 6 *d.* and Dollars at 4 *s.* and got an equal number of each; What is that number? Add 4 *s*: 6 *d.* to 4 *s.* and say, If the Sum 8 *s*: 6 *d.* buy 1 of each Species, how many times 1 of each Species will 300 *l.* buy? The *Answer* is, $705\frac{90}{102}$, (for, dividing 300 *l.* by 8 *s*: 6 *d.* the whole is 705, and 90 remains) so he receiv'd 705 Dollars, and as many Crowns, with $\frac{90}{102}$ parts of 1 of each. And because there is a Remainder in the Division, therefore this shews that the Exchange cannot be made exactly, by a certain number of Crowns and Dollars; so that 90 *d.* remaining, the 300 *l.* is 90 *d.* better than the Sum of 705 Dollars, and 705 Crowns: Wherefore he who receives only 705 Dollars and 705 Crowns, must give 90 *d.* less than 300 *l.*

Again; If the Proportion of the number of Crowns and Dollars is suppos'd to be any other than Equality (for example, 2 Dollars for every 3 Crowns) then add the Value of 2 Dollars and 3 Crowns, and divide by that Sum: The Quote shews how many times 2 Dollars and 3 Crowns are to be receiv'd; and if there is a Remainder, 'tis to be consider'd as so many Units of the Denominator of the Divisor; and so much the Dividend is of more Value than the number of Crowns and Dollars found.

If

If there are more than two different Species, as, *Crowns*, *Dollars*, *Ducats*, *Pistoles*, the manner of working is the same; for, if an equal number of each is suppos'd, then add the value of an Unit of each, and by that Sum divide: If their numbers are not equal, then either (1°) the correspondent Numbers of each that are equal Numbers of Times taken is given, as, for every 2 Dollars 3 Ducats, 5 Crowns, and 1 Pistole: And here we add the value of 2 Dollars, 3 Ducats, 5 Crowns, and 1 Pistole, and by that Sum divide. (2°) If the Proportion of the Numbers are given, but not in one Series, as, suppose for 3 Dollars 2 Ducats, for 3 Ducats 4 Crowns, and for 7 Crowns 1 Pistole; then we must reduce these Proportions to one Series of correspondent Numbers of each, *thus*; Keep the first two Numbers, *viz.* 3 Dollars 2 Ducats, then find how many Crowns for 2 Ducats (at 3 Ducats for 4 Crowns) and how many Pistoles for that number of Crowns last found (at the rate of 1 Pistole to 7 Crowns) then proceed as before, by adding the Values of these correspondent Numbers of the different Species; and, to go thro' the reduction of the Proportions more orderly, set the Species and their proportional Numbers down as here:

Doll.	:	Duc.	:	Crowns	:	Pistoles.
3	:	2				
		3	:	4		
				7	:	1

Qu. 23*d.* If I receive 11 Crowns and 7 Dollars for 4 *l* : 10 *s* : 10 *d.* or 4 Crowns and 3 Dollars for 1 *l* : 15 *s.* the Value of 1 Crown and 1 Dollar being the same in both, What is that Value?

This Question may be solv'd two Ways; (1°) Reduce the Money all to Pence; then, to make the same number of Dollars in both Cases, multiply the 1090 *d.* and its equivalent number of Crowns and Dollars by 3, also the 420 *d.* and its equivalent number of Crowns and Dollars by 7, the Products must still be of equal value: And if the one Line of Products be taken from the other, it's manifest that the Remainders will also be equal; and, because the Dollars are equal in both, therefore there are none in the Remainders; and so we have found that 5 Crowns are equal to 330 *d.* consequently 1 Crown is 5 *s* : 6 *d.* Then, to find the Value of 1 Dollar, multiply 5 *s* : 6 *d.* by 4, the Product is 22 *s.* the Value of 4 Crowns; which taken from 35 *s.* the Value of 4 Crowns and 3 Dollars, there remains 13 *s.* the Value of 3 Dollars; wherefore 4 *s* : 4 *d.* is the Value of 1 Dollar.

			Cr.		Dol.
4 *l* : 10 *s* : 10 *d.*	or	1090 *d.* =	11	+	7
1 *l* : 15 *s.*	or	420 *d.* =	4	+	3
		3270 *d.* =	33	+	21
		2940 *d.* =	28	+	21
		330 *d.* =	5 Cro.		

(2°) We may also solve it *thus*; say, If 1090 *d.* buy 18 Pieces (*viz.* 11 Crowns and 7 Dollars) how many will 420 *d.* buy? The Answer is, $6\frac{102}{109}$: Then I divide this Number into Crowns and Dollars, in the same proportion as 18 is to 11 Crowns and 7 Dollars, *thus*; As 18 to 11 Crowns, so $6\frac{102}{109}$ to $4\frac{26}{109}$ Crowns: And this taken from $6\frac{102}{109}$ there remains $2\frac{76}{109}$ Dollars. And, because the same 420 *d.* buy 4 Crowns and 3 Dollars,

therefore

$4\frac{26}{109}$ Cr. $+ 2\frac{76}{109}$ Dol. $= 4$ Cr. $+ 3$ Dol.
$\frac{26}{109}$ Cr. $= \frac{33}{109}$ Dol. or 26 Cr. $= 33$ Dol.

therefore $4\frac{26}{109}$ Crowns and $2\frac{76}{109}$ Dollars are of equal Value with 4 Crowns and 3 Dollars: And if we cast equal numbers of the same Species out of both sides, the Remainders will still be equal: So cast out 4 Crowns and $2\frac{76}{109}$ Dollars, the Remainders are $\frac{26}{109}$ Crowns, equal to $\frac{33}{109}$ Dollars; and consequently 26 Crowns equal to 33 Dollars. Then say, If 33 Dollars are worth 26 Crowns, how many Crowns are 3 Dollars worth? It is $2\frac{4}{11}$; then, Consequently, 4 Crowns and $2\frac{4}{11}$ Crowns (which are worth 3 Dollars) are worth 420 *d.* because 4 Crowns and 3 Dollars are worth 420 *d.* Lastly, If $6\frac{4}{11}$ Crowns are worth 420 *d*, 1 Crown is worth 66 *d.* or 5 *s* : 6 *d.* By which find the Value of the Dollar; as before.

Of the Reduction *of* Weights *and* Measures.

The *Reduction* of *Weights* and *Measures* of different Places is done after the same manner as Money and Coins. The Proportions being known either immediately or mediately, thro' several different Places, therefore I shall give only two Examples.

Qu. 1*st.* If 1 Eln of *Amsterdam* is equal to $1\frac{1}{5}$ of *London*, how many Elns of *Amsterdam* are in 1000 *English* Elns? Say, If $1\frac{1}{5}$ give 1, what will 1000? It is $833\frac{1}{3}$.

Qu. 2*d.* If 3 *lb* weight at *A* are equal to 2 *lb* at *B*, and 5 *lb* at *B* equal to 2 *lb* at *C*, and 7 *lb* at *C* equal to 8 *lb* at *D*; What is the Proportion betwixt *A* and *D*?

$$\begin{array}{cccc} A & : B & : C & : D \\ 3 = & 2 & & \\ & 5 = & 2 & \\ & & 7 = & 8 \\ \hline 3 = & 2 = & \frac{4}{5} = & \frac{32}{35} \end{array}$$

Set down the Names and Numbers given, as in the Margin; and work as directed in *Qu.* 5*th*, whereby you will find not only the Proportion of the first and last Places, but of all the Places to one another: so here 3 *lb* at *A* is equal to 2 *lb* at *B*, $\frac{4}{5}$ at *C*, and $\frac{32}{35}$ at *D*.

CHAP.

CHAP. X.

Of Intereſt *and* Annuities.

§. 1. *Of* Intereſt.

DEFIN. 1. INTEREST is the Præmium or Money paid for the Loan or Uſe of Money; and is diſtinguiſh'd into two Kinds, *Simple* and *Compound*.

2. *Simple Intereſt* is that which is paid for the *Principal*, or Sum lent, at a certain Rate or Allowance made by Law, (or Agreement of Parties) whereby ſo much as 5 *l.* or 6 *l.* or any other Sum, is paid for 100 *l.* lent out for one Year; and more or leſs proportionally for greater or leſſer Sums; and for more or leſs time. For *Example*: If it's 6 *l.* to 100 for one Year, it's 3 *l.* for half a Year, and 12 *l.* for two Years. Alſo 12 *l.* for one Year of 200 *l.* and 6 *l.* for half a Year; and ſo on for other Sums and Times.

3 *Compound Intereſt* is that which is paid for any principal Sum, and the ſimple Intereſt due upon it for any time, accumulated into one principal Sum. *Example*: If 100 *l.* is lent out for one Year at 6 *l.* and if at the End of that Year the 6 *l.* due of Intereſt be added to the Principal; and the Sum 106 *l.* conſider'd as a new Principal bearing Intereſt for the next Year, (or whatever leſs time it remains unpaid) this is called *Compound Intereſt*, becauſe there is Intereſt upon Intereſt, which may go on, by adding this ſecond Year's Intereſt of 106 *l.* to the Principal 106 *l.* and making the Whole a Principal for the next Year.

SCHOLIUM. Our Law allows only *Simple Intereſt*: But abſtracting from the Reaſon of the Law, [which may be the encouraging of Trade, by employing Money that way rather than upon Intereſt] if taking Intereſt be at all juſt, *Compound Intereſt* cannot be unreaſonable. For if I can demand my Intereſt when it is due, I may take that Intereſt Money, and lend it out again upon Intereſt to any other Perſon; why then may I not lend it out alſo to the Perſon who has my principal Sum? And, in point of Right and Juſtice, it is the ſame thing if I continue or leave that Intereſt in his Hands: there is the ſame Reaſon that it ſhould bear Intereſt after it becomes due, as that the original Sum ſhould do ſo.

PART I. *Of* Simple Intereſt.

We have already ſeen in the Rule of Five, how, from any ſuppoſed principal Sum, with the ſuppoſed Intereſt of it for any ſuppoſed time, we can find at that rate, or upon that Suppoſition, the Intereſt of any other principal Sum for any time; or the Principal correſponding to any Sum of Intereſt and Time; or laſtly, the Time in which any Principal gives any Intereſt.

We ſhall now conſider the Application of that Rule more particularly; by limiting the Queſtions to the more common Circumſtances of Buſineſs. Thus: As the Law, or Agreement of Parties, fixes a certain Ratio, or, as we call it, Rate of Intereſt, which is ſo much on the 100 *l* for one Year; from this we can eaſily find the proportional Intereſt on 1 *l.* for one Year, being plainly the $\frac{1}{100}$ Part of the Intereſt of 100 *l*; ſo if this is 5 *l.* that is .05 *l.*, if this is 6 *l.* that is .06 *l.*, and if this is 5 *l.* 10 *s.* or 5.5 *l.* that is .055 *l.* Wherefore

fore if we underſtand the Rate of Intereſt to be the Intereſt of 1 *l.* for one Year, the more common Queſtions about ſimple Intereſt will relate to theſe four things, *viz.* any principal Sum, its Intereſt, the Time in which it gives that Intereſt, and the Rate (or Intereſt of 1 *l.* for one Year) according to which that Principal, Intereſt and Time are adjuſted to one another.

From which we have four *Problems*: In the Rules whereof, I ſuppoſe the Principal and Intereſt expreſſed in the Denomination of Pounds; by reducing what is leſs than 1 *l.* to a Decimal of 1 *l.* and the Time to be expreſſed in Years, and decimal Parts of one Year.

PROBLEM I. Having any principal Sum, and Time, with the Rate of Intereſt given, to find the Intereſt of that Sum for that Time and Rate.

Rule. Multiply the Principal, Rate, and Time continually into one another, the Product is the Intereſt ſought.

Obſerve; If we expreſs the Principal by p, the Intereſt by n, the Time by t, and the Rate by r; then this Rule is thus repreſented, $n=ptr$.

Example. The Rate of Intereſt being .05 *l.* what is the Intereſt of 85 *l.* for 4 Years and 3 quarters, or 4.75 Years? Anſwer, 20 *l.* 3 *s.* 9 *d.* $= 20.1875\ l. = 85 \times 4.75 \times .05$.

DEMONSTR. If we ſtate this Queſtion by the Rule of Five, it ſtands thus: If 1 *l.* in 1 yr gives .05 *l.* what 85 *l.* in 4.75 yr, and the two firſt Terms are the Diviſors in the two ſimple Proportions, but theſe being both Units, the Anſwer is the Product of the other three. The Reaſon is the ſame in all Caſes, by putting r, p, and t in place of theſe particular Numbers. Or we ſhall repeat the Reaſoning thus: If 1 *l.* give r, p will give pr in the ſame time. Again; if p in one Year give rp, in t Years it muſt give $t \times rp$ or trp.

PROB. 2. Having the Rate, Principal and Intereſt to find the Time.

Rule. Divide the Intereſt by the Product of the Rate and Principal, the Quote is the Time; thus, $t=\frac{n}{rp}$.

Example. The Rate .05 *l.* Principal 85 *l.* Intereſt 20 *l.* 3 *s.* 9 *d.* or 20.1875 *l.* The Time is 4.75 Years, or 4 Years and $\frac{3}{4}$. Thus; $4.75=\frac{20.1875}{85 \times .05}$ or $\frac{20.1875}{4.25}$.

DEMONSTR. This Rule is deduced from the former; thus, Since $n=trp$; then dividing both Sides by rp, it is $\frac{n}{rp}=t$. Or it may be deduced from the Rule of Five, as the former.

PROB. 3. Having the Principal, Intereſt, and Time, to find the Rate.

Rule. Divide the Intereſt by the Product of Principal and Time, the Quote is the Rate. Thus $\frac{n}{tp}=r$.

Example. $n=20.1875\ l.$ $t=4.75$ yr. $p=85\ l.$ then is $r=.05\ l.=\frac{20.1875}{4.75 \times 85}$ or $\frac{20.1875}{403.75}$.

DEMONSTR. Since $n=trp$, divide both by tp; it is $\frac{n}{tp}=r$.

PROB. 4. Having the Rate, Time and Intereſt, to find the Principal.

Rule. Divide the Intereſt by the Product of Rate and Time, the Quote is the Principal; thus, $\frac{n}{tr}=p$.

Exam.

Exam. $n = 20.1875\, l.$ $t = 4.75$ yr. $r = .05\, l.$; then is $p = 85\, l. = \frac{20.1875}{4.75 \times .05}$ or $\frac{20.1875}{.2375}$.

DEMONSTR. Since $n = trp$, divide both by tr, the Quotes are $\frac{n}{tr} = p$.

SCHOLIUM. If the Intereſt of any Sum for any time is added to the Principal, this Total or Sum is called the *Amount*, (*viz.* of the Principal and its Intereſt for that time.) And then from theſe four things, *viz.* the *Amount*, (which we ſhall call a) the Principal, the Time and Rate, ariſe other four *Problems*; for each of theſe may be found from the other three. Thus:

PROB. 5. Having the Principal, Time and Rate, to find the Amount.

Rule. Find the Intereſt by *Prob.* 1. Add it to the Principal, the Sum is the Amount. Thus, by *Prob.* 1. the Intereſt is rtp; therefore the Amount is $a = rtp + p$. The Reaſon is evident.

And obſerve, Becauſe $rtp = rt \times p$, and $p = 1 \times p$; therefore $rtp + p = \overline{rt + 1} \times p = a$. And ſo the Rule may be expreſſed thus: To the Product of the Rate and Time add Unity; and multiply the Sum by the Principal, the Product is the Amount.

Example. What is the Amount of $246\, l.$ Principal in 2 Years and $\frac{1}{2}$, or 2.5 yr, the Rate of Intereſt being $.05\, l$? Anſwer, $246\, l. + 30.75\, l. = 276\, l.$ 15 *s.* for the Intereſt is $= 246 \times .05 \times 2.5 = 30.75\, l.$ Or thus; $.05 \times 2.5 = .125\, l.$ to which add 1, it is $1 + .125\, l.$ which multiplied by 246, produces $276.75\, l.$

PROB. 6. Given the Principal, Amount and Time, to find the Rate.

Rule. Take the Difference betwixt the Principal and Amount, and divide it by the Product of the Time and Principal, the Quote is the Rate. Thus, $r = \frac{a - p}{tp}$.

Exam. Suppoſe $a = 276.75\, l.$ $p = 246$, $t = 2.5$ yr; then is $r = .05\, l. = \frac{276.75 - 246}{2.5 \times 246} = \frac{30.75}{615}$.

DEMONSTR. Since by *Prob.* 5. $a = trp + p$; take p from both Sides, it is $a - p = trp$; then divide both by tp, it is $\frac{a-p}{tp} = r$.

Or we may deduce it thus; $a - p$ is the Intereſt of p for the Time t, and Rate r; then by the Rule of Five, find what is the Intereſt of it for one Year, when the Principal p gives $a - p$ in t Years; the State of which is thus, $p \,.\, t \,.\, \overline{a - p} \,..\, 1 \,.\, 1$. By that Rule it is $\frac{a - p}{tr}$; for the 4th and 5th Terms are Units, and the middle Term is to be multiplied into theſe two, whoſe Product is nothing but $\overline{a - p}$.

PROB. 7. Given the Amount, Principal and Rate, to find the Time.

Rule. Take the Difference of the Amount and Principal, and divide it by the Product of the Principal and Rate, the Quote is the Time. Thus, $t = \frac{a - p}{rp}$.

Exam. Suppoſe $a = 276.75\, l.$ $p = 246$. $r = .05$; then is $t = 2.5$ yr. $= \frac{276.75 - 246}{246 \times .05} = \frac{30.75}{12.3}$.

DEMONSTR. In the last we saw $a - p = trp$. Divide both by rp, it is $\frac{a-p}{rp} = t$. Or by the Rule of Five, thus, If 1 *l.* give r in 1 y^r. in what Time will p give $\overline{a-p}$, following the common Rule, the Time sought is $\frac{a-p}{rp}$ as before.

PROB. 8. Given the Amount, Rate and Time, to find the Principal.

Rule. Add 1 to the Product of the Rate and Time, and by that Sum divide the Amount, the Quote is the Principal; thus, $p = \frac{a}{rt+1}$.

Exam. $a = 276.75$ *l.* $r = .05$ *l.* $t = 2.5$ y^r.; then is $p = 246 = \frac{276.75}{2.5 \times .05 + 1} = \frac{276.75}{1.125}$.

DEMONSTR. By *Prob.* 5. it is $a = \overline{rt+1} \times p$; divide both by $\overline{rt+1}$, it is $\frac{a}{rt+1} = p$. Or from the Rules of Proportion; thus, Find the Interest of 1 *l.* for the Time given, at the given Rate, it is rt for 1 y^r. : r :: t : rt; then find a Principal so proportioned to a, as 1 *l.* is to $1 + rt$; thus, $1 + rt : 1 :: a : \frac{a}{1+rt}$.

SCHOLIUM. *Concerning the* Rebate *or* Discompt *to be allowed for paying of Money before it falls due.*

IN the last *Problem*, we have the Foundation of the Rule for *Discompt*, or the Allowance to be made for the paying of Money before it falls due; and which is supposed to bear no Interest till after it is due: for in that Case there is no Reason for Discompt, as there is in the other; which is this, That the Debtor can employ his Money upon Interest, (or as he pleases) till the time of Payment comes; and to pay it before it is due, is to communicate that Benefit to the Creditor, who ought therefore to pay for it; and the Question is, What Discompt or Allowance is to be made? The Debtor will be apt to argue in this manner: By paying the Debt before it falls due, I lose the Interest I could make of it till then, and therefore all that Interest must be discompted. But this Argument is false: For tho' it be true, that from this time to the time the Debt falls due, he would make so much Interest; yet it is not true, that by paying the whole Debt without Discompt, he loses so much at this time as that Interest; because he can't be said to have lost it, till the time come at which he should receive it; therefore he can be said to lose no more at this time than such a Sum, as being laid out upon Interest from this time, till the time of Payment of the Debt would amount to the Interest of the Debt for the same time; therefore such a Sum being found by *Prob.* 8. is the Discompt. In order to which, find the Amount of 1 *l.* for the given time; then say, If that Amount give 1 *l.* what will the Interest of the Debt for the time give.

This I shall also confirm by another Rule for finding the Discompt; which is this: Find by *Prob.* 8. a Principal, which being laid out at a certain Rate of Interest, for so long as a Debt is paid before it was really due, will then amount to that Debt; that Sum is the thing which truly satisfies or clears the Debt, and so is called the present Worth of it; the Difference betwixt this and the Debt being the true *Discompt*. The Reason is this; If the Money paid is such, that being laid out at Interest from the time it is paid to the time the Debt falls due, would amount to the Debt, then neither Party is wronged; for if he who receives it, or he who pays it, lays it but upon Interest so long, he will have then as much to receive as the Debt. For *Example*: If Interest is at 5 *per Cent.* 100 *l.* laid out for one Year

Year is worth 105 *l.* at the Year's End; therefore 105 *l.* due at a Year's End is worth only 100 *l.* presently paid, and the Discompt is 5 *l.* which is one Year's Interest of 100 *l.* not of 105 *l.*

Now I shall demonstrate, that both the Rules given for finding of *Discompt* are in effect the same. Thus: The Interest of 1 *l.* for the time t (*viz.* the time of the Discompt) at the Rate r is rt, therefore the Amount of 1 *l.* for that time is $1+rt$; and the Principal corresponding to rpt (the Interest of p for the same time) in the same Proportion as 1 to $1+rt$ is $\frac{rpt}{1+rt}$, for $1+rt : 1 :: rpt : \frac{rpt}{1+rt}$; which is the Discompt of the Debt by the first Rule. Again; the Principal corresponding to p (the Debt) in the same Proportion as $1+rt$ to 1, is $\frac{p}{1+rt}$, for $1+rt : 1 :: p : \frac{p}{1+rt}$; which is the true Payment to be made; which subtracted from p, the Remainder is $p - \frac{p}{1+rt} = \frac{prt}{1+rt}$, the Discompt by the former Rule.

To sum up this, let p be any Debt payable after the time t, and r the Rate of Interest; then

The RULES *for* Present Worth *and* Discompt *are*

1. *Present-Worth* $= \frac{p}{1+rt}$

i.e. Divide the Debt by the Sum of 1, and the Product of the Rate and Time.

Discompt $= \frac{prt}{1+rt}$

i.e. Divide the continual Product of the Debt, Rate and Time, by the Sum of 1, and the Product of Rate and Time.

Observe, The Discompt upon any Debt for one Year being found, (which will be $\frac{pr}{1+r}$, for t being here one, has no Effect). Dr. *Harris* teaches to find the Discompt of the same Debt for any Part of one Year, in proportion to the Time; so the Discompt for $\frac{1}{2}$ a Year is $\frac{1}{2}$ of the Discompt for one Year. But the Error of this Rule will appear by comparing it with the preceding general one, thus; One Year's Discompt is $\frac{pr}{1+r}$; Let t be any Time less than 1 Year, expressed fractionally, then the Doctor's Rule is $1 : \frac{pr}{1+r} :: t : \frac{prt}{1+r}$ = the Discompt sought: Whereas by the Rule above demonstrated, it is $\frac{prt}{1+rt}$: Which will be greater than the other if t is a Fraction; because then $1+rt$ is less than $1+r$; but if t is a whole or mixt Number, $1+rt$ is greater than $1+r$; whence $\frac{prt}{1+rt}$ is less than $\frac{prt}{1+r}$.

PART II. *Of* Compound Interest.

THE Calculation of *Compound Interest* supposes a certain stated Time for which Interest is at a certain determinate Rate; after which that Interest becomes a Principal bearing Interest. For *Example*: If Simple Interest is payable yearly, so is the Compound; and if Simple Interest is payable Quarterly and Monthly, so is the Compound: But then for the Interest of any Time less than that to which the Rate is determined, there are different Opinions about the Way of calculating it, which I shall explain in the following *Problem*; where I shall first consider the Rate of Interest as determined to one Year, and the

the Time of the Queſtion limited to whole Years, and afterwards conſider other Suppoſitions about the Rate and Time.

Obſerve, again; That the finding of the whole Improvement of Compound Intereſt depends upon the Rule for finding the Amount; therefore the Queſtions in which the Amount is concerned muſt be explained before theſe in which the Intereſt alone is concerned, contrary to what we did in Simple Intereſt.

In the following *Problems*, by the *Rate* is underſtood the Amount of 1 *l.* and 1 Year's Simple Intereſt. So Intereſt being at 5 *per Cent.* the Rate or Amount of it is 1.05 *l.* which I ſhall mark R, the Time t being whole Years.

PROBL. I. Given the Principal, Rate, and Time, to find the Amount.

Rule. Find ſuch a Power of the Rate whoſe Index is the Time (*i.e.* multiply the Rate by it ſelf, and this Product by the Rate, and ſo on, multiplying the laſt Product by the Rate, till the Number of Multiplications be equal to the Time leſs 1; ſo for 2 Years multiply once; for 3 Years multiply twice, *&c.*) This Power of the Rate is the Amount of 1 *l.* for the Time and Rate given; which, multiplied by the given Principal, the Product is the Amount ſought.

Univerſally; $A = p \times R^t$: And *obſerve*, that the Logarithms will be convenient for finding R^t, if t is a great Number.

Example: What is the Amount of 160 *l* at Compound Intereſt for 4 Years, the Rate being 1.06 *l.* (*i.e.* 6 *per Cent.*)? *Anſwer*: 201.9963136 *l*; which is 201 *l.*: 19 *ſh.* 11 *d.* and a Fraction leſs than 1 Farthing. Found thus:

$R = 1.06$. and $R^t = \overline{1.06}^4$, or $1.06 \times 1.06 \times 1.06 \times 1.06 = 1.26247696$, then $p \times R^t = 160 \times 1.26247696 = 201.9963136$.

DEMONSTR. If 1 *l.* amount in 1 Year to R, then all this forborn another Year, the Amount is $R \times R$ or R^2; for all principal Sums have neceſſarily the ſame Proportion to their Amounts for the ſame Time and Rate, and $1 : R :: R : R^2$, which is therefore the Amount of R for 1 Year; R being the Amount of 1. For the ſame Reaſon R^2 forborn another Year will amount to $R \times R \times R$, or R^3; for $1 : R :: R^2 : R^3$, and ſo on, to any Number of Years, which being called t, the Amount of 1 *l* for t Years, is R^t: conſequently the Amount of any other Principal R for t Years, is $p \times R^t$; for $1 : R^t :: p : p \times R^t$. Or thus: R being the Amount of 1 *l.* for 1 Year, pR is that of p. for $1 : R :: p : pR$. Again; $1 : R :: pR : pR^2$, the Amount of p for 2 Years, and ſo on; *i.e.* Univerſally it is pR^t.

SCHOLIUMS.

1. The ſucceſſive Amounts of any Principal for 1, 2, 3, *&c.* Years, make a Geometrical Progreſſion, whoſe Ratio is the Amount of 1 *l.* for 1 Year. Thus: The Series of the Amounts of 1 *l.* is $R : R^2 : R^3 : R^4$: *&c.* and of any other Principal p, it is $pR : pR^2 : pR^3 : pR^4$, *&c.* And if we make the Principal the 1ſt Term, the whole is a Geometrical Progreſſion, *viz.* $1 : R : R^2 : R^3$, *&c.* or $p : pR : pR^2 : pR^3$, *&c.*

2. If the Rate of Intereſt is determined to any other Time than a Year, as $\frac{1}{2}$, or $\frac{1}{4}$ of a Year, the Rule is the ſame; and then t repreſents that ſtated Time.

But whatever the ſtated Time is, it remains to be explained, how the Intereſt or Amount of any Sum is to be calculated for a leſſer time; I ſhall ſuppoſe 1 Year the ſtated Time, becauſe it is ſo in Law; and whatever the Rules are for the Parts of a Year, they are equally applicable to the Parts of any other Time, to which the Rate of Intereſt may be ſuppoſed to be determined.

Of the Compound Intereſt *or* Amount *of any Sum for the Parts of a Year; the Rate of Intereſt being determined to* 1 *Year.*

Method 1. Some will have it done in Simple Proportion to the Time, or, as ſimple Intereſt; becauſe, ſay they, Compound Intereſt muſt ſuppoſe a certain Time for which Intereſt is at a ſimple ſtated Rate, and after that becomes a Principal bearing Intereſt; ſo that this Time being 1 Year, there can be no ſuch thing as Compound Intereſt for any Part of a Year.

Wherefore, following this Rule, If the Time of a Queſtion is leſs than 1 Year, the Amount is found by the Rule of Simple Intereſt; and if there are whole Years, and part of a Year, then, having found the Amount for the whole Years (which is only Simple Intereſt if there is but 1 Year), take that Amount, and find what it amounts to at Simple Intereſt for the remaining Time leſs than 1 Year.

Method 2 Others proceed upon this Principle, *viz.* That ſince the Rate of Intereſt is determined to 1 Year, this, ſay they, ſuppoſes that all the Improvement that can be made of any Sum by Intereſt in 1 Year, is the ſtated Rate of 5 or 6, *&c. per Cent.* But if Money is Lent for any Time leſs than 1 Year, and Intereſt received for it in ſimple Proportion to the Time, then, by lending out the Whole again, more will be made by it than 5 or 6 *per Cent.* in 1 Year. Therefore they would have the Amount for any Part of a Year calculated ſo, that if this is again conſidered as a Principal bearing Intereſt, it ſhall, after ſo much Time as the former wants of 1 Year, amount, at the ſame Rate, only to the Sum of the Principal and its Simple Intereſt for 1 Year. For *Example:* The Amount for $\frac{1}{4}$ of a Year is ſuch, that being put out to Intereſt for the remaining $\frac{3}{4}$, and ſuppoſed to bear Compound Intereſt from Quarter to Quarter, at the Rate of Simple Intereſt allowed for the 1ſt Quarter, the Amount at the End of the $\frac{3}{4}$ ſhall only be the Sum of the Principal and 1 Year's Simple Intereſt, at the allowed Rate of 5 or 6, &c. *per Cent.*

How this may be done, I ſhall preſently explain, after this general Reflection, *viz.* That as the Law determines the Intereſt of Money lying in the hands of the ſame Perſon to be at Simple Intereſt, ſo any Perſon who puts out Money at Intereſt, and calling it in as oft as poſſible, puts out the Whole again, improves the Money by Compound Intereſt without Breach of Law: Therefore the ſame Reaſon that juſtifies Compound Intereſt from Year to Year, ſeems equally to juſtify it from Quarter to Quarter, or from Day to Day; And ſince the Law allows Intereſt for Parts of a Year in Proportion to the time, it would ſeem alſo by this that the ſtated Intereſt for 1 Quarter or 1 Day, ought to be the proportional Part of one Year's Intereſt, and then the Compound Intereſt to be calculated accordingly from Day to Day. The only Objection to this is, that it is impoſſible for one to lend Money, and be paid from Day to Day, or even by Months or Quarters; and therefore that Foundation is unreaſonable, and conſequently one of the other two Methods muſt be taken; but which of them is to be choſen for the Parts of a Year, let every one determine for themſelves; [for it ſeems to me to depend all upon a Suppoſition] this only I ſhall further obſerve, that as the Simple Intereſt for the Parts of 1 Year is greater than that found by the 2d *Method,* it ſeems reaſonable that he who receives only Simple Intereſt for whole Years, ſhould have the Advantage of the proportional Part for a leſſer Time; and he who has compound Intereſt for whole Years ſhould have Intereſt for the Parts of a Year at a leſſer Rate, by the 2d *Method.*

Now if Intereſt for the Parts of a Year is taken by the 1ſt *Method,* the Rule is already given; but if it's taken upon the other Foundation, then to make the Rule more clear, I ſhall firſt ſuppoſe 1 *l.* Principal, and then apply it to other Caſes. Alſo, we muſt diſtinguiſh according as the Time is or is not an aliquot Part of a Year.

Rule 1. If the Time is an aliquot Part of a Year, as $\frac{1}{2}$, $\frac{1}{3}$, or, universally, $\frac{1}{n}$ Part, take the Amount of 1 *l.* for 1 Year, and from it extract the nth Root (*i. e.* the Square Root if it's $\frac{1}{2}$ a Year, the Cube Root if it's $\frac{1}{3}$, the Biquadrate, or 4th Root if it's $\frac{1}{4}$); that Root is the Amount sought.

Example. At 5 *per Cent.* the Amount of 1 *l.* for $\frac{1}{2}$ a Year, is 1.02469, *&c*, this being nearly the Square Root of 1.05

DEMONSTR. $1 : \overline{1.05}^{\frac{1}{2}} :: \overline{1.05}^{\frac{1}{2}} : 1.05$, therefore $\overline{1.05}^{\frac{1}{2}}$ (or 1.02469) put out at Interest for $\frac{1}{2}$ a Year amounts to 1.05, according to the Rate allowed for the 1st Half Year, which is 1.02469, the Principal for the 2d Half Year.

If the Time is $\frac{1}{3}$ of a Year, then, if we suppose two Geometrical Mean Proportionals betwixt 1 and R, thus, $1 : a : b : R$, then must a be the Cube Root of R, from the Nature of Geometrical Progressions; for whatever a is b is equal to a^2, for $1 : a :: a : a^2$, but it is supposed that $1 : a :: a : b$, therefore $a^2 = b$. Again; R is $= a^3$, for $1 : a :: b : R$, that is $1 : a :: a^2 \ (= b) : R$, but also $1 : a :: a^2 : a^2 \times a = a^3$, consequently, $R = a^3$, and hence $a = R^{\frac{1}{3}}$. The same Reason is good in all Cases, that is, however many, as n, Terms there be in a Geometrical Progression from 1 to R, as $1 : a : b : c$, &c. : R, or $1 : a : a^2 : a^3$, &c. : R, the first of them, a, is such a Root of R whose Denominator is the Number of Terms, as $R^{\frac{1}{n}}$. Again; According to the supposed Foundation, a, (or $R^{\frac{1}{n}}$) is a Principal, which bearing Compound Interest. at the Rate of $a - 1$ Interest for the nth Part of 1 Year, will amount to R at the Year's End.

2. If the Time is not an aliquot Part of a Year, reduce it to Days, and the 365th Root of R is the Amount for one Day; which Amount raise to that Power whose Index is the Number of Days in the Question, and it is the Amount sought. The Reason is plain from the preceding Case; but the Practice difficult, because of the Difficulty of finding the Root required.

Now for all other principal Sums, their Amount is found from that of 1 *l.* thus; As 1 *l.* to its Amount, so is any other Principal to its Amount; which will be the Product of its Principal, and the Amount of 1 *l.*

Before I leave this Subject, I must observe, That the Extraction required for any aliquot Part of a Year more than $\frac{1}{2}$ or $\frac{1}{3}$ will be tedious by the common Rules, and next to impossible for one Day, which requires the 365th Root. The Algebraic Art furnishes easier Rules for these Extractions; but they go beyond my Limits. The Method of Logarithms will be tolerable exact, which is this, *viz.* Take the Logarithm of the Rate R, divide it by the Denominator of the given aliquot Part of a Year, the Quote is the Logarithm of the Root sought; which therefore is found in the *Table* against that Logarithm.

I shall only add, That if the Time is $\frac{1}{4}$ it may be done at two Extractions of the Square Root, *viz.* Take the Square Root of R, and then the Square Root of the former Root; because the Square Root of the Square Root is the 4th Root, since $2 \times 2 = 4$. Again, from the 4th Root extract the Square Root, it is the 8th Root, and is the Amount for $\frac{1}{8}$ of a Year, and going so on, we may find by the Square Root the Amount for $\frac{1}{16}$, or $\frac{1}{32}$, or $\frac{1}{64}$ *&c.* Part of a Year: But yet these will be tedious above two Extractions. Again, for $\frac{1}{12}$ Part, or one Month, extract the Cube Root of R, then of this the Square Root, and of this again the Square Root, and this last is the 12th Root, or the Amount for one Month, because $3 \times 2 \times 2 = 12$

And this I presume will be going near enough for *Compound* Interest; so that what Time there is less than one Month, or over any Number of Months, take the Interest of the last Amount for that Time at Simple Interest, the Difference will be inconsiderable in any Case that can occur in common Affairs. But, lastly, I must observe, That the great and

and allowed Uſe of Compound Intereſt being in the Purchaſe of Annuities, which ſeldom, if ever, go lower than Quarters: It is enough that we have a Rule eaſy enough for that, *viz.* by two Extractions of the Square Root.

PROBL. 2. Having the Amount, Rate, and Time, to find the Principal.

Rule. Divide the given Amount by the Amount of 1 *l.* for the given Time and Rate, *i. e.* by ſuch a Power of the Rate whoſe Index is the Number of Years; the Quote is the Principal ſought.

Univerſally, $\frac{A}{R^t}=p$.

Example. What principal Sum will amount to 201.9963136 *l.* (or 201 *l.* 19 *ſh.* 11 *d.* nearly) in 4 Years, at the Rate of 6 *per Cent.* Compound Intereſt? *Anſwer:* 160 *l.* Thus; The Rate is 1.06, and $\overline{1.06}^4=1.26247696$; then $\frac{201.9963136}{1.26247696}=160$.

DEMONSTR. By *Probl.* 1. $A=p\times R^t$, hence, dividing both by R^t it is $\frac{A}{R^t}=p$.

Or thus, by Proportion; $R^t:1::A:\frac{A}{R^t}=p$, becauſe of the Proportion of Principals and their Amounts.

SCHOLIUMS.

1. This *Problem* is the ſame thing as finding the preſent Worth of a Debt due at the End of a certain Number of Years, diſcompting Compound Intereſt: For that preſent Worth muſt be a Sum, which, conſidered as a Principal, will, at the End of the given Number of Years, amount to the Debt; therefore the Difference of the Principal and Amount is the *Diſcompt.*

2. To divide by R^t, is the ſame thing in effect as to divide firſt by R, and this Quote again by R, and ſo on, ſtill dividing by R, till the Number of Diviſions be equal to t; and from this, (or alſo from the Nature of Compound Intereſt, which is again a Proof of this) it follows, that any Debt or Sum due after a certain Number of Years, wsth the preſent Worths of it for 1, 2, 3, *&c.* Yearts before it falls due, make a Goometrical Progreſſion decreaſing from the given Sum in the Ratio of R to 1. Thus, in the Series $A:\frac{A}{R}:\frac{A}{R^2}:\frac{A}{R^3}$, *&c.* if A is a Debt payable after 1, or 2, *&c.* Years, then is $\frac{A}{R}$ the preſent Worth of it, diſcompting for 1 Year, $\frac{A}{R^2}$ the preſent Worth, diſcompting for 2 Years, and ſo on; and this is alſo another *Demonſtration* of the *Rule.*

3. If the Time given is leſs than 1 Year, the preſent Worth is to be found either by the Rule of Simple Intereſt; or (if it's ſo agreed upon, or thought more reaſonable) upon the ſame Principle with the other Method of finding the Amount for Time leſs than 1 Year, thus; Find the Amount of 1 *l.* for the given Time; then as that Amount is to 1 *l.* ſo is the given Amount to the preſent Worth ſought; which is plainly the Quote of the given Sum by the Amount of 1 *l.*

PROBL. 3. Having the Principal, Amount, and Rate, to find the Time.

Rule. Divide the Amount by the Principal, then multiply the Rate by itſelf continually till the Product is equal to the preceding Quote; the Index of that Power of the Rate thus produced, or the Number of Multiplications more 1, is the Time.

Example: At 5 *per Cent.* Compound Intereſt, in what Time will 50 *l.* amount to 60 *l.* 15 *ſh.* $6\frac{3}{40}$ *d.* or 60.7753125 *l.*? *Anſwer:* 4 Years: For $\frac{60.7753125}{50}=1.21550625=\overline{1.05}^4$ or $1.05\times1.05\times1.05\times1.05$.

DEMONSTR. By *Probl.* 1. $A = p \times R^t$; divide both by p, and it is $\frac{A}{p} = R^t$; whence the reſt of the Rule is manifeſt.

SCHOLIUM. If none of the Powers of R is found exactly equal to $\frac{A}{p}$, this ſhews that the given Principal cannot make the given Amount in any whole Number of Years; but that beſides the Number of Years expreſſed by the Index of that Power of R, which is next leſs than $\frac{A}{p}$, there muſt be allowed moreover ſome Part of a Year; and to find what that is, multiply that Power of R, ſuppoſe R^t, by the Principal p, the Product pR^t is the Amount of p for t Years (by *Probl.* 1); wherefore take the Difference of pR^t, and the given Amount A, *viz.* $A - pR^t$; find in what Time pR^t will amount to A, reckoning by Simple Intereſt, *i. e.* in what Time it will give $A - pR^t$ of Intereſt at the given Rate, and that is the additional Time ſought. Or if you chuſe the other Method, then, to do it to the greateſt Exactneſs, we muſt know the 365th Root of R; and having found a 4th Proportional to theſe, *viz.* $pR^t : A :: 1$, which is $\frac{A}{pR^t}$; take the 365th Root of R, and multiply it by it ſelf continually till the Product is equal to $\frac{A}{pR^t}$ or next leſs, the Number of Multiplications more 1 is the Number of Days ſought. But we may more eaſily do it within 10 or 11 Days by two Extractions of the Square Root, and two of the Cube Root, thus; Take the Square Root of the Square Root of R; then of this take the Cube Root of the Cube Root, and you have the 36th Root, (for $2 \times 2 \times 3 \times 3 = 36$); multiply this by it ſelf continually till the Product is equal to or next leſs than $\frac{A}{pR^t}$, the Number of Multiplications more 1, ſhew nearly how many times 10 Days are contained in the Time ſought, becauſe $36 \times 10 = 360$, which is nearly a Year. Or I ſhall propoſe another Method: Find the 8th Root of R, (*i. e.* the Square Root of the Square Root of the Square Root); involve it till it be equal to $\frac{A}{pR^t}$, or next leſs: If you find a Product equal, then the Number of Multiplications, which cannot exceed 7, ſhew how many times $45\frac{5}{8}$ Days is in the Time ſought (for $365 \div 8 = 45\frac{5}{8}$): But if there is none of the Products equal to $\frac{A}{pR^t}$, take the next leſs, and let us repreſent it by $R^{\frac{n}{8}}$ (it is the Amount of 1 *l.* for ſo many times $45\frac{5}{8}$ Days as the Number of Multiplications); take the Difference of this Product and $\frac{A}{pR^t}$, *viz.* $\frac{A}{pR^t} - R^{\frac{n}{8}}$; then find by Simple Intereſt in what Time $R^{\frac{n}{8}}$ will amount to $\frac{A}{pR^t}$, that is, in what Time it will yield a Sum of Intereſt equal to $\frac{A}{pR^t} - R^{\frac{n}{8}}$. The *Anſwer* of this will give a Number of Days, which, added to the Number of Days in as many Times $45\frac{5}{8}$ as the Number of Multiplications that produced $R^{\frac{n}{8}}$ (which are $n - 1$) gives the whole Number of Days nearly; for tho' the laſt Part of the Work is by ſimple Intereſt, yet being within 45 Days it can make no conſiderable Difference unleſs the Principal were very large. Or, to add no more, If we find the 12th Root, and uſe it as above directed for the 8th, we ſhall find the Time by Compound Intereſt within $30\frac{5}{12}$ Days, and then find the reſt by Simple Intereſt.

In all theſe Methods the Logarithms will be very convenient for the Extractions of the Roots of R, and again involving this Root into it ſelf, as the Rules preſcribe.

PROBL. 4. Having the Principal, Amount, and Time, to find the Rate.

Rule.

Rule. Take the Quote of the Amount divided by the Principal, and extract a Root of it whose Denominator is the Number of Years; that is the Rate.

Example: At what Rate of Compound Interest will 50 *l*. amount to 60.7753125 *l*. in 4 Years? *Answer*: 5 *per Cent*. for $\frac{60.7753125}{50} = 1.21550625$, whose 4th Root is 1.05, the Rate or Amount of 1 *l*. for 1 Year.

DEMONSTR. Since $\frac{A}{p} = R^t$ by the last, and the t Root of R^t is R, therefore the same Root of $\frac{A}{p}$ is R.

Observe, As the first two *Problems* are the most useful, so their Answers are more easily found, and also more determinate.

SCHOLIUM. As the Amount of any Principal is the Sum of the Principal and Inteaest, so if in any of the preceding *Problems* the Interest is sought or given instead of the Amount, the *Answer* is easily found from the preceding; which I shall briefly explain, thus:

PROBL. 5. Having the Principal, Rate, and Time, to find the Interest.

Rule. Find the Amount by *Probl*. 1. the Difference of this and the Principal is the Interest.

PROBL. 6. Having the Interest, Time, and Rate, to find the Principal.

Rule. Find the Amount of 1 *l*. for the given Time and Rate, then the Difference of 1 *l*. and that Amount being the Interest of 1 *l*. say as that Interest is to 1 *l*. so is the given Interest to its Principal sought.

PROBL. 7. Having the Principal, Interest, and Rate, to find the Time.

Rule. The Sum of Principal and Interest is the Amount; by which, with the Rate and Principal, find the Time by *Probl*. 3.

PROBL. 8. Having the Principal, Interest, and Time, to find the Rate.

Rule. Find the Amount, as in the last, and then apply *Probl*. 4.

§. 2. *Of* ANNUITIES.

DEFIN. 1. AN Annuity is a Sum of Money payable every Year for a certain Number of Years, or for ever. And tho' it be divided into half Years or quarters Payments, it still goes under the general Name of Annuity, because the whole Payments make so much in a Year.

I shall first consider the Supposition of yearly Payments, and then other Cases; and also first consider Annuities to continue for a certain determined Number of Years.

2. An Annuity is said to be in *Arrears*, when the Debtor keeps it in his hands for a certain Number of Years, paying the whole at last with Interest for every Year after it falls due; and the Total of the several Years with the Interest due upon each, is called the *Amount* of the Annuity forborn for that time. Again; If an Annuity is to be bought off, or paid all at once, at the very Beginning of the first Year, the Price which ought to be paid for it, discompting for the Advancement, is called the present Worth of it for so many Years.

But then, as either Simple or Compound Intereſt may be made a ſuppoſed Condition in the Queſtion, we muſt accordingly diſtinguiſh; tho' Simple Intereſt, eſpecially in the purchaſing of Annuities, is very unjuſt, as ſhall be fully demonſtrated.

PART. I. *Of* Annuities *at Simple Intereſt.*

1. *Of* Annuities *in Arrears at Simple Intereſt.*

PROB. I. Having the Annuity, Time, and Rate of Intereſt (*i. e.* the Intereſt of 1 *l.* for one Year) to find the Amount.

Rule. Take the natural Series of Numbers 1, 2, 3, *&c.* to the Number of Years leſs 1. Multiply the Sum of this Series by one Year's Intereſt of the Annuity, (which is the Rate, if the Annuity is 1 *l.*; but it is the Product of the Rate and Annuity, if the Annuity is not 1 *l.*) this Product is the whole Intereſt due upon the Annuity. To which add the whole Annuities, (*i. e.* the Product of the Annuity and Time) the Sum is the Amount ſought. And *obſerve,* That as the Series 1, 2, 3, *&c.* is an Arithmetical Progreſſion, if you find its Sum by the Rule of *Prob.* 5 *Chap.* 2. *Book* IV. (which is this, multiply the Sum of the Extremes by the Number of Terms, the half of the Product is the Sum) the Work is thereby eaſier.

Example. What is the Amount of 50 *l.* Annuity for 7 Years, allowing Simple Intereſt at 5 *l. per Cent* for every Year after it falls due? Anſwer, 402 *l.* 10 *s.* Found thus: $1+2+3+4+5+6=21$; then $.05\times 50=2.5$, and $2.5\times 21=52.5$. Again, $50\times 7=350$, and $350+52.5=402.5$, or 402 *l.* 10 *s.* the Amount ſought.

DEMONSTR. Whatever the Time is, there is due upon the firſt Year's Annuity as many Years Intereſt as the whole Number of Years leſs 1; and gradually 1 leſs upon every ſucceeding Year, to the laſt but one; upon which there is due one Year's Intereſt, and none upon the laſt; wherefore in whole there is due as many Years Intereſt of the Annuity, as the Sum of the Series 1, 2, 3, *&c.* to the Number of Years leſs 1. Conſequently one Year's Intereſt multiplied by this Sum muſt be the whole Intereſt due. To which the whole Annuities added, the Sum is plainly the Amount.

SCHOL. This *Problem* may be ſolved alſo in this manner, *viz.* Take an Year's Intereſt of the Annuity for the leaſt Term, and alſo the common Difference of an Arithmetical Progreſſion carried to as many Terms as the Number of Years leſs 1, (*i. e.* take one Year's Intereſt of the Annuity, then double it; then take it three times, and ſo on, till you take it as oft as the Number of Years leſs 1) its Sum is the whole Intereſt due, as is plain by what's already ſhewn; or appears by this, that it is the ſame thing to multiply by the Sum of the Series 1, 2, 3, *&c.* as in the 1ſt *Method,* or by each of the Terms ſeparately; and then add the Products, which is the other *Method.* Again; It will come to the ſame thing, if we take an Arithmetical Progreſſion, whoſe leaſt Term, and alſo the common Difference, is the Rate, and carry it to the ſame Number of Terms as before; the Sum of this Series is the whole Intereſt due upon 1 *l.* Annuity, (as appears by the laſt *Method*); wherefore if we multiply this Sum by any other given Annuity, the Product is the whole Intereſt due upon this other Annuity. Or take the Reaſon of it thus: Since one Year's Intereſt of the Annuity is the Product of the Rate and Annuity, (as is plain by the common Rules) it is the ſame thing to multiply each Term of the Series 1, 2, 3, *&c.* by that Product; and then add all the Products, which is the 2d *Method*; or to multiply them firſt by the Rate, and then each of theſe Products, or their Sum, by the Annuity; which is the laſt *Method.*

2. But I ſhall again more briefly ſhew the Coincidence of all theſe three *Methods* by the *univerſal Method* of Expreſſion, (which is neceſſary for the ſake of the following *Problems*) and alſo bring them all into one general Rule. Thus:

Let

Let r be the Rate, n the Annuity, t the Time, and A the Amount; then is rn one Year's Interest of the Annuity, (for $1:r::n:rn$) and so the 1st Rule may be expressed thus: $\overline{1+2+3,\ \&c.\ +\overline{t-1}}\times rn+tn=A$. Again; The 2d Rule is $rn+2rn+3rn,\ \&c.\ +\overline{t-1}\times rn+tn=A$, which coincides with the first, because $rn+2rn+3rn,\ \&c.\ +\overline{t-1}\times rn=\overline{1+2+3,\ \&c.\ +t-1}\times rn$.) Lastly, the 3d Rule is, $\overline{r+2r+3r,\ \&c.\ +\overline{t-1}\times r}\times n+tn=A$, (which coincides with the 2d, because $rn+2rn+3rn,\ \&c.\ +\overline{t-1}\times rn=\overline{r+2r+3r,\ \&c.\ +\overline{t-1}\times r}\times n$.)

In the last place, to bring all these into one general Rule. 1. The Sum of the Series $1+2+3,\ \&c.\ +\overline{t-1}$, is by the Rules of Progression $\overline{t-1}+1\ (=t)\times\frac{t-1}{2}=\frac{t^2-t}{2}$; and this multiplied by rn, is $\frac{t^2rn-trn}{2}$ the Sum of the Interest due. 2. In the 2d Rule, the Sum of the Series $rn+2rn+3rn,\ \&c.\ +\overline{t-1}\times rn$ is $\overline{t-1}\times rn+rn\ (=trn)\times\frac{t-1}{2}=\frac{t^2rn-trn}{2}$ the Sum of the Interest as before. 3. In the 3d Rule, the Sum of the Series $r+2r+3r,\ \&c.\ +\overline{t-1}\times r$ is $\overline{t-1}\times r+r\ (=tr)\times\frac{t-1}{2}=\frac{t^2r-tr}{2}$; which multiplied by n produces $\frac{t^2rn-trn}{2}$ the whole Interest as before. If now to this Sum of the Interest we add the whole Annuities, the Sum is $\frac{t^2rn-trn}{2}+tn=A$; which is the universal Rule. In Words thus:

Multiply the Time, Rate and Annuity continually. Then multiply this Product by the Time; subtract the first Product from the 2d, and take the half of the Difference; to which add the Product of the Time and Annuity, the Sum is the Amount.

PROB 2. Having the Amount, Rate and Time, to find the Annuity.

Rule. Take any Annuity at pleasure, and, by *Prob.* 1. find its Amount; then, by the Rule of 3, as this Amount is to its Annuity, so is the given Amount to its Annuity. So that if you take 1 *l.* Annuity, by its Amount divide the given Amount, the Quote is the Annuity sought.

Example. What Annuity will in 7 Years amount to 402 *l.* 10 *s.* allowing 5 *per Cent.* Simple Interest? Answer, 50 *l.* Found thus: The Amount of 1 *l.* Annuity is 8.05 *l.* for by *Prob.* 1. $1+2+3+4+5+6=21$, and $21\times.05=1.05$; then $1.05+7=8.05$, the Amount of 1 *l* Annuity. Lastly, As $8.05:1::402.5:\frac{402.5}{8.05}=50$ the Annuity sought.

DEMONSTR. The Reason of this Rule is plain, since to every Part of an Annuity there must necessarily correspond a proportional Part of the Amount.

SCHOLIUM. The *Problem* may be also solved thus: Take the Sum of the Series 1, 2, 3, *&c.* to the Number of Years less 1; this Sum multiply by the Rate, and to the Product add the Years; and by this Sum divide the Amount, the Quote is the Annuity.

For, by *Problem* 1. $A=\overline{1+2+3,\ \&c.\ +\overline{t-1}}\times rn+tn$; which is $=\overline{\overline{1+2+3,\ \&c.\ t-1}\times r+t}\times n$; and dividing both by $\overline{1+2+3,\ \&c.\ +\overline{t-1}}\times r+t$, it is $\frac{A}{\overline{1+2+3,\ \&c.\ t-1}\times r+t}=n$.

Again;

Again; Because $1+2+3$, &c. $+\overline{t-1}=\frac{t^2-t}{2}$; this multiplied by r is $\frac{t^2r-tr}{2}$; to which add t, the Sum is $\frac{t^2r-tr+2t}{2}$. By which dividing A, the Quote is $\frac{2A}{t^2r-tr+2t}=n$, which is another *Rule*; and this may also be immediately deduced from the general Rule of *Prob.* 1. *viz.* from this Equation $A=\frac{t^2rn-trn}{2}+tn$, or $\frac{t^2rn-trn+2tn}{2}$. For multiplying both by 2, it is $2A=t^2rn-trn+2tn$; and dividing by $t^2r-tr+2t$, it is $\frac{2A}{t^2r-tr+2t}=n$.

PROB. 3. Having the Annuity, Amount, and Time, to find the Rate.

Rule. Take the Difference betwixt the Amount, and the Product of the Annuity and Time; the Difference divide by the Product of the Annuity multiplied into the Sum of this Series 1, 2, 3, &c. to the Number of Years less 1, the Quote is the Rate.

Example. At what Rate of Interest will an Annuity of 50 *l.* amount to 402 *l.* 10 *s.* in 7 Years? Answer, 5 *per Cent.* or .05 to 1 *l.* Thus, $50\times7=350$, then $402.5-350=52.5$. Again, $1+2+3+4+5+6=21$, and $21\times50=1050$. Then, lastly, $\frac{52.5}{1050}=.05$.

DEMON. By *Prob.* 1. $A=\underline{1+2+3, \&c. +\overline{t-1}}\times rn+tn$. Take tn from both, and it is $A-tn=\underline{1+2+3, \&c. +\overline{t-1}}\times rn$. Divide both by $\underline{1+2+3, \&c. +\overline{t-1}\times n}$, and it is $\frac{A-tn}{\underline{1+2+3, \&c. +t-1\times n}}=r$, which is the Rule.

SCHOLIUM. $1+2+3$, &c. $+\overline{t-1}=\frac{t^2-t}{2}$; and this multiplied by n is $\frac{t^2n-tn}{2}$. By which divide $A-tn$, the Quote is $\frac{2A-2tn}{t^2n-tn}=r$, which is another *Rule*; and this may be deduced from the general Rule of *Prob.* 1. thus, $A=\frac{t^2rn-trn}{2}+tn$. Hence $A-tn=\frac{t^2rn-trn}{2}$, and $2A-2tn=t^2rn-trn$; and lastly, $\frac{2A-2tn}{t^2n-tn}=r$.

PROB. 4. Having the Annuity, Amount and Rate, to find the Time.

Rule. Take the Product of the Annuity and Rate, as the first Term of a Progression, whereof the same Product is the common Difference; at every Step from the very first Term, take the Sum of the Series, and to it add the Product of the Annuity multiplied into a Number more by 1 than the Number of Terms summed. Go on in this manner till you find a Sum equal to the given Amount, and the Number multiplied into the Annuity in that last Step is the time sought.

Example. In what time will 50 *l.* Annuity amount to 402 *l.* 10 *s.* at 5 *per Cent.*? Answer, 7 Years. Found thus:

$50\times.05=2.5$: then for the rest of the Operation, see it in the Margin.

	rn	$2rn$	$3rn$	$4rn$	$5rn$	$6rn$
	2.5 +	5 +	7.5 +	10 +	12.5 +	15.
Sums,		7.5	15	25	37.5	52.5.
Products of 50,	150	200	250	300	350 (=50×7)	
2d Sums,	157.5	215	275	337.5	402.5.	

DEMONSTR. By *Prob.* 1. we have $\underline{A}=rn+2rn+3rn$, &c. $+\overline{t-1}\times rn+tn$; and if the Time is but 1 Year, then is $A=n$. If it is 2 Years, then is $A=rn+2n$; if 3 Years, then is $A=rn+2rn+3n$, and so on; which makes the Rule manifest.

SCHO-

SCHOLIUMS.

1. If you never find a Sum equal to the Amount, then the *Problem* is impoſſible in whole Years.

2. The preceding *Problem* will be tedious if there are many Years.

There is another Rule eaſier in Practice, tho' not ſo ſimple in its Demonſtration. It is deduced from *Problem* 1. thus, $A = \frac{rnt^2 + 2tn - trn}{2}$, whence $2A = rnt^2 + 2tn - trn = rnt^2 + \overline{2n - rn} \times t$; divide by rn, and it is $\frac{2A}{rn} = t^2 + \frac{2n - rn}{rn} \times t$: Call $\frac{2n - rn}{rn} = d$, then it is $\frac{2A}{rn} = t^2 + dt$, and by *Probl.* 6. *Ch.* 2. *Book* III. $t = \overline{\frac{2A}{rn} + \frac{dd}{4}}\big|^{\frac{1}{2}} - \frac{d}{2}$.

2. *Of the Purchaſe of* Annuities *at Simple Intereſt.*

In the following *Problems*, the Things concerned are the Annuity, it's preſent Worth, the Time of Continuance, and Rate of Intereſt allowed to the Purchaſer for the Advancement of his Money.

As before, we ſhall expreſs the Annuity by n, the Time t, Rate r, and the preſent Worth p.

PROBL. 5. Having the Annuity, Rate, and Time, to find the preſent Worth.

For the Solution of this *Problem*, there are two different Rules given by different Authors; but as they give different Anſwers, they cannot be both right. I ſhall firſt lay down the *Rules*, and then examine which is the right one.

Rule 1. Find the preſent Worth of each Year by it ſelf, diſcompting from the Time it falls due (by *Probl.* 8. *Part* 1. §. 1.); the Sum of all theſe is the preſent Worth ſought.

Example: What is the preſent Worth of an Annuity of 100 *l.* to continue 5 Years, diſcompting at the Rate of 6 *per Cent.* or .06 to 1 *l.*? *Anſwer:* 425 *l.* 18 *s.* 9 *d.* 2 *f.* neareſt; in Decimals 425.93932, *&c.* Found thus:

		as		to		ſo is		to	
The Amount of 1 *l.* for	1 Year,	1.06	:	1	::	100	:	94.33962	
	2 Years,	1.12	:	1	::	100	:	89.28571	
	3 Years,	1.18	:	1	::	100	:	84.74576	*&c.*
	4 Years,	1.24	:	1	::	100	:	80.64516	
	5 Years,	1. 3	:	1	::	100	:	76.92307	
								425.93932	

Obſerve, The Work will be a little eaſier, if you find the preſent Worth of 1 *l.* Annuity for the given Time, and then multiply that by the given Annuity, the Product is the preſent Worth of it, becauſe of the Proportionality of Annuities and their preſent Worths.

Rule 2. Find what each Year's Annuity would amount to, being forborn to the End of the laſt Year, allowing Simple Intereſt from the time it falls due; *that is,* find the Amount of the Annuity as forborn the whole Time (by *Probl.* 1. of this §.) then find the preſent Worth of that Amount as a Sum due at the End of the given Time.

In

In the preceding *Example* the Amount of 100 *l.* Annuity for 5 Years at 6 *per Cent.* is 560 *l. viz.* the 5 Years Annuities 500 *l.* and 60 *l.* which is 10 Years Interest of 100 *l.* Then the Amount of 1 *l.* forborn 5 Years is 1.3; and as 1.3 is to 1, so is 560 to 430.76923, *&c.* or 430 *l.* 15 *sh.* 4 *d.* 2 *f.* nearest the present Worth by this Rule.

SCHOLIUM. The first *Rule* is Mr. *Kersey*'s in his *Appendix to* Wingate's *Arithmetick:* The Second has for its Author or Defender Sir *Samuel Moreland*, in his *Doctrine of Interest*, who wonders how such gross Mistakes, as he calls *Kersey*'s Rule, could pass thro' the Hands of so many learned and ingenious Artists. After him, others have taken the same Method, and particularly Mr. *Ward*, who makes this Remark, *That Sir* Samuel Moreland *has detected several material Errors in* Kersey *and others in the Business of Interest:* But to my Apprehension the Error lies all upon Sir *Samuel*'s Side.

Before I enter upon the Examination of the Reasons of these Rules, I must first *observe*, That by saying, *Kersey*'s Rule gives the true Price of the Annuity, I mean only, That it is true in consistence with the supposed Condition or Agreement of allowing Simple Interest, and not absolutely so: For if we enquire what, in strict Equity and Justice, ought to be paid for the Annuity, then neither of these Rules shew it; for they both give too much, and the true Price must be found by discompting Compound Interest (as *Moreland* acknowledges), and then both the Methods give the same Answer. Nor is this contrary to Law: For tho' when an Annuity is in Arrears the Law forbids taking Compound Interest, yet in the Purchase of an Annuity, if the Buyer offers such a Price as allows him Compound Interest for the Advancement of his Money, he does nothing contrary to Law; because in Buying one may offer what Price he thinks fit: And he has this good Reason for it, that by putting out his Money, and *lifting* it at every Year's End, he can improve it by Compound Interest.

But to shew further how unjust Simple Interest is in the Purchase of Annuities, take this *Example:* An Annuity of 50 *l.* is to be bought for 40 Years, discompting Simple Interest at 5 *per Cent*: The Price according to *Moreland*'s Rule is 1316 *l.* 13 *sh.* 4 *d.* a Sum of which one Year's Interest exceeds the Annuity: Would not then one think he had made a pretty Bargain, to give for an Annuity to continue only 40 Years, a Sum which would yield him a greater Yearly Interest for ever? If it's also calculated by the other Rule, the same will happen, as I have actually found; tho' it is much less than the other, for it does not exceed 1100 *l.*

This *Example* may, I think, be sufficient to shew the Absurdity of discompting for Annuities at Simple Interest, and consequently to put that Practice quite out. And this too might perhaps be a good enough Reason for not troubling you any further with the Comparison of the preceding *Rules*, and so to pass on to Compound Interest. But as it is a Question that belongs to Arithmetick, to find the Price of an Annuity upon the Supposition of Simple Interest, which is to be found in every Book upon this Subject, and since these two Rules are become Matter of Dispute among Authors, I thought I could not reasonably omit the Examination of these, and shewing why I prefer *Kersey*'s.

The Reasons *of the preceding* Rules *examined.*

Mr. *Kersey* takes the Reason of his Rule for a thing of it self manifest from the Nature and Rules of Discompt; for if that is right, he considers every Year by it self, as so many single and independent Debts, due after 1, 2, 3, *&c.* Years, so that the present Worth of each being found, the Sum of these must be the present Worth of the Whole; which seems to be the plain state of the Question.

Sir *Samuel Moreland* says this Rule is grosly wrong, because there is no Consideration had of the *Forbearance* of Interest, (*i. e.* of the Annuity); and therefore he proposes the considering what the Annuity will amount to, being forborn for the whole Time of the Question; and then he supposes that all must agree, That whatever be the present Worth of the Annuity, it must be such a Sum, as, put out to Interest for the Number of Years

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the Annuity continues, will amount to the same Sum the Annuity does; because the Seller ought to have a Price by which he can make as much at Simple Interest during the Continuance of the Annuity, as he could have made by *lifting* every Year of his Annuity as it falls due, and putting it out to Interest for the remaining Years of the Question.

This is the first and chief Argument which Sir *Samuel* uses; he adds something further concerning the present Worth of the several Years, which I shall consider afterwards: But as to this first Argument, I am so far from agreeing to his Supposition, that I think the very contrary of what he objects to *Kersey*'s Rule is a just Objection to his own; because I think the Consideration of the Forbearance of the Annuity is a thing altogether out of the Question: For in purchasing an Annuity, does the Purchaser any other thing than buy several Sums of Money to be paid to him at several Times, for which he advances ready Money all at once? And this being plainly the Case, what has he to do with the Consideration of the Forbearance of the Annuity? Does he not fairly pay for the Whole when he pays for each Sum separately, discompting from the Time it falls due, or payable to him? For thus he pays for each Year such a Sum as will amount to that Year when it falls due: And each Year's Price being considered as a Principal Sum bearing Interest in the Seller's Hands, will make him as much Debtor to the Buyer at each Year's End, as the Annuity then due and received by the Buyer makes him in the Seller's Debt; and consequently the Buyer is never in Arrears with the Seller, and so has no Business with the Consideration of Forbearance or Amount of the Annuity: The Seller being thus cleared at every Year's End, let him make the best he can of it.

Again; The Argument from the Amount of the Annuity would be just as good if you extend it to 7 or any Number of Years after the last Year in the Question: For the Whole being in Arrears 7 Years longer, would have a greater Amount; and this considered as a Debt payable after so many Years, will have a greater present Worth: And since the Annuity in the Question will make such an Amount, being forborn so long after the Time of the Annuity, why should not the Seller insist upon a Price that will Amount to the same Sum in the same Time? But if this would be ridiculous, the other is equally so; for I have no more to do with the Consideration of any Year being in Arrears after it is payable (and indeed actually paid) to the End of the last Year, than with that Year, or all the Years being forborn after the last Year in the Question.

I shall in the next place shew the Fallacy of *Moreland*'s Argument from his own Concessions. He owns that the Present Worth of a single 100 *l.* due at one Year's End, of another single 100 *l.* due after 2 Years, and another 100 *l.* due after 3 Years, and so on, are all justly found according to *Kersey*'s Rule: And if so, pray where is the Difference betwixt an Annuity of 100 *l.* and so many single 100 *l.*'s due after 1, 2, 3, *&c.* Years? It is beyond Question, the Cases are the same if these several Debts are all owing by one Person. But perhaps it will be said, that the Concession is made only upon Supposition that these several 100 *l.*'s are due by different Persons; which seems to be *Moreland*'s Meaning by calling it a single 100 *l.* (*i.e.* as I now take it, 100 *l.* in one Man's Hands, and another 100 *l.* in another Man's Hands, *&c.*) Now to shew that this can make no Alteration of the Case: Suppose 5 Men owe you each 100 *l.* payable one at 1 Year's End, another at 2 Years End, *&c.* the Present Worths to be paid by the several Debtors for their several Debts, are, it is own'd, according to *Kersey*'s Rule: Suppose next, that I would buy the Right of these Debts, paying them *per Advance*, can there be any more justly asked of me than of the several Debtors? And is there any manner of Difference, either as to the Buyer or Seller, betwixt this Case, and an Annuity of 100 *l.*? For in either Case I buy 100 *l.* to be paid me after 1 Year, another 100 *l.* to be paid after 2 Years, and so on. And on the other hand, the Seller has the same Argument with me from the Amount of these Debts forborn till the Time the last is payable, as from the Amount of an Annuity payable by one Debtor; and yet it is destroyed in this Case, by acknowledging that the several Debtors ought to pay by *Kersey*'s Rule. Again; Take this other *Example*: Suppose

I owe to each of 5 different Perſons 100 *l.* payable at 1, 2, 3, 4, 5 Years End; if I were to pay off theſe Debts *per Advance*, the Preſent Worths are acknowledged to be according to *Kerſey*'s Rule: Now if one Perſon gets a Right to all theſe Debts, it is plain I become Debtor to him in an Annuity of 100 *l.* to continue 5 Years; and if I propoſe to pay them off at once, by what Reaſon ought I to pay him more than would have juſtly ſatisfied his Conſtituents?

It remains that I conſider *Moreland*'s Argument from the Preſent Worths of the ſeveral Years, from whence he thought the Demonſtration perfected; tho' his way of calculating theſe Preſent Worths, affords, I think, another ſtrong Argument againſt the Truth of his Rule.

He ſuppoſes an Annuity of 100 *l.* The Preſent Worth of the 1ſt Year he allows to be according to *Kerſey*'s Rule: For the 2d Year he takes the Amount of the firſt 2 Years, *viz.* 206 *l.* (ſuppoſing 6 *per Cent.*) then finds the Preſent Worth of 206 *l* due after 2 Years, which he conſiders as the Preſent Worth of the firſt 2 Years together; from that he takes the Preſent Worth of the firſt Year, and calls the Difference the Preſent Worth of the 2d Year; becauſe the preſent Worth of the 1ſt Year taken from the Preſent Worth of the firſt two Years, muſt leave the Preſent Worth of the 2d Year alone. Then for the 3d Year, he takes the Amount of the firſt 3 Years, *viz.* 318 *l.* and finding its Preſent Worth as a Debt due after 3 Years, from this he takes the Preſent Worth of the firſt 2 Years, (or of 206 *l.* due after 2 Years) and the Difference he calls the Preſent Worth of the 3d Year alone; and in this Manner he goes on; then concludes, that becauſe what he calls the Preſent Worths of the ſeveral Years make up his total preſent Worth, therefore his general Rule is right.

For *Anſwer* to this, I do acknowledge that the Preſent Worth of all the Years together, is exactly equal to the Sum of the Preſent Worths of the ſeveral Years conſidered ſeparately: But then to ſhew that his Preſent Worth of the Whole is right, he ought firſt to have proved, that his Preſent Worths of the ſeveral Years are right; which I deny, and ſhall prove to be falſe.

In order to which, *Obſerve*, That by theſe Preſent Worths his Argument requires that he mean the Prices the ſeveral Purchaſers of the ſeveral Years ought to pay for them: For you give the Buyer of the Whole no Reaſon why he ought to pay the Sum of all theſe Preſent Worths, unleſs you ſhew him that theſe are the Prices to be paid by ſeveral Purchaſers of the ſeveral Years; and as he muſt be ſatisfied that he ought to pay no leſs, ſo there is no manner of Reaſon why he ſhould pay more. Let us then examine the Juſtice of theſe ſeveral Preſent Worths.

The firſt Year is the ſame as by *Kerſey*'s Rule. But he who buys the 2d Year pays the Difference betwixt the Price of the firſt Year and the Preſent Worth of 206 *l.* due after 2 Years; and this 6 *l.* is one Year's Intereſt of the 1ſt Year's Annuity, conſidered as forborn 1 Year; ſo that the Purchaſer of the 2d Year pays moſt unreaſonably for the Conſideration of the Forbearance of the 1ſt Year, with which he has no Concern; and the further from the 1ſt Year, theſe Preſent Worths are the further wrong. It had been more agreeable to his fundamental Suppoſition of Forbearance, to make the Price of every Year the Preſent Worth of its Amount, conſidered as forborn to the End of the whole Years; for the Sum of theſe is alſo equal to his Preſent Worth of the Whole: But this ſeemed too groſs when applied to ſingle Purchaſers of the ſeveral Years; And is it not rather more abſurd to make them pay for the Forbearance of what they have not purchaſed?

Tho' this Argument is tedious enough already, yet I muſt take notice of one Caſe wherein *Moreland*'s Rule would take place; that is, If we ſuppoſe the Debtor of an Annuity is obliged to keep it in his Hands, paying Intereſt to the End of the whole Time; then if he would pay it off, the Creditor has Reaſon to inſiſt on the Conſideration of the Amount of the Annuity as a Debt payable at the End of the Years of the Annuity; but

if

if the Annuity is payable as it falls due, he cannot justly insist on that Consideration; and in this Case the Price ought to be by *Kersey's* Rule. So that where the Condition of paying the Annuity is the easiest (as it is certainly more advantageous for the Debtor to keep it in Arrears at Simple Interest, than pay it every Year), the Purchase of it is the dearer; And where the Condition of paying it is the hardest, (*viz.* paying it as it falls due) the Purchase is the easiest; which is something absurd and contradictory: But then this proves neither of the Rules to be wrong; and it arises from the Injustice of Simple Interest; for if Compound Interest is allowed, then to pay the Annuity as it falls due, or keep it in Arrears at Compound Interest, are the same thing; and so also the Present Worths by both Methods are the same. Again; If we suppose either the Creditor or Debtor has it in his Choice, that the Annuity be paid as it falls due, or kept in Arrears at Simple Interest, then the Advantage is upon the same Side both in Paying and Buying: For as the Creditor would certainly chuse to have it paid Yearly, so in selling it he would have Reason to insist on the other side of the Choice, and demand a Price agreeable to that Obligation on the Debtor to keep it in Arrears: Again, As the Debtor would certainly chuse to keep it in Arrears, so in buying it off he would reasonably insist on the other side of the Choice, *viz.* paying it Yearly, because the Creditor cannot oblige him to keep it in Arrears: Yet with Compound Interest there would be no Advantage in having the Choice.

Observe, in the last place, That if another than the Debtor of an Annuity proposes to buy it; then, if the Debtor can keep it in Arrears at Simple Interest, (or is obliged to do so) as this is a real Disadvantage to the Purchaser, so the Seller cannot reasonably insist on that Consideration with him in calculating the Price of it, as he might do if the Debtor buys it; which is another Contradiction, *viz.* that the Price should be different to different Buyers. But this also arises from simple Interest.

SCHOLIUM 2. Tho' I think it's made evident that *Kersey*'s Rule is the right one, yet I shall give the Solution of the following three *Problems* upon both Suppositions; and in order to it, I shall here give you the universal Expressions of both these Rules.

The 1st *Rule* is $p = \frac{n}{1+r} + \frac{n}{1+2r} + \frac{n}{1+3r}$, &c. to $\frac{n}{1+tr}$.

The *Reason* is this: r being 1 Year's Interest of 1 *l*, $1+r$ is the Amount of 1 *l.* for 1 Year; and therefore the Present Worth of the 1st Year's Annuity, n, is $\frac{n}{1+r}$; for $1+r : 1 :: n : \frac{n}{1+r}$. For the like Reason $\frac{n}{1+2r}$ is the Present Worth of the 2d Year; for $1+2r$ is the Amount of 1 *l.* for 2 Years; and so of the rest; which are the Present Worths of the several Years of the Annuity.

The 2d Rule is $p = \frac{t^2rn + 2tn - trn}{2+2tr}$.

The *Reason* is this: By *Probl.* 1. the Amount of the Annuity is $\frac{t^2rn - trn}{2} + tn$ equal to $\frac{t^2rn - trn + 2tn}{2}$: Then the Present Worth of this, as a Debt payable after t Years, is a 4th Proportional to $1+tr$ (the Amount of 1 *l.* for t Years) 1 *l.* and the Amount of the Annuity, which is according to the *Rule*: For $1+tr : 1 :: \frac{t^2rn + 2tn - trn}{2} : \frac{t^2rn + 2tn - trn}{2+2tr}$.

PROBL. 6. Having the Present Worth, Rate, and Time, to find the Annuity.

Rule. Take any Annuity at pleasure, and find its Present Worth by either of the preceding Rules you please; then, as that present Worth is to its Annuity, so is the given Pre-

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ſent Worth to its Annuity: So that if you take 1 *l* Annuity, then the given Preſent Worth divided by the Preſent Worth of 1 *l.* Annuity, quotes the Annuity ſought.

Example: What Annuity, to continue 5 Years, is worth 220 *l* Preſent Worth, allowing Simple Intereſt at 5 *per Cent.*? *Anſwer:* 50 *l.* 8 *ſh.* nearly, by *Kerſey*'s Rule, and by *Moreland*'s Rule it is preciſely 50 *l.*

Thus, the Preſent Worth of 1 *l.* Annuity for 5 Years, at 5 *per Cent.* is, by *Kerſey*'s Rule, 4.3641 *l.* nearly; then, as 4.3641 to 1, ſo is 220 to 50.399 *l* nearly, which is nearly 50 *l.* 8 *ſh.*

The Preſent Worth of 1 *l.* Annuity by *Moreland*'s Rule is 4.4 *l*; and as 4.4 : 1 :: 220 : 50 preciſely.

The *Reaſon* of this Rule is manifeſt, becauſe all Annuities and their Preſent Worths are proportional.

PROB. 7. Having the Annuity, preſent Worth, and Rate, to find the Time.

***Rule* 1.** If the preſent Worth is according to *Kerſey*'s Rule, then take the Series of the Amounts of 1 *l.* Principal for 1, 2, 3, *&c.* Years. Divide the given Annuity by each Term of that Series ſucceſſively; and at every Step take the Sum of all theſe Quotes, (*i. e.* add the 1ſt Quote to the 2d, and this Sum to the 3d, and ſo on) going on thus till you find a Sum equal to the given preſent Worth; and the Number of Quotes added is the Number of Years ſought. The Reaſon of which is manifeſt from *Prob.* 5. *Schol.* 2.

Example. What time muſt an Annuity of 50 *l.* 8 *s.* continue, to be worth 220 *l.* ready Money at the Rate of 5 *per Cent.*? Anſwer, 5 Years.

For dividing 50 *l.* 8 *s.* or 50.4 *l.* ſucceſſively by theſe 5 Diviſors, 1.05, 1.1, 1.15, 1.2, 1.25, (the Amounts of 1 *l.* for 1, 2, 3, 4, 5 Years) the Sum of the Quotes make 219 *l.* 19 *s.* 3 *d.* nearly; which is near to 220 *l.* The Reaſon why it makes not preciſely 220 *l.* is that 50 *l.* 8 *s.* is not preciſely the Annuity which has 220 *l.* for its preſent Worth for 5 Years, as we ſaw in the laſt *Problem*: and beſides, the Quotes are here determined only to a certain Degree; but as three of them are compleat, ſo the other two don't want 1 Farthing each.

Rule 2. If the preſent Worth is according to *Moreland*'s Rule, then the Time is to be found thus: Divide 2 by the Ratio; alſo divide the Sum of the Annuity, and twice the preſent Worth by the Annuity. Then take the Difference betwixt theſe two Quotes; and next, to the 4th Part of the Square of that Difference, add the Quote of twice the preſent Worth divided by the Product of the Rate and Annuity. Of this Sum extract the Square Root: From which Root take half of the firſt mention'd Difference, in caſe the Quote of 2 divided by the Ratio be greater than the other Quote; but if it's leſſer, add the half of that Difference to the Root; this laſt Difference or Sum is the Number of Years.

Example. What Time muſt an Annuity of 50 *l.* continue to be worth 220 *l.* at 5 *per Cent.*? Anſwer, 5 Years. Thus:

$2 \div .05 = 40$; then $2 \times 220 = 440$, $50 + 440 = 490$. $490 \div 50 = 9.8$; then $40 - 9.8 = 30.2$, and the Square of $30.2 = 912.04$, of which $\frac{1}{4}$ is 228.01 Next $.05 \times 50 = 2.5$, and $440 \div 2.5 = 176$; then $228.01 + 176 = 404.01$, whoſe Square Root is $= 20.1$; then $30.2 \div 2 = 15.1$; and laſtly, $20.1 - 15.1 = 5$, the Years ſought.

DEMONSTR. By *Problem* 5. (ſee *Schol.* 2. *Rule* 2.) it is $p = \frac{t^2 r n + 2 t n - t r n}{2 + 2 t r}$. Whence $2p + 2trp = t^2 r n + 2 t n - t r n$; and again, $2p = t^2 r n + 2 t n - t r n - 2 t r p$; and, dividing by rn, 'tis $\frac{2p}{rn} = t^2 + \frac{2tn - 2trp - trn}{rn}$, or $t^2 + \frac{2n - 2rp - rn}{rn} \times t$.

But

But $\frac{2n-2rp-rn}{rn}=\frac{2n}{rn}\left(=\frac{2}{r}\right)-\frac{2rp}{rn}\left(=\frac{2p}{n}\right)-\frac{rn}{rn}(=1)$; therefore $\frac{2p}{rn}=t^2+\overline{\frac{2}{r}-\frac{2p}{n}-1}\times t=t^2+\overline{\frac{2}{r}-\frac{2p+n}{n}}\times t$; and if for brevity we call $\frac{2}{r}-\frac{2p+n}{n}=d$; then is $\frac{2p}{rn}=t^2+dt$, and $t=\overline{\frac{2p}{rn}+\frac{dd}{4}}\Big|^{\frac{1}{2}}-\frac{d}{2}$, (by *Prob.* 6. *Chap* 2. *Book* III.) But if $\frac{2}{r}$ is less than $\frac{2p+n}{n}$, let also $\frac{2p+n}{n}-\frac{2}{r}$ be called d, and then is $\frac{2p}{rn}=t^2-dt$, and $t=\overline{\frac{2p}{rn}+\frac{dd}{4}}\Big|^{\frac{1}{2}}+\frac{d}{2}$, by the same *Problem*.

PROB. 8. Having the Annuity, present Worth and Time, to find the Rate.

1. If the present Worth is taken according to *Kersey*'s Rule, there is no Method within my Limits that will solve the *Problem*, except for that one Case where the Time is two Years. The Rule for which may be easily deduced from *Prob.* 5. *viz.* from this, $p=\frac{n}{1+r}+\frac{n}{1+2r}$. Thus, Adding these two Quotes or Fractions, the Sum is $p=\frac{2n+3rn}{1+3r+2r^2}$. Hence $p+3rp+2pr^2=2n+3rn$, and $2pr^2+3rp=2n+3rn-p$, and $2pr^2+3rp-3rn=2n-p$, and $r^2+\frac{3p-3n}{2p}\times r=\frac{2n-p}{2p}$; and calling $\frac{3p-3n}{2p}=d$, it is $r^2+dr=\frac{2n-p}{2p}$. Whence (by *Problem* 6. *Chapter* 2. *Book* III.) $r=\overline{\frac{2n-p}{2p}+\frac{dd}{4}}\Big|^{\frac{1}{2}}-\frac{d}{2}$.

2. If the present Worth is according to *Moreland*'s Rule, then the Solution of thi *Problem* is deduced from his Rule for *Prob.* 5. *viz.* $p=\frac{nrt^2+2tn-trn}{2+2tr}$. Whence $2p+2ptr=nrt^2+2tn-trn$, and $2ptr-nrt^2+trn=2tn-2p$, and $r=\frac{2tn-2p}{2pt+tn-nt^2}$; that is, take double of the Difference betwixt the present Worth and the Product of Time and Annuity; divide this by the Difference betwixt the Product of Annuity into the Square of the Time, and the Sum of the Product of Time and Annuity, and twice the Product of present Worth and Time; the Quote is the Rate.

For Annuities *payable in half yearly or quarterly Payments.*

In the preceding *Problems*, let t represent the Number of half Years or quarters that an Annuity continues; r the Interest of 1 *l.* for $\frac{1}{2}$ or $\frac{1}{4}$ of a Year; and n the $\frac{1}{2}$ Years or quarters Payment: Then all the preceding Rules are applicable to half yearly or quarterly paid Annuities the same way as to yearly Payments.

PART II. *Of* Annuities *at Compound Interest.*

Observe; *In the following* Problems, *the Amount of* 1 *l.* *and* 1 *Year's Interest is called the* Rate. *For Example:* 1.05, *if Interest is at* 5 per Cent.

1. *Of* Annuities *in* Arrears *at Compound Interest.*

PROB. 1. Having the Annuity, Rate and Time (in whole Years) to find the Amount.

Rule

Rule. Make 1 the least Term of a Geometrical Progression, the Rate the 2d Term, (which consequently is the Ratio of the Progression by which every Term is to be multiplied to produce the next) carry it to as many Terms as the Number of Years; its Sum is the Amount of 1 *l.* Annuity for the given Time, [and to find the Sum most easily, multiply the last Term by the Rate, or Ratio, which produces a Power of the Rate whose Index is the Time; and from the Product take 1 the 1st Term; then divide the Remainder by the Rate less 1; the Quote is the Sum] which Sum multiplied by the given Annuity, the Product is the Amount sought.

Example. What is the Amount of an Annuity of 40 *l.* to continue 5 Years, allowing Compound Interest at 5 *per Cent.*? Answer, 221.02525 *l.* which is 221 *l* 0 *s* 6.06 *d.* *Thus:* Of a Geometrical Progression beginning with 1, whose Ratio is 1.05, the 5th Term is $\overline{1.05}^4 = 1.21550625$, and the Sum of the Series is 5.52563125. For $\overline{1.05}^5 = 1.2762815625$, and $\overline{1.05}^5 - 1 = .2762815625$; which divided by $1.05 - 1$, or .05, the Quote is 5.52563125, the Amount of 1 *l.* Annuity for 5 Years. Which multiplied by 40, the Product is 221.02525 = 221 *l.* 0 *s.* 6.06 *d.* the Amount of 40 *l.* Annuity.

DEMONSTR. It is plain that upon the 1st Year's Annuity there will be due as many Years Compound Interest, as the given Number of Years less 1; and gradually one Year less upon every succeeding Year to that preceding the last, which has but one Year's Interest, the last having no Interest due: But the Amount of 1 *l.* for 1 Year being 1.05, the Amounts of it for 2, 3, *&c.* Years are (by *Prob.* 1. *Part* 2. §. 1.) the several Powers or Products of 1.05 multiplied continually by itself 2, 3, *&c* times; consequently $\overline{1.05}^4$, $\overline{1.05}^3$, $\overline{1.05}^2$, 1.05, 1 are the Amounts of the 1st, 2d, 3d, 4th and 5th Years Annuity of 1 *l.* whose Sum is therefore the whole Amount of the Annuity of 1 *l.* for 5 Years. But 1 *l.* is to the Amount of 1 *l.* as any other Annuity to its Amount. Wherefore the Amount of 1 *l.* *Ann.* multiplied by another *Ann.* gives its Amount.

Universally. Let R be the Rate, or Amount of 1 *l.* with 1 Year's Interest; then the Series of Amounts of several Years of 1 *l.* Annuity from the last to the first is 1, R, R^2, R^3, *&c.* R^{t-1}. And the Sum of this, according to the Rule of a Progression Geometrical, is $\frac{R^t - 1}{R - 1}$, the Amount of 1 *l.* Annuity for *t* Years. And this multiplied by any other Annuity *n* gives the Amount of that Annuity, *viz.* $\frac{R^t - 1}{R - 1} \times n$, or $\frac{nR^t - n}{R - 1}$, (the universal Expression of the Rule) because all Annuities are proportional with their Amounts, and $1 : \frac{R^t - 1}{R - 1} :: n : \frac{R^t - 1}{R - 1} \times n$.

SCHOLIUM. As $1 : R : R^2$, *&c.* R^{t-1} is the Series of Amounts of 1 *l.* Annuity forborn for *t* Years; so is $n : nR : nR^2$, *&c.* nR^{t-1} the Amounts of the Annuity *n*. Wherefore the Amount of any Annuity for *t* Years is the Sum of a Geometrical Progression, whose least Term is the Annuity, and Number of Terms equal to the Years, the Ratio being the Sum of 1 *l.* and one Year's Interest expressed here by R. And by the Rule of Geometrical Progressions, the Sum of this Series is $\frac{nR^t - n}{R - 1}$ as before.

Another Rule. Find a principal Sum, of which 1 Year's Interest is equal to the given Annuity; then find the Amount of that Principal for the given Time and Rate, (by *Prob.* 1. *Part* 2. §. 1.) from which Amount subtract the Principal, the Remainder is the Amount of the Annuity.

The *Reason* is plain; For the Amount of the Principal is the Sum of the Principal, and every Year's Simple Interest, (which make the several Years Annuities) with all the Compound Interest arising from these; so that the Principal taken from the Amount, leaves the Sum

Sum of the several Years Simple Interest with all their Compound Interest; which is plainly the whole Amount of the Annuity.

In the preceding *Example*; I say, as .05 to 1, so is 40 to 800 a Principal, which gives 40*l.* Interest in 1 Year at 5 *per Cent.*; then the Amount of this Principal forborn 5 Years at Compound Interest is 1021.02525; from which take 800, the Remainder is 221.02525 as before.

SCHOLIUM. This *Rule* may be proved from its Coincidence with the former, thus; R being the Sum of 1 *l.* and 1 Year's Interest; then is $R-1$ the Year's Interest, and $R-1:1::n:\frac{n}{R-1}$ the Principal, whose Year's Interest is n. Then the Amount of this Principal for t Years is $\frac{nR^t}{R-1}$ (by *Prob.* 1. *Part* 2. §. 1.) from which take the Principal $\frac{n}{R-1}$, the Remainder is $\frac{nR^t-n}{R-1}$ the Amount by the former *Rule.*

PROB. 2. Having the Amount, Rate and Time, to find the Annuity.

Rule. Take any Annuity at pleasure, and find its Amount by the last. Then as that Amount is to its Annuity, so is the given Amount to its Annuity. And *observe*, If you chuse 1 *l.* Annuity, you have nothing to do but divide the given Amount by the Amount of 1 *l.* Annuity.

Example. What Annuity will amount to 221 *l.* 0 *s.* 6 *d.* in 5 Years at the Rate of 5 *per Cent.*? Answer, 40 *l.* For the Amount of 1 *l.* Annuity in 5 Years is 5.52563125; by which dividing the given Amount 221.025, the Quote is 39.999 = 39 *l.* 19 *s.* 11 *d.* 3 *f.* *&c.* or 40 *l.* nearly; which it would have been precisely, had we taken 221 *l.* 0 *s.* 6.06 *d.* (or 221.02525) for the Amount, as in the 1st *Problem.*

DEMONSTR. The Reason of this Rule is plain: For all Annuities and their Amounts must be proportional.

SCHOLIUM. The Amount of 1 *l.* Annuity is, by *Prob.* 1. expressed $\frac{R^t-1}{R-1}$; and calling the given Amount A, then this Rule is $n = A \div \frac{R^t-1}{R-1} = \frac{AR-A}{R^t-1}$; that is, multiply the Rate by the Amount, and from the Product take the Amount; the Difference divide by the Difference of 1, and the t Power of the Rate. The Use of this Expression of the Rule you'll find in the following *Problems.* And we may also deduce it thus, by *Prob.* 1. $A = \frac{nR^t-n}{R-1}$; hence $AR-A = nR^t-n$, and $\frac{AR-A}{R^t-1} = n$.

PROBL. 3. Having the Annuity, Rate, and Amount, to find the Time.

Rule. Find a corresponding Principal (as in the 2d *Rule* of *Probl.* 1.), the Sum of this and the given Amount, is the Amount of that Principal for the given Rate and Time sought: Wherefore find the Time (by *Probl.* 3. *Part* 2. §. 1.) thus; divide that Amount by its Principal, then multiply the Rate continually by it self till the Product or Power produced be equal to the former Quote, the Index of that Power, or Number of Multiplications more 1, is the Number of Years sought.

Example. In what Time will 40 *l.* Annuity amount to 221.02525 *l.* at the Rate of 5 *per Cent*? Answer, 5 Years. Thus, .05 : 1 : : 40 : 800 the corresponding Principal, then 800 + 221.02525 = 1021.02525; which divided by 800, the Quote is 1.2762815625, equal to the 5th Power of 1.05, or $\overline{1.05}^5$; so that 5 is the Number of Years sought.

DEMON-

DEMONSTR. By what's shewn in *Prob.* 1. *Rule* 2. the Difference of the Principal found and its Amount, is the Amount of the Annuity; wherefore that Principal added to the Amount of the Annuity gives the Amount of the Principal: The rest of the *Rule* is demonstrated in the *Problem* referred to.

SCHOLIUM. The corresponding Principal is expressed $\frac{n}{R-1}$ (as in *Schol.* to *Prob.* 1.) and this added to the given Amount A, the Sum is $A+\frac{n}{R-1}=\frac{n+AR-A}{R-1}$; which divided by the Principal $\frac{n}{R-1}$, the Quote is $\frac{n+AR-A}{n}=R^t$ by the Rule, and is another Expression of it; which may also be deduced thus, by *Prob.* 2. $\frac{AR-A}{R^t-1}=n$; hence $AR-A=nR^t-n$, and $n+AR-A=nR^t$; and lastly, $\frac{n+AR-A}{n}=R^t$.

PROB. 4. Having the Annuity, Amount and Time, to find the Rate.

There is no Rule within my Limits that will solve this *Problem*, except for that one Case where the Time is 2 Years. And to come at that Rule, I must deduce the general Rule from what precedes, and so leave it with the Application to this Case.

In the last *Problem* we saw $AR-A=nR^t-n$. Whence $AR=nR^t-n+A$; and again, $AR-nR^t=A-n$; and dividing by n, it is $\frac{A}{n}\times R-R^t=\frac{A-n}{n}$. Now if t is above 2, we can make nothing of it by common Methods: But if t is $=2$, then it's $\frac{A}{n}\times R-R^2=\frac{A-n}{n}$. And calling $\frac{A}{n}=d$, then (by *Prob.* 6. *Chap.* 2. *Book* III.) $R=\frac{d}{2}+$, or $-\overline{\frac{A-n}{n}-\frac{da}{4}}|^{\frac{1}{2}}$.

2. *Of the Purchase of Annuities at* Compound Interest.

PROBL. 5. Having the Annuity, Rate, and Time, to find the Present Worth.

Rule 1. Find the Present Worth of each Year by itself (by *Probl.* 2. *Part* 2 §. 1.); the Sum of all these is the Present Worth sought.

Rule 2. Find the Amount of the Annuity (by *Probl.* 1) then find the Present Worth of this Amount as a Sum due at the End of the whole Time (by *Probl.* 2. *Part* 2. §. 1.); it is the Present Worth sought.

Rule 3. Find a Principal Sum whereof the Annuity is 1 Years Interest; and find the Present Worth of it as a Sum due at the End of the Time; subtract this Present Worth from its Principal; the Remainder is the Present Worth of the Annuity.

Observe, The last *Rule* is the easiest, and therefore I apply it in the following *Example*; but there was a Necessity to deliver them all, because the 1st is the fundamental Rule, which has a Reason in it self; the 2d having its Reason only in its Coincidence with the 1st, as to the *Answer*; and the 3d depends upon the 2d.

Example. What is the Present Worth of an Annuity of 40*l.* to continue 5 Years, discompting at 5 *per Cent.*? *Answer:* 173 *l.* 3 *sh.* 7 *d.* Found thus; as .05 to 1, so is 40 to 800, a Principal Sum whereof the Present Worth, discompting Compound Interest for 5 Years, at 5 *per Cent.* is 626 *l.* 16 *sh.* 5 *d.* [for the Amount of 1 *l.* in 5 Years is $\overline{1.05}^5=$ 1.276281, *&c.* and 800 ÷ 1.276281 = 626.8212, *&c.* = 626 *l.* 16 *sh.* 5 *d.* nearly] then from 800 take 626 *l.* 16 *sh.* 5 *d.* the Remainder is 173 *l.* 3 *sh.* 7 *d.*

DEMON-

DEMONSTR. 1. That the 1st *Rule* gives the true *Answer* agreeable to the plain Meaning of the *Question*, is manifest, and is also confirm'd by what has been said upon the like Question with Simple Interest; so that every other Rule must coincide with this in the *Answer*, else it cannot be true; and that the other two *Rules* give the same *Answer*, I shall also demonstrate.

2. For the 2d Rule. The Present Worth of any single Year amounts to the Annuity when it becomes payable. [For *Example*: the Present Worth of the 3d Year is a Principal, which in 3 Years Time will make an Amount equal to the Annuity] and therefore the Amount of that Year from the Time it falls due to the End of the given Time, is the same as the Amount of the Present Worth of it from the Time of the Purchase to the End of the Annuity. Consequently, the Present Worth of any Year, discompting from the Time it falls due, is the same as the Present Worth of the Amount of that Year, summed up to the End of the Annuity, and then discompted to the Time of the Purchase. But the Amount of the Annuity at the End of the whole Years is the Sum of the Amounts of the several Years; and consequently the Present Worth of this Sum is the Sum of the Present Worths of these particular Amounts; which being equal to the Present Worths of the several Years discompted from the Times they fall due: Therefore both *Rules* give the same *Answer*.

3. For the 3d *Rule*, It follows from the 2d thus. The Principal found by the *Rule*, will make an Amount from the Time of the Purchase to the End of the Annuity, equal to the Sum of it self, and the Amount of the Annuity (as we saw in the last *Problem*); but the Present Worth of any Sum is equal to the Sum of the Present Worths of any two (or more) Parts of that total Present Worth; consequently, the Present Worth of one Part taken from the Present Worth of the Whole, leaves the Present Worth of the other Part; *that is*, in the present Case, The Present Worth of the Principal found (which is a Part of its own Amount) taken from the same Principal (which is the total Present Worth of its total Amount), leaves the Present Worth of the Annuity (which is the other Part of the Amount of that Principal).

SCHOLIUM. By *Probl.* 1. the Amount of the Annuity is expressed thus, $A = \frac{nR^t - n}{R - 1}$ and (by *Probl.* 2. *Part* 2. §. 1.) the Present Worth of this is $p = \frac{nR^t - n}{R^{t+1} - R^t}$, which is therefore the Expression of the 2d *Rule*: And that the 1st *Rule* resolves into the same Expression, may be thus shewn: The Present Worths of the several Years are (by *Probl.* 2. *Part* 2. §. 1.) thus expressed, $\frac{n}{R} : \frac{n}{R^2} : \frac{n}{R^3}$, &c. to $\frac{n}{R^t}$, which is a Geometrical Progression, in the Ratio of R to 1, whose Sum is, by the Rule of a Geometrical Progression, $\frac{nR^t - n}{R^{t+1} - R^t}$, as before. The Coincidence of the 3d *Rule* with this may also be easily shewn after the same Manner; wherefore, instead of the preceding three *Rules*, we may take it according to this universal Expression, thus;

Take two Powers of the Rate whose Indexes are the Time, and the Time more 1; multiply the 1st of these Powers by the Annuity, and take the Annuity from the Product; then divide this Difference by the Difference of these two Powers; the Quote is the Present Worth.

Or thus: Take the Difference of 1 and the lesser Power, which divide by the Difference of the Powers, and then multiply the Quote by the Annuity; because $\frac{nR^t - n}{R^{t+1} - R^t} = \frac{R^t - 1}{R^{t+1} - R^t} \times n$.

COROLL. Hence, knowing the Time any Annuity continues, and the Rate allowed in the Purchase, we can find how many Years and Parts of a Year's Purchase it is worth, without knowing the Annuity: For since the Price is $\frac{R^t - 1}{R^{t+1} - R^t} \times n$, therefore, whatever n is, $\frac{R^t - 1}{R^{t+1} - R^t}$ does universally express the Number of Times the Annuity is contained in the Price.

PROBL. 6. Having the Present Worth, Rate, and Time, to find the Annuity.

Rule. Take any Annuity at pleasure, and find its Present Worth by the last; then, as that Present Worth is to its Annuity, so is the given Present Worth to the Annuity sought. *Observe,* If you take 1 *l.* Annuity, then there is no more to be done but divide the given Present Worth by that of 1 *l.* Annuity.

Example: What Annuity, to continue 5 Years will be purchased for 173 *l.* 3 *sh.* 7 *d.* allowing Compound Interest at 5 *per Cent.*? *Answer:* 40 *l.* Found thus:
The Present Worth of 1 *l.* Annuity to continue 5 Years is 4.329, and 173 *l.* 3 *sh.* 7 *d.* or 173.1788, *&c.* ÷ 4.329 = 40.004, *&c.* which is 40 *l.* nearly, for the Decimal is less than a Farthing.

DEMONSTR. The Reason of this *Rule* is plain: For all Annuities are proportional with their Present Worths for the same Time and Rate.

SCHOLIUM. By *Schol.* to the preceding *Problem,* the Present Worth of 1 *l.* Annuity is expressed $\frac{R^t - 1}{R^{t+1} - 1}$; wherefore the Annuity answering to any Present Worth p (for the same Time and Rate) is $p \div \frac{R^t - 1}{R^{t+1} - 1} = p \times \frac{R^{t+1} - 1}{R^t - 1}$, which is an universal Rule. We may also deduce it from the preceding *Problem,* thus; $p = \frac{nR^t - n}{R^{t+1} - 1}$, whence $pR^{t+1} - p = nR^t - n$, and dividing by $R^t - 1$, it is $n = \frac{pR^{t+1} - p}{R^t - 1}$, or $p \times \frac{R^{t+1} - 1}{R^t - 1}$.

PROBL. 7. Having the Annuity, Present Worth, and Rate, to find the Time.

Rule. Find a Principal whereof 1 Year's Interest is the Annuity, from which subtract the given Present Worth; the Remainder is the Present Worth of that Principal considered as a Sum due at the End of the Annuity: Therefore, by *Probl.* 3. *Part* 2. §. 1. find in what Time this Present Worth will amount to the Principal found; it is the Time sought.

Example: What Time must an Annuity of 40 *l.* continue, to be worth of ready Money 173 *l.* 3 *sh.* 7 *d.* allowing 5 *per Cent.* Compound Interest. *Answer:* 5 Years
For 05 : 1 :: 40 : 800 the Principal sought; from which take 173 *l.* 3 *sh.* 7 *d.* the Remainder is 626 *l.* 16 *sh.* 5 *d.* or 626.82 *l.* nearly, the Present Worth of 800 due at the End of the Time sought; then 800 ÷ 626.82 = 1.27678, *&c.* = $\overline{1.05}^{5}$ nearly, (for this is 1.276281) so the Time sought is 5 Years.

DEMONSTR. It was shewn in *Probl.* 5. that the Difference of the Principal found and its Present Worth, is the Present Worth of the Annuity; consequently the Difference of

of the Present Worth of the Annuity, and the same Principal, must be the Present Worth of that Principal. The rest of the *Rule* is demonstrated in the *Problem* referred to.

SCHOLIUM. The corresponding Principal being expressed $\frac{n}{R-1}$ (for $R - 1 : 1 :: n : \frac{n}{R-1}$) and the given Present Worth p, then by this Rule $\frac{n}{R-1} - p = \frac{n+p-pR}{R-1}$ is the present Worth of $\frac{n}{R-1}$: And by *Probl.* 3. *Part* 2. §. 1. it is $\frac{n}{R-1} \div \frac{n+p-pR}{R-1}$ $\left(= \frac{n}{n+p-pR}\right) = R^t$, which is the universal Expression of the Rule; to be deduced also from *Probl.* 5. thus, $p = \frac{nR^t - n}{R^{t+1} - R^t}$; hence $pR^{t+1} - pR^t = nR^t - n$, and $pR^{t+1} = nR^t - n + pR^t$, then $pR^{t+1} + n = nR^t + pR^t$; also $n = nR^t + pR^t - pR^{t+1}$ $= \overline{n+p-pR} \times R^t$; and lastly, $\frac{n}{n+p-pR} = R^t$.

PROBL. 8. Having the Annuity, Present Worth, and Time, to find the Rate.

This *Problem* can be solved by no Method within my Limits, except for that Case where the Time is 1 Year; which I shall shew by deducing the general Rule from the preceding; thus, in the *Scholium* to the last *Problem* we see $n = nR^t + pR^t - pR^{t+1} =$ $\overline{n+p} \times R^t - pR^{t+1}$, and $\frac{n}{p} = \frac{n+p}{p} \times R^t - R^{t+1}$; now if t is any Number greater than 1, to find R we have an adfected Root, above the Square, to extract; for which we have no Rules in this Work: But if the Time is 1 Year, then the Rule is $\frac{n}{p} = \frac{n+p}{p} \times R -$ R^2; call $\frac{n+p}{p} = d$, and then, by *Probl.* 6. *Ch.* 2. *Book* III. $R = \frac{d}{2} +$ or $- \left(\frac{n}{p} - \frac{dd}{4}\right)^{\frac{1}{2}}$

Of Annuities *payable Half-yearly or Quarterly, and other Questions, wherein the Time is some Part of a Year, or whole Years with Part of a Year.*

Part I. IF Annuities are payable in Half-years or Quarters (the Half-year or Quarter's Payment being the $\frac{1}{2}$ or $\frac{1}{4}$ of the whole Years Payment), then if in the preceding *Problems* t express the Number of Half-Years or Quarters, and n the Half-Year's or Quarter's Payment, also R the Sum of 1 *l.* with $\frac{1}{2}$ or $\frac{1}{4}$ of a Year's Interest of it, taken according to any Supposition (*i. e.* By either of the *Methods* explained in *Schol.* 2. *Probl.* 1. *Part* 2. §. 1. and here the 2d *Method* seems to be the fairest), then all the preceding *Rules* are applicable to Half-yearly and Quarterly Payments the same way as for Years; yet the Labour of the Calculation is hereby increased.

Part II. But whether Annuities are payable Yearly or Quarterly, the Time sought may perhaps not come out in whole Years or Quarters precisely; which is a Sign that the Question is impossible in whole Years or Quarters: So that if what is wanting in the Number found, to make it whole Years or Quarters, be very little, it may be neglected; but if it is great, then we may find some Part of the Year or Quarter's Payment corresponding to it, as

I ſhall briefly explain with reſpect to Annual Payments; and, to make this Part compleat, I ſhall alſo conſider thoſe Queſtions wherein the Time is given in Years and Parts of a Year. And *obſerve* too, That tho' the Payments are by Half-years or Quarters, the Calculations may juſtly be made as if they were by Years.

1. In *Probl.* 1. If there is any Part of a Year, then, having found the Amount for the whole Years, find the Intereſt of that Amount for the Parts of a Year in the Queſtion, by any of the *Methods* explained in *Schol.* 2. *Probl.* 1. *Part* 2. §. 1. as you think fit, and add that to the Amount for the whole Years.

2. In *Probl.* 2. If the Time is whole Years and Parts of a Year, the Solution depends upon the laſt.

3. In *Probl.* 3. Where the Time is ſought, the laſt Part of the Solution depends upon *Probl.* 3. *Part* 2. §. 1. And for the Parts of a Year in the Time ſought, ſee the *Schol.* to that *Problem.*

4. For *Probl.* 4. we can make nothing of it; and it is indeed of little Uſe.

5. In *Probl.* 5. if the Time is whole Years and Parts of a Year, the Solution will be different according as you ſuppoſe Intereſt for the Parts of a Year to be taken. But however that be, this is firſt to be done, *viz.* to find the Preſent Worth, diſcompting for the Whole Years in the Queſtion; then find the Preſent Worth of this Preſent Worth, diſcompting for the Parts of a Year in the Queſtion. See *Schol.* 3. to *Probl.* 2. *Part* 2. §. 1.

6. For *Probl.* 6. the Solution depends upon the laſt.

7. For *Probl.* 7. the laſt Part of the Solution depends upon *Probl.* 3. *Part* 2. §. 1. See the *Schol.* to that *Problem.*

8. For the 8. *Probl.* we can make nothing of it; and it's of as little Uſe as the 4th.

§. III. *Of the* Purchaſe *of* Freehold Eſtates, *or* Annuities *to continue for ever, allowing Compound Intereſt.*

Probl. To find the Price or Preſent Worth of a Freehold Eſtate or Annuity to continue for ever, diſcompting Compound Intereſt.

Rule. Find a Principal Sum of which one Year's Intereſt is the Rent or Annuity given; it is the Price ſought.

Example: What is the Price of a perpetual Annuity of 40 *l.* diſcompting at 5 *per Cent.* Compound Intereſt? *Anſwer:* 800 *l.* for .05 : 1 : : 40 : 800 *l.*

Demonstr. The *Reaſon* of this *Rule* ſeems of it ſelf obvious; for it's plain, that ſince a Year's Intereſt of the Price is the Annuity, therefore there can neither more nor leſs be made of that Price than of the Annuity, whether we employ it by Simple or Compound Intereſt.

SCHO-

SCHOLIUMS.

1. But left any body fhould doubt whether the Difcompt is here made at Simple or Compound Intereft, let them confider the *Rule* of *Probl.* 2. *Part* 2. § 1. for the Prefent Worth of an Annuity, which is thus expreffed, $p = \frac{n}{R} + \frac{n}{R^2} + \frac{n}{R^3}$, *&c.* ($n$ being the yearly Rent or Annuity). Now if the Annuity continues for ever, then this Geometrical Progreffion goes on, decreafing for ever; and the Sum of it, by *Probl.* 1. *Ch.* 3. *Book* V. is $\frac{n}{R-1}$, which is a Principal Sum of which one Year's Intereft is the Annuity n; for $R-1 : 1 :: n : \frac{n}{R-1}$.

2. Again: Of thefe three things, the Annuity to continue for ever, the Price, and Rate of Intereft, any two of them being given, the third may be found eafily, from the preceding *Rule*, which being expreffed thus, *viz.* $p = \frac{n}{R-1}$, hence it follows that $n = pR - p$, and $R - 1 = \frac{n}{p}$, or $R = \frac{n}{p} + 1$.

3. The Unreafonablenefs of purchafing Annuities at Simple Intereft is further confirmed by this *Problem*: For the Price of a perpetual Annuity, difcompting Simple Intereft, as it would be greater than when Compound Intereft is difcompted, it would therefore be a Principal Sum of which 1 Year's Intereft is greater than the Annuity: And indeed the Price would be infinitely great, or greater than any affignable Number.

§. IV. *Of* Annuities, *or* Leafes *in* Reverfion.

Hitherto I have confidered Annuities as immediately entered upon, *i. e.* That the 1ft Year commences prefently or at payment of the Money: But if it is in *Reverfion*, then we muft confider the Time betwixt payment of the Money and the beginning of the 1ft Year of the Annuity; and therefore,

1. In *Probl.* 5. §. 3. having found the Prefent Worth as if the Annuity were immediately entred upon, find again the Prefent Worth of that Prefent Worth, rebating for the Time betwixt the Purchafe and Commencement of the Annuity; and this is the Anfwer.

2. In *Problems* 6. and 7. take the given Prefent Worth, and find its Amount for the Time of *Reverfion*, and take this Amount as the Prefent Worth paid at the Commencement of the Annuity, and proceed with it.

Obferve, Annuities, and Rents of Houfes or Lands, are of the fame Nature, wherein the fame Queftions occur as to their being in Arrears or being purchafed: But with refpect to *Leafes* there arife Queftions with fome different Circumftances, owing to the Practice of taking what they call *Fines*, which is a Sum of Money paid at the Beginning of the Leafe, befides the Yearly Rent. I fhall give you a few of thefe Queftions, with a general Direction for the Solution; which will be fufficient upon this Subject.

Queftion 1. There is a Piece of Land worth 20 *l.* yearly Rent, and 100 *l.* of Fine for a Leafe of 21 Years: The Mafter is willing to quit the Fine, and increafe the Rent; What

What ought the Rent to be? *Rule.* Find what Rent or Annuity, to continue 21 Years, 100 *l.* will purchafe, difcompting at the agreed Rate of Intereft; the Sum of that and the former Rent is the Rent fought. *Obferve,* If the whole Fine is not to be taken away, find the Annuity anfwering to the Part taken away.

Queft. 2. A Piece of Land is worth 12 *l.* Yearly Rent, and a Fine of 30 *l.* for 19 Years: The Farmer is willing to pay more Fine, and reduce the Rent to 20 *l.* What ought the Fine to be? *Rule.* Take the Difference of the two Rents (30 and 20), and find the Prefent Worth of an Annuity equal to the Difference for the feme Time (19 Years); that is the additional Fine to be paid. *Obferve,* The fame way the whole Rent may be taken away.

Queft. 3. There is a Farm to be let for 21 Years, at 10 *l.* Yearly Rent and 20 *l.* of Fine; if the fame be lett for 30 Years at the fame Rent, What ought the Fine to be? *Rule.* Find what Annuity 20 *l.* will purchafe for 21 Years, at the agreed Rate (that added to the Rent 10 *l.* is the true Rent with no Fine), therefore find the Prefent Worth of that Annuity to continue 30 Years; it is the Fine fought.

Queft. 4. A Perfon has 7 Years to run of a Leafe of 21 Years, for which he paid 40 *l.* Fine and 15 *l.* Yearly Rent: He would renew the Leafe to 16 Years from this Time (*i. e.* for 12 Years after the firft Leafe expires): What Fine ought he to pay? *Rule.* Find what Rent for 21 Years the given Sum 40 *l.* will purchafe; then find the Prefent Worth of this Yearly Rent to continue 12 Years; laftly, find the Prefent Worth of this laft Prefent Worth, rebating for 7 Years, the Time that remains of the old Leafe; this is the Fine to be paid.

General SCHOLIUM, *with an* Abftract *of the preceding* Problems *where Compound Intereft is concerned.*

It is thought the moft convenient Method for the Calculation of Annuities, to have *Tables* ready made, extending to the greateft Number of Years that ordinarily occur in that Bufinefs, and for feveral Rates of Intereft that are moft likely to occur; by Means of which the Anfwers of the more ufeful of the preceding *Problems* may be eafily found.

The *Tables* which are common upon this Subject are limited to 1 *l.* Thus, we have 1. A Table of the Amounts of 1 *l.* for 30 or 50 Years, at feveral Rates Compound Intereft. 2. Of the Prefent Worths of 1 *l.* due after any Number of Years from 1 to 30 or 50. 3. The Amounts of 1 *l.* Annuity. 4. The Prefent Worths of 1 *l.* Annuity. 5. The Annuity to be purchafed for 1 *l.* By Means of which the Anfwer to any of the preceding *Problems* may be eafily got for any other Sum than 1 *l.* by one Multiplication or Divifion.

But here I muft *obferve,* That thofe who are employed in fuch Calculations ought to underftand the Rules at large, and fo be able to examine and make Tables for themfelves: And in my Opinion, it is not fit in Queftions of Confequence to truft to any Tables but what one has examined, or made for himfelf. Now all that is propofed by the Tables is the Eafe and Expeditioufnefs of the Calculation; and the greateft Burden of the Work is the finding the Power or Product of the Rate, (or Sum of 1 *l.* and 1 Year's Intereft) multiplied continually into itfelf, as the Rules direct; which, if the Number of Years in the Queftion is great, becomes very tedious; for this being found, the Reft of the Work, in the more ufeful of the preceding *Problems,* is no more than one or two Operations of Multiplication or Divifion; with a fimple Addition or Subtraction in fome Cafes: And therefore, whoever can readily do Multiplication or Divifion with Integers or Decimals (and no other can perform thefe Calculations, even with the common Tables, which require at leaft one Multiplication or Divifion) needs no more than a Table of the Powers of

of several Rates of Interest carried to a convenient Length; which are the Amounts of 1 *l.* Principal for 1, 2, *&c.* Years at Compound Interest (for I speak of no other here), according to the *Rule* of *Probl.* 1. §. 1. *Part* 2. The following *Table* at 5 *per Cent.* to 31 Years is an Example which may be extended to more Years, and the like made for other Rates of Interest. Whoever understands the preceding *Rules* will know the Uses to be made of this *Table*; yet it will be convenient along with this *Table* to point out the Use of it (for it will be needless to make *Examples*, since the preceding are sufficient), by which you'll have the preceding *Rules* drawn together in one short and clear View.

TABLE *of the Powers of* 1.05, *or the* Amounts *of* 1 l. *at* 5 per Cent.

Yrs.		
1	1.05 *l.*	$=$ R the Rate.
2	1.1025	$= R^2$
3	1.157525	$= R^3$
4	1.215506	$= R^4$
5	1.276281	$= R^5$
6	1.340096	$= R^6$
7	1.407100	$= R^7$
8	1.477455	$= R^8$
9	1.551328	$= R^9$
10	1.628895	$= R^{10}$
11	1.710339	$= R^{11}$
12	1.795856	$= R^{12}$
13	1.885649	$= R^{13}$
14	1.979932	$= R^{14}$
15	2.078928	$= R^{15}$
16	2.182874	$= R^{16}$
17	2 292018	$= R^{17}$
18	2.406919	$= R^{18}$
19	2.526950	$= R^{19}$
20	2.653298	$= R^{20}$
21	2.785962	$= R^{21}$
22	2.925261	$= R^{22}$
23	3.071524	$= R^{23}$
24	3 225100	$= R^{24}$
25	3.386355	$= R^{25}$
26	3.555673	$= R^{26}$
27	3 733456	$= R^{27}$
28	3.920129	$= R^{28}$
29	4.116135	$= R^{29}$
30	4.321942	$= R^{30}$
31	4.538039	$= R^{31}$

The Use *of this* Table *in the Solution of the preceding* Problems.

1. *Of* Interest.

1. To find the Amount of any Principal for any Time, at 5 *per Cent.* Compound Interest. *Rule.* Take the tabular Number against the Number of Years, its Product by the Principal gives the Amount, which, in *Probl.* 1. §. 1. *Part* 2. is expressed thus, $A = p \times R^t$.

2. To find the Principal (or Present Worth) answering to any Amount (or Debt payable) after certain Years. *Rule.* Divide the Amount by the tabular Number against the Number of Years; the Quote is the Principal; expressed thus, $p = \frac{A}{R^t}$, as in *Probl.* 2. § 1. *Part* 2.

3. To find in what Time any Principal will make a certain Amount. *Rule.* Divide the Amount by the Principal, and seek the Quote, or the next lesser Number in the *Table*, and against it is the Number of Years sought. Because $\frac{A}{p} = R^t$, as in *Probl.* 3. §. 1. *Part* 2.

Observe, For finding the Rate from the Principal Amount and Time, we should have Tables at different Rates of Interest; then taking the Quote of the Amount, divided by the Principal, seek it against the Number of Years in the Tables: The Rate of that Table where it is found is the Rate sought. But if it is not exactly found, take the next Number lesser and also the next greater (if both a greater and lesser canbe found among the Tables) and the Rates of these two *Tables* are Limits betwixt which the Rate sought lies

2. *Of* Annuities.

1. To find the Amount of an Annuity. *Rule.* From the tabular Number against the Number of Years take 1, and divide the Remainder by the Rate less 1; multiply the Quote by the Annuity; the Product is the Amount; expressed thus, $\frac{R^t - 1}{R - 1} \times u = A$, as in *Probl.* 1. §. 2. *Part* 2.

2. To

2. To find the Annuity which makes a certain Amount. *Rule.* Take the Quote directed in the last *Rule*, and by it divide the Amount; this Quote is the Annuity sought. thus, $A \div \frac{R^t - 1}{R - 1} = n$, as in *Probl.* 2. §. 2. *Part* 2.

3. To find in what Time any Annuity will make a certain Amount. *Rule.* Divide the Annuity by the Rate less 1, to the Quote add the Amount, and divide the Sum by the first Quote; seek this last Quote, or the next lesser Number in the *Table*; against it stands the Number of Years. Thus, $\overline{\frac{n}{R - 1} + A} \div \frac{n}{R - 1} = R^t$, as in *Probl.* 3. §. 2. *Part* 2.

4. To find the Present Worth of an Annuity. *Rule.* Divide the Annuity by the Rate less 1; this Quote divide by the tabular Number against the Years, and subtract this last Quote from the first Quote; the Remainder is the Present Worth sought. Thus, $\frac{n}{R - 1} - \frac{n}{R - 1} \div R^t = p$, as in *Rule* 3. *Probl.* 5. §. 2. *Part* 2.

5. To find what Annuity a given Sum will purchase for certain Years. *Rule.* Take 1 from the tabular Number against the Years; divide it by the Difference betwixt that tabular Number and the next greater, and by this Quote divide the Present Worth; this last Quote is the Annuity sought, as in *Schol.* to *Problem* 6. §. 2. *Part* 2. thus expressed, $n = p \div \frac{R^t - 1}{R^{t+1} - R^t}$.

6. To find what Time an Annuity must continue to be worth a certain Price. *Rule.* Divide the Annuity by the Rate less 1, and from the Quote take the Price; by this Remainder divide the first Quote, and seek this last Quote in the *Table*; against it is the Number of Years. Thus, $\frac{n}{R - 1} \div \overline{\frac{n}{R - 1} - p} = R^t$, as in *Schol.* to *Probl.* 7. §. 2. *Part* 2.

Observe, That the preceding *Table* was made by compleating the Multiplication at every Step, and then taking only the first 6 decimal Places. Also, when the Figure in the 7th Place exceeded 5, 1 was added to the 6th Place, which makes the Error less, only it makes it Excessive instead of Defective.

§. V. *Of the* Equation of Payments.

DEFIN. WHEN several Debts are payable at several Terms (bearing no Interest till after the Term of Payment), to find a Term at which if they are all paid neither Debtor or Creditor loses any thing, is called, *Equating* the Terms of Payment, *i. e.* reducing them to one.

PROBL. To find the *equated* Term at which several Debts payable at different Terms, may be paid at once, without Loss to Debtor or Creditor, allowing Simple Interest.

For the Solution of this *Problem* there are different Methods, which I shall explain and compare.

Method 1. The Debts being expressed in Periods and decimal Parts, the Times in Years and Decimals, or Months and Decimals, if there are no Years, multiply each Debt by its Time; add all these Products into one Sum, and divide it by the Sum of the Debts; the Quote is the Time sought [reckoned from the Day of this Calculation, as the other Times are also supposed to be].

Exam-

Example: If 40 *l.* is due after 6 Months, 70 *l.* after 4 Months, the equated Time is 4 Months 21 Days: Found thus; $40 \times 6 = 240$. $70 \times 4 = 280$. $40 + 70 = 110$. $240 + 280 = 520$. then $520 \div 110 = 4.72$, *&c.* which is 4 Months 21 Days nearly, allowing 30 Days to a Month.

This *Rule* has Mr. *Cocker* and Mr. *Hatton* (among others) for its Defenders; who endeavour, each in his own way, to demonſtrate the Truth and Juſtice of it. How they have ſucceeded, I'll here examine.

Mr. *Hatton* ſuppoſes that the Intereſt of the Sum of the Debts from the Time of the Queſtion to the equated Time, ought to be equal to the Sum of the Intereſts of the ſeveral Debts from the Time of the Queſtion to their ſeveral Terms of Payment; and then, by an Example, ſhews that the Rule agrees to this Suppoſition. But this, inſtead of being a *Demonſtration,* is a plain begging of the Queſtion; for the Equity of that Suppoſition ought to have been firſt proved; againſt which there lies this obvious Reaſon, that the Debts bearing no Intereſt till after their Terms of Payments, the Conſideration of theſe Intereſts ſeems to be out of the Queſtion; and therefore this Suppoſition cannot be granted without Demonſtration. I ſhall preſently ſhew you other Reaſons againſt it; and in the mean time *obſerve,* that tho' it were right, yet Mr. *Hatton* has not demonſtrated his Rule; becauſe it is not enough to ſhew that a Rule gives a true Anſwer in one particular Caſe; a Demonſtration muſt comprehend all Caſes; and that the Rule is univerſally agreeable to the Suppoſition, I thus demonſtrate.

Take A for a Debt payable at the End of the Time t, and B for a Debt payable after the Time u: The equated Time muſt lie betwixt the two, and, by this *Rule,* it is $\frac{At + Bu}{A + B}$. Now the Intereſt of A for the Time t, is, (by *Probl.* 1. §. 1. *Part* 1.) Atr (r being the Rate of Intereſt for 1 *l.* in 1 Year, or Month, as the given Time t is). The Intereſt of B for the Time u is Bur, and the Sum of both is $Atr + Bur$. Alſo the Intereſt of $A + B$ for the equated Time $\frac{At + Bu}{A + B}$, is $\overline{A + B} \times r \times \frac{At + Bu}{A + B} = r \times \overline{At + Bu} = Atr + Bur$, as before: Therefore the Rule is univerſally agreeable to the Suppoſition.

Mr. *Cocker* ſuppoſes that the equated Time is right, when the Sum of the Intereſts of the ſeveral Debts that are payable before the equated Time, from their Terms to that Term, is equal to the Sum of the Intereſts of the Debts payable after the equated Time, from that Time to their Terms. The Agreement of the Rule with this Suppoſition, will eaſily appear by comparing it with the preceding Suppoſition: For they are manifeſt Conſequences one of another; and therefore the Rule that agrees to one muſt alſo agree to the other. Or we may eaſily prove the Agreement of the Rule with this laſt Suppoſition, after the ſame manner as I have done the former; but this I ſhall paſs over, and demonſtrate the Injuſtice of the Suppoſition, which Mr. *Cocker* endeavours to prove to be right by this Argument, *viz.* That what is gained by keeping ſome of the Debts after they are payable, is loſt by paying of others before they are due: But this is falſe; for tho' by keeping a Debt unpaid after it is due there is gained the Intereſt of it for that Time (becauſe the Creditor has a juſt Demand for that Intereſt), yet by paying a Debt before it is due the Payer does not loſe the Intereſt for that Time, but only the Diſcompt he could juſtly demand, which is leſs than the Intereſt (as former Rules ſhew). Thus we have a ſecond and more poſitive Demonſtration of the Error of this Rule. And the Fallacy in this laſt way of founding the Rule does plainly lead us to the true Foundation for a Solution of the *Problem.* But the explaining of that I ſhall leave to the laſt, and firſt explain the Methods of ſome other Authors.

Method 2. Mr *Kerſey* was the firſt, as I know, who obſerved the Error of the preceding Rule, and has given us another in its place, with a more probable Foundation, tho' ſtill wrong.

The Subſtance of Mr. *Kerſey*'s Argument againſt the former Rule, is this, That there being no Conſideration had of different Rates of Intereſt, the Anſwer will ſtill be the ſame, whatever Rate be ſuppoſed; which he affirms to be wrong for this Reaſon, That to equal the Time ſo as neither Debtor or Creditor be a Loſer, does imply ſome Rate of Intereſt; ſince otherwiſe any Day may be aſſigned for one entire Payment. His Meaning I take to be this, *viz.* that the Loſs which either Party can ſuſtain, muſt ariſe from the Intereſt of Money paid before or after it is really due; ſo that if there is no ſuch thing as Intereſt, there can be no Loſs, at whatever Time the Whole is paid; and if the Loſs depends upon the Conſideration of Intereſt, then different Rates of Intereſt muſt have different Effects, and occaſion different Solutions to the Queſtion: Therefore, concludes he, if ſome Rate of Intereſt be implyed, Equity requires that the Preſent Worth of the total Sum payable at one entire Payment, Diſcompt being made at that Rate of Intereſt, be equal to the Sum of the Preſent Worths of the particular Debts, diſcompting at the ſame Rate; upon which Foundation he gives us this Rule.

Kerſey's Rule. Find the Preſent Worth of each Debt, diſcompting from the Time at which it is payable; then find in what Time the Total of theſe Preſent Worths would amount to the Total of the Debts; that is the Time ſought.

Example: Suppoſe 300 *l.* due after 4 Months, 100 *l.* after 6 Months, and 100 *l.* after 12 Months: The equated Time diſcompting 6 *per Cent.* is 5.952, *&c.* Months. Found thus: Preſent Worth of 300 *l.* for 4 Months is 294.117, *&c. l.*; of 100 *l.* after 6 Months it is 97.087, *&c. l.*; and of 100 *l.* after 12 Months it is 94.339, *&c.* the Total is 485.544, *&c.* which will amount to 500 *l.* in 5.952, *&c.* Months, which is a little leſs than the Anſwer the preceding *Rule* would give.

The Argument for this *Rule* may be formed thus: If the Total of the Preſent Worths preſently paid, is equivalent to the Sum of all the Debts paid at once, at the Time found, or alſo to the Sum of theſe Debts paid ſeverally at their Terms, then theſe two laſt Methods of making Payment muſt be equal: Conſequently the *Rule* is true, which is founded upon that Equality.

Now tho' this *Rule* has a more apparent and probable Ground of Equity than either of theſe alledged for the former, if the Rule for the Diſcompt of Simple Intereſt be allowed to be juſt and true; yet the Foundation is not ſtrictly good, as will fully appear from the laſt *Method* after explained, whoſe Foundation is unexceptionable.

In the mean time, I make this Objection to it, *viz.* That the Anſwer will be different, if the Queſtion is propoſed at different Times, before the Term at which the firſt payable Debt falls due; *i. e.* if you alter all the given Times, ſo as there be the ſame Diſtances or Differences among them. Now this is a manifeſt Sign that the Rule is not good; becauſe there is no Reaſon in the Nature of the thing, why the Anſwer ſhould be different. For *Example*; One Debt is payable 3 Years hence, and another 7 Years hence: I ſuppoſe again, that the Calculation of an equated Time for the ſame Debts is propoſed 2 Years after this (ſo that the one Debt is then payable after 1 Year, and the other after 5 Years), it is plain that the equated Term ought to fall upon the ſame Day as it would have done in the other Caſe (*i. e.* the Time laſt found muſt be 2 Years leſs than before), becauſe the Debtor and Creditor continue in the ſame Situation to one another, neither of them gaining or loſing by the two Years that are paſt ſince the Queſtion was formerly propoſed, there being nothing payable all that Time; ſo that the Determination of the *Problem* depends only upon the Conſideration of the Diſtances of Time among the Terms of Payment of the ſeveral Debts: The Conſideration of any Preſent Worths, except of thoſe Debts whoſe Terms are beyond the equated Time, being out of the Queſtion.

Method 3. Sir *Samuel Moreland,* after his pretended Correction of *Kersey's* Rule for the Present Worth of Annuities at Simple Interest, gives us, in consequence of that, another Rule than *Kersey's* for the Equation of Payments, which is this;

Rule. Find the Amount of each Debt, with Interest from its Term of Payment to the Term at which the Debt last payable falls due; then find in what Time the Sum of all the Debts will amount to the Sum of the Amounts found; the Difference of that Time and the Time to which these Amounts were carried, is the Time sought.

As to this Rule, I *observe,* 1. That, being founded on his Reasoning about Present Worths of Annuities, which I have already shewn to be false, it must therefore also be false; and indeed more so than *Kersey's* Rule.

But, 2. *observe,* That tho' it is differently expressed, yet it is in effect the very same as the 1st *Rule*; as will easily appear by comparing it with *Hatton's* Way of founding that *Rule.* Or you may also prove its Coincidence with that *Rule* after the same Manner as I have done the Agreement of *Hatton's* Supposition with it. These things I leave to your own Exercise.

Before I explain the true Method of solving this *Problem,* I must take notice of what Mr. *Ward,* in his *Key to Interest,* has said upon it. After he had mention'd the first *Rule* or *Method* above explained, he says,

" I shall pass over all the Arguments made use of, *pro* and *con,* by Mr. *John Kersey* and " Sir *Samuel Moreland,* and other Authors, about the Erroneousness of this Rule, as also " the Rules they lay down instead of it, and shall only proceed to shew how the true " equated Time may be found from what hath been already done and proved."

And his Rule is this: Find the Discompts of all the particular Debts separately, according to the Distances of their Terms of Payments; divide their Sum by the Product of the Present Worth of all the Debts, multiplied by the Rate of Interest (or Interest of 1 *l.* for 1 Year); the Quote is the equated Time.

Now from what is premised we had good Reason to expect a new Rule, quite different in its Effects from either *Kersey's* or *Moreland's*; and yet its the very same in effect as *Kersey's,* only differently expressed; which may be easily demonstrated. But this I leave also to your own Exercise, and come to the true Solution.

Method 4. I come now to explain a Method, which no body, as I know, has hitherto published, and which has strict Equity upon its Side, as far as Simple Interest has it in any Case.

To explain this Method, we must reflect upon the Reason brought against *Cocker's* Foundation of the 1st *Rule,* which was this, *viz.* That the Sum of the Interest of Debts kept in the Debtors Hands after they are due to the equated Time, and the Sum of the Interests of Debts paid before they are due from the equated Time to the several Terms, is not a just Balance of Gain and Loss: Because, tho' the Debtor gains the Interest of those he keeps after they are due, yet he loses only the Discompt upon those he pays before they are due, which is less than the Interest; and therefore the Creditor has a just Exception to this Rule. Hence we see that as the true Foundation upon which the Time can be equated, is an Equality of Gain and Loss, so the Gain consists in the *Interests* of the Debts kept after they are due, and the Loss in the *Discompts* of those that are paid before they fall due: Wherefore, such a Time must be found, as that the Sum of the Interests of the Debts due before the equated Time from their Terms to that Time, be equal to the Sum of the Discompts of those Debts that are payable after the equated Time, from this to their several Terms. For which this is the

Rule. 1. Suppose two Debts: Then expressing the Debts in one Denomination (as *l.* and decimal Parts), find 1 Year's Interest of the Debt that is first payable (expressed in the

ſame Denomination) by which divide the Sum of the Debts, and add the Quote to the Sum of the Times, (taken in Years and decimal Parts, from the time of the Queſtion to the Terms of Payment of the two Debts): Call this the 1ſt Number found, Again, multiply each Debt by its Time, and divide the Sum of the Products by 1 Year's Intereſt of the 1ſt payable Debt, which Quote add to the Product of the two Times, and call this Sum the 2d Number found; Subtract this 2d Number found from ¼ of the Square of the 1ſt Number found; and out of the Difference extract the Square Root; which Root being added to or ſubtracted from the half of the 1ſt Number found, the Sum or the Difference will be the Time ſought, in the ſame Denomination: And to know which is the Anſwer, you muſt apply both according to the Conditions of the Queſtion. Thus, If you take the Sum, then, if that is a Time greater than the Time to the laſt payable Debt, the Difference will be the Time ſought: Or if you take the Difference, and that be leſs than the Time to the Term of the firſt payable Debt, the Sum is the Time ſought. But having tried either, and found it betwixt the Terms of Payment of the two Debts, you may try if the other does not caſt it beyond the laſt, or within the firſt Term; for in this Caſe, that which was firſt tried is the Anſwer: But if both give Times betwixt the two given Terms, then you muſt examine which of them will make an Equality of Intereſt and Diſcompt (in the Manner above explained).

Example: There is 100 *l.* payable 1 Year hence, and 105 *l.* payable 3 Years hence: What is the equated Time, allowing Simple Intereſt at 5 *per Cent.* per Annum? ***Anſwer***: 2 Years.

The *Operation* according to the *Rule*.

Debts $\left\{ \begin{array}{l} 100\ l. \\ 105 \end{array} \right.$ Intereſt of 100 *l.* for 1 Year is 5 *l.*

Sum 205

One Debt 100
Its Time 1
Product 100

The other Debt 105
Its Time 3
Product 315

205 divided by 5, the Quote, is 41
which added to the Sum of the Times 4
The Sum is 45
the 1ſt Number found.

The Sum of theſe Products is 415, which divided by 5, quotes 83, which added to 3 (the Product of the two Times 1 and 3) the Sum is 86, the 2d Number found.

Then the Square of 45 (the 1ſt Number found) is 2025, whoſe ¼ Part is 506.25; from which taking 86 (the 2d Number found) the Difference is 420.25, whoſe Square Root is 20.5 Now this being added to 22.5 (the ½ of the 1ſt Number found) the Sum is 43, which cannot be the Anſwer of the Queſtion, becauſe it is greater than the Diſtance of the laſt Term in the Queſtion; wherefore I take the Difference of 20.5 and 22.5; it is 2 Years, the true Anſwer: Which we prove alſo by Application: For this Time being exactly in the middle betwixt the two given Times, the Intereſt of 100 *l.* for 1 Year is equal to the Diſcompt of 105 *l.* for 1 Year; each of them being 5 *l.*

2. If there are more Debts than two, Find an equated Time for the two that are firſt payable: then conſider their Sum as a Debt payable at that equated Time, and find another equated Time for that Debt, and the next of the given Debts, and ſo on, through them all.

For the *Demonſtration* of the 1ſt Part of this *Rule*, thoſe who have not ſtudied the Method of Demonſtration uſed in the preceding Parts of this Work cannot poſſibly underſtand

ſtand it; and to thoſe who have, I need only lay down the general Steps of the Demonſtration, and leave Particulars to themſelves. Thus,

Let the Debt firſt payable be called	d	The laſt payable Debt	D
The Diſtance of its Term of Payment	t	The Diſtance of its Term	T

The Diſtance of the equated Time - - - - - - - - x
The Rate of Intereſt, or 1 Year's Intereſt of 1 *l.* r

The Diſtaance of the Time t and x is $x-t$
The Diſtance of the Time T and x is $T-x$ } For x lies betwixt them.

Then the Intereſt of d, for the Time $x-t$, is $dr\times\overline{x-t}$, or $drx-drt$, (dr being 1 Year's Intereſt of d) and $\frac{DTr-Drx}{1+Tr-rx}$ is the the Diſcompt of D for the Time $T-x$ ($Tr-rx$ being the Intereſt of 1 *l.* for that Time, which is conſequently the *Diſcompt* of $1+Tr-rx$ for the ſame Time): Wherefore we have, from the Nature of the Queſtion, this *Equation, viz.* $drx-drt=\frac{DTr-Drx}{1+Tr-rx}$; which being reduced according to the Common Rules of *Algebra*, comes at laſt to this Equation, $\overline{T+t+\frac{D+d}{dr}}\times x-x^2=\frac{DT+dt}{dr}+Tt$. Now $T+t+\frac{D+d}{dr}$ is what in the *Rule* I have called the 1ſt Number found, and $\frac{DT+dt}{dr}+Tt$ the 2d Number found: Call the 1ſt, a, and the 2d, s, then will the Equation ſtand thus, $ax-x^2=s$; and, by *Problem* 6. *Chap.* 2. *Book* III. we have $x=\frac{a}{2}+\text{ or }-\overline{\frac{aa}{4}-s}|^{\frac{1}{2}}$, which is the preſent *Rule*.

Some will be ready to think I have taken too much Pains about a Queſtion of no great Moment in Buſineſs; and that in common Affairs any of the *Rules* may do without any conſiderable Error. I do freely own the laſt Part: Yet I believe what is done may prove a very uſeful Exerciſe for a Student; and in all Caſes I think Truth is worth the knowing.

ADDITION to *Chap.* 8. *Book* VI.

Of the Alligation, *or* Mixture *of Bodies, with reſpect to their Specifick Gravities.*

DEFIN. WHEN of Bodies of different kinds of Matter there are equal *Maſſes* or *Bulks*, for Example, a Sphere of the ſame Diameter (or an equal Cube) their different Weights are called their *Specifick Gravities*: And tho' the Weight is greater or leſſer, as the Bulk is, yet being ſo always proportionably, the Weights under any equal Bulk whatever are called the *Specifick Gravities*; ſo that the more ſtrict Definition is, the *Proportion of Weights under the ſame Bulk.*

Some Bodies of different Specifick Gravities are capable of being mixt and incorporated with one another, as are ſome Liquors and Metals; and as the Mixture muſt neceſſarily make the compound Body of a Specifick Gravity different from that of all, or all but one, of the Bodies mixt; ſo the ſame Problems ariſe about their *Mixture*, with reſpect to their Specifick Gravities, and the Proportion of the Quantities mixt, as before with reſpect to the

the Quantities and Prices of Liquors, and other things; and which are to be ſolved by the ſame Methods of Operation.

But in order to underſtand the Application, there is one Propoſition to be firſt explain'd, which is this, *viz.*

The *Bulks* (or Numbers of cubical Inches) of two Bodies of equal Weight (which I ſhall call their *Specifick Bulks*), are reciprocally proportional to their Specifick Gravities.

Thus: Call two Species of Bodies A and B; if the Specifick Gravity of A is to that of B, as 1 to 2, then the Bulk of A is to that of B of the ſame Weight as 2 to 1. For Example; If a Body of a cubical Inch of one Metal is 2 Ounces, and a Body of a cubical Inch of another Metal is 1 Ounce, then it's plain that the Half Bulk of the firſt will be as heavy as the Whole of the other. Or if the Specifick Gravities are as 5 to 3, then $\frac{3}{5}$ of the firſt is as heavy as the Whole of the other. And ſo in any other Proportion.

Hence again; If the Specifick Gravities of 2 or more Bodies are expreſſed by any Numbers whatever, their *Specifick Bulks* are expreſſed by the Reciprocals of theſe Numbers, thus; If the *Specifick Gravities* of 3 Bodies are as 5 . 7 . 8, their *Specifick Bulks* are as $\frac{1}{5} \cdot \frac{1}{7} \cdot \frac{1}{8}$; and if the *Specifick Gravities* are as $\frac{2}{3} \cdot \frac{5}{7} \cdot \frac{3}{4}$, the Bulks are as $\frac{3}{2} \cdot \frac{7}{5} \cdot \frac{4}{3}$; becauſe any two Numbers are proportional with their Reciprocals, taken reciprocally. For the ſame Reaſon, however the Reciprocal Bulks are expreſſed, the Specifick Gravities are expreſſed by the Reciprocal Numbers.

Theſe things being premiſed, I ſhall next apply the Rules of *Alligation* to the *Mixture* of Bodies, with reſpect to their *Specifick Gravities* (or Bulks, which is in effect the ſame thing), thus:

Caſe 1. If there are given the *Specifick Bulks* of ſeveral Bodies (or *Specifick Gravities*, by which the Bulks may be found), together with the Quantities or Weights of each put into a Mixture; to find the *Specifick Bulk*, or *Gravity* of the *Mixture*; it is done after the manner of *Caſe* 1. *Chap.* 8 thus:

Rule. Find the total Bulk (or Numbers of cubical Inches) of each Quantity, by the Specifick Bulk; add all theſe Bulks into one Sum, and alſo the Quantities taken, and ſay, as the total Quantity to the total Bulk, ſo is an Unit of the Denomination by which the Specifick Bulks are expreſſed, to the Specifick Bulk ſought.

Example: Suppoſe 10 Ounces of Metal, whoſe Specifick Bulk is 3 (*i. e.* 3 cubical Inches to 1 Ounce) are mixed with 14 Ounces of another whoſe Specifick Bulk is 2: What is the *Specifick Bulk of the Mixture? *Anſwer*, $\frac{12}{29}$: Which I find thus: If 1 Ounce makes 3 cubical Inches; then (by the *Rule of Three*) 10 Ounces make 30; and if 1 Ounce makes 2, then 14 Ounce make 28; and, laſtly, if 24 (*i. e.* 10 and 14) Ounces make 58 (*i. e.* 30 and 28) cubical Inches, 1 Ounce makes $\frac{12}{29}$. Then the Specifick Gravities of the Metals that are mixed, and of the Compound, are $\frac{1}{10}, \frac{1}{14}, \frac{29}{12}$.

The *Reaſon* of the *Operation* is obvious.

Obſerve, If the given Quantities have all one Denomination, or are reduced to one, whatever that is, they may then be conceived to be all of that Denomination to which the *Specifick Bulks* are ſuppoſed to refer; ſince the Proportion is ſtill the ſame: So hereby the Work becomes eaſier; for we have nothing to do but multiply the Quantity of each Simple (after they are reduced to one Denomination) by its Specifick Bulk, and divide the Sum of the Products by the Sum of the Quantities; as it is in the preceding *Example*.

Caſe 2. Having the *Specifick Bulks* (or Gravities by which the Bulks are found) of ſeveral Bodies to be mixt, and the Specifick Bulk (or Gravity) to which a Mixture of theſe Bodies is to be reduced; To find the proportional Quantities to be taken of each, that the Mixture may bear the given Rate of Specifick Bulk (or Gravity).

Rule. Take the given Specifick Bulks of the Bodies to be mixt, link them together, then take and place their Differences from the Specifick Bulk of the Mixture, the ſame way as taught in Alligation alternate, and you have the Anſwer. *Exam-*

Cub. Inch.		lb Weight.
	2	2
3		
	5	1

Example. Suppose two Metals, whereof the one has 2 cubical Inches to 1 lb Weight, and the other 5; What Proportion of Weight of each must be taken to make the Mixture 3 cubical Inches to 1 lb *Ans.* 2 lb of the 1st to every 1 of the other, as in the Margin.

To shew the *Reason* of this, we must premise this Truth, *viz.* That of three Numbers, if the Difference of the least Extreme and middle Term is multiplied by the greater Extreme; and the Difference of the greater Extreme and middle Term is multiplied by the lesser Extreme; the Sum of their Products is equal to the Product of the Sum of the two Differences by the middle Term; as is plain in the annexed Scheme, where the three Numbers being a, b, c, their Differences, according to the *Rule*, are $c-b$, and $b-a$, whose Sum is $c-a$; then the Products mentioned are $ac-ab$, and $bc-ac$, whose Sum is $bc-ab$ equal to $c-a$ multiplied by b.

$$\begin{array}{lll} a & \times\overline{c-b} & = ac-ab \\ b & & \\ c & \times\overline{b-a} & = bc-ac \\ \hline b & \times\overline{c-a} & = bc-ab \end{array}$$

Now to apply this to the *Demonstration* of the *Rule*, Consider in the above *Example*, that 2 lb, whereof each is 2 cubical Inches, make 4 cubical Inches; and 1 lb of 5 cubical Inches, is 5 cubical Inches; the Sum is 9 cubical Inches; so that the Mass of Mixture is 3 lb, and the Bulk is 9 cubical Inches; therefore, the Specifick Bulk, or Bulk of 1 lb of the Mixture is the 3d Part of 9, *viz.* 3 cubical Inches, the proposed Specifick Bulk of the Mixture; which is therefore justly proportioned: Or thus; the Product of the total Weight mixt 3 lb multiplied by the given Specifick Bulk 3, is 9, equal to the Sum of the Products of the particular Quantities multiplied by their respective Specifick Bulks; therefore the Solution must be true; and, by the universal Truth premised, it must be so in all Cases of two Bodies mixt; and consequently of any Number.

Observe, 1. The same Method of *Demonstration* may be applied to the Mixture of any other things, as Liquors, *&c.*

2. All the same kind of Questions made above with respect to Liquors, may also be made with respect to Specifick Gravities, or Bulks of Bodies.

3. By this Rule is the famous Question solved about *Hiero*, King of *Syracuse*'s Crown: He gave a certain Quantity of pure Gold to make a Crown; but suspecting the Goldsmith had mixt Silver with it, he desired *Archimedes* to discover it if possible; who did it by this Means: When he went into a bathing Tub, he reflected that every Body immersed in Water must put as much Water out of its Place as is equal in Bulk to it self: Therefore he took a Quantity of pure Gold, and another of Silver, each of the Weight of the Crown; or, as some say, caused a Crown to be made of pure Gold, and another of Silver, each of the same Weight with the 1st Crown, and measured their Specifick Bulks by the Bulks of the Quantities of Water put out of its Place by the Immersion of each of the 3 Crowns (or of the suspected Crown, and each of the Masses of pure Gold and Silver; which would put out the same Quantity of Water, whatever Shape they were in) and by Comparison of these 3 Specifick Bulks, he found how much Gold and Silver was in the mixt Crown; which may be done after the manner of *Qu* 3. *Page* 569. thus: We have the Specifick Bulks of the Mixture, and of the two Simples; by which, finding the Proportion of Weights that brings them to that Specifick Bulk, if their Sum is equal to the Weight of the mixt Crown, then the Proportion of Gold and Silver in it is discover'd; by which the Gold and Silver separately are easily found: And if the Sum is not the same as the Weight of the Crown, yet that Weight being divided into the same Proportion as the Parts of that Sum, solves the Question.

FINIS.

ERRATA.

Page	Line	For	Read
101.	42.	$a+bc$	$\overline{a+bc}$
111.	10.	$\frac{1}{54}$	$\frac{18}{54}$
131.	8.	$\frac{3}{5}$	$\frac{5}{3}$
239.	12.	$-\frac{2a}{2}$	$=\frac{2a}{2}$
288.	45.	*Theor.* IV.	*Theor.* VII.
383.	34.	after *oddly even only*, add, *above* 2.	
416.	22.	$-\frac{n^{m+2}+n^{m+1}}{2}$	$-\frac{n^{m+2}+n^{m+1}}{2}$
424.	20.	3	8
472.	11.	$10m\times\overline{10m-1}$	$10^{m}\div\overline{10^{m}-1}$
ib.	ib.	$l\times l^{2}$	*Extreme.*
486.	11.	2d or 3d	2 or 3.
ib.	14.	$a^{n}\times a^{m}$	$a^{m}\times a^{m}$
ib.	ib.	$a^{2m}\div a$	$a^{2m}\div a^{n}$
488.	31.	$\overline{n-1a}$	$\overline{n-1}\times a$
ib.	40.	$n-10$	$\overline{n-10}$
489.	To the Scheme of Numbers, after *Line* 18. add this Line of Numbers, *viz.* 0 : 1 : 2 : 3 : 4 : 5 : 6 : 7 : 8		
499.	In the Scheme, after *Line* 25, for 2.9332209, *read* $\bar{2}$.9332209.		
514. 515.	25. 13.	$\times B-\overline{A-1}$	$\times\underline{B-\overline{A-1}}$
ib.	ib.	$\times B-A$	$\times\overline{B-A}$
ib.	33.	$A+1$	$\underline{A+1}$
517.	14.	$\frac{n-2}{1}$	$\frac{n-1}{2}$
545.	2.	3 *oz.* 14 *dr.*	1205 ℔ 3 *oz.* 14 *dr.*
551.	21.	18148	18338
559.	12.	*what*	*which*
616.	35.	*Periods*	*Pounds.*

In the *Preface*, at the Foot of *Page* ix. for *Chap.* 11. read *Chap.* 10. §. 5.